QCD, top and LHC
Lecture I: QCD, Feynman rules, and asymptotic freedom

ICTP Summer School on Particle Physics, June 2013

Keith Ellis

ellis@fnal.gov

Fermilab,

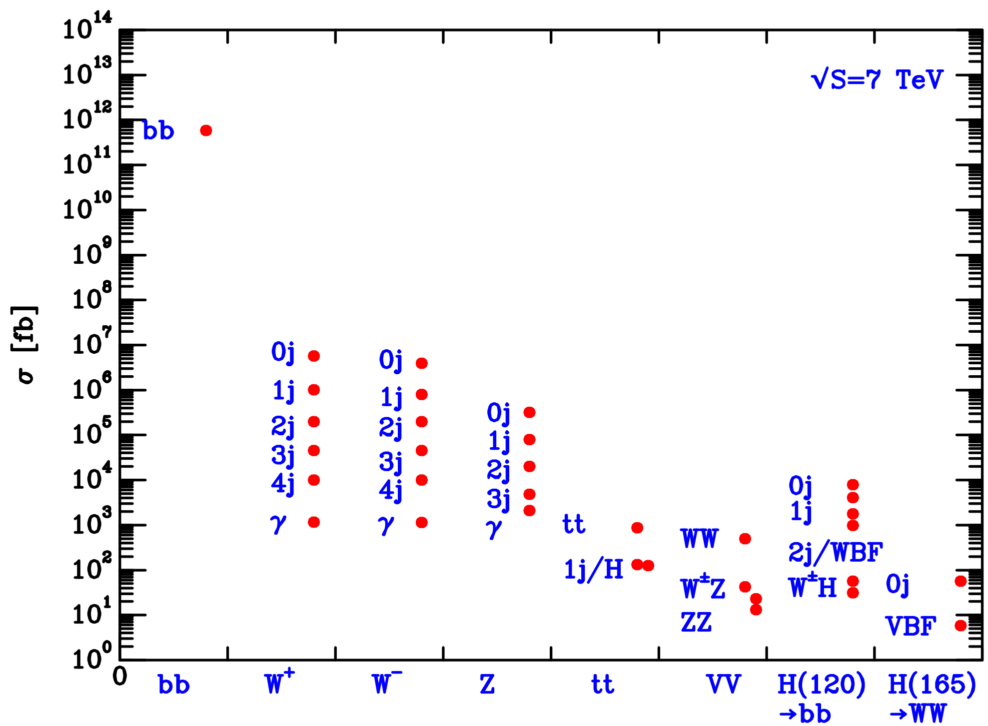
Slides available from http://theory.fnal.gov/people/ellis/Talks/

QCD, Feynman rules, and asymptotic freedom

- Overview of Cross sections
- Motivation for colour
- Lagrangian of QCD
  - Local gauge invariance
  - Derivation of Feynman rules
- Running coupling
  - Running coupling
  - Asymptotic freedom
  - Measurements of α_S
  - Infrared slavery

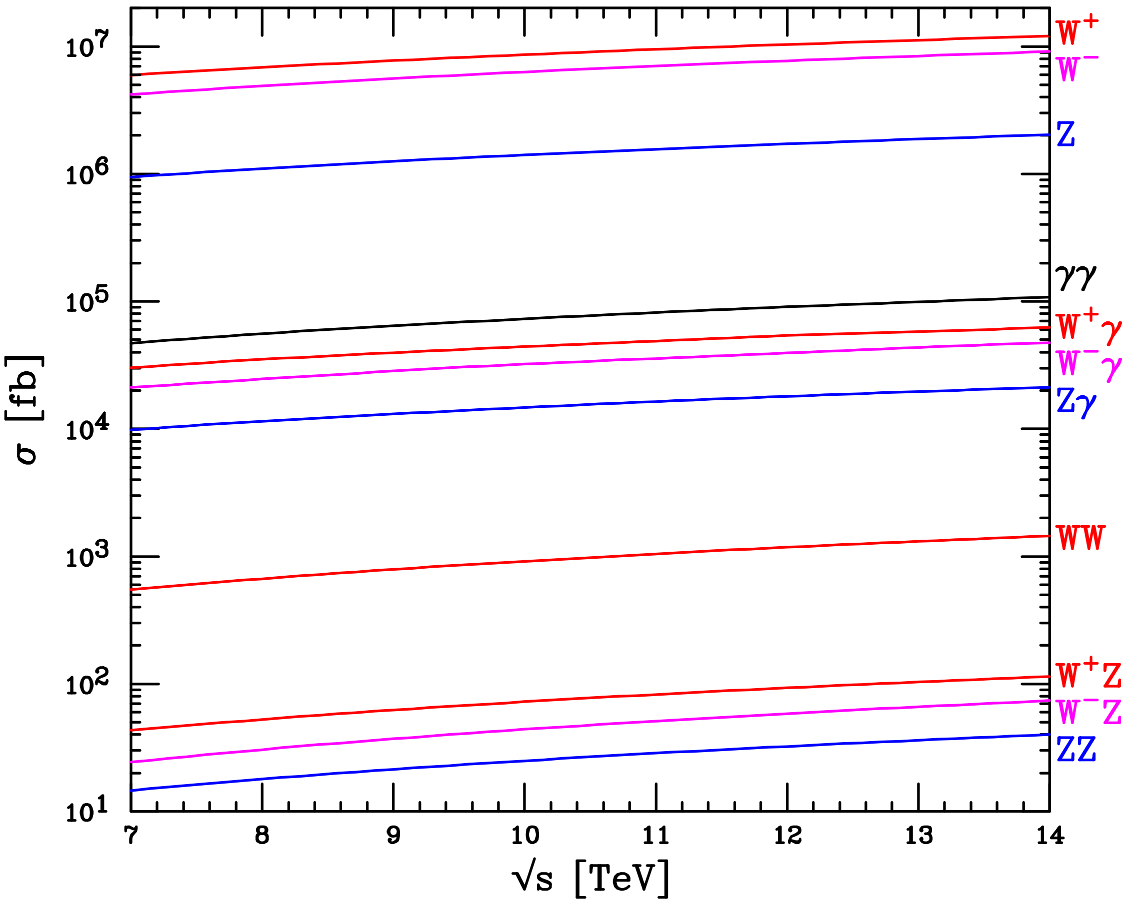
Overview

- These lectures illustrate how we can make the best possible predictions for QCD processes at the LHC.
- Both CMS and ATLAS at the LHC have accumulated about 23fb^{-1} at $\sqrt{s} = 8$ TeV.
- The number of events produced is the product of the integrated luminosity and the cross section.
- They will restart in 2015 at $\sqrt{s} = 13$ TeV.
- Eventual goal of 3000 fb^{-1} at $\sqrt{s} = 14$ TeV by about 2030.
- No branching ratios to observable states are included in the plot below



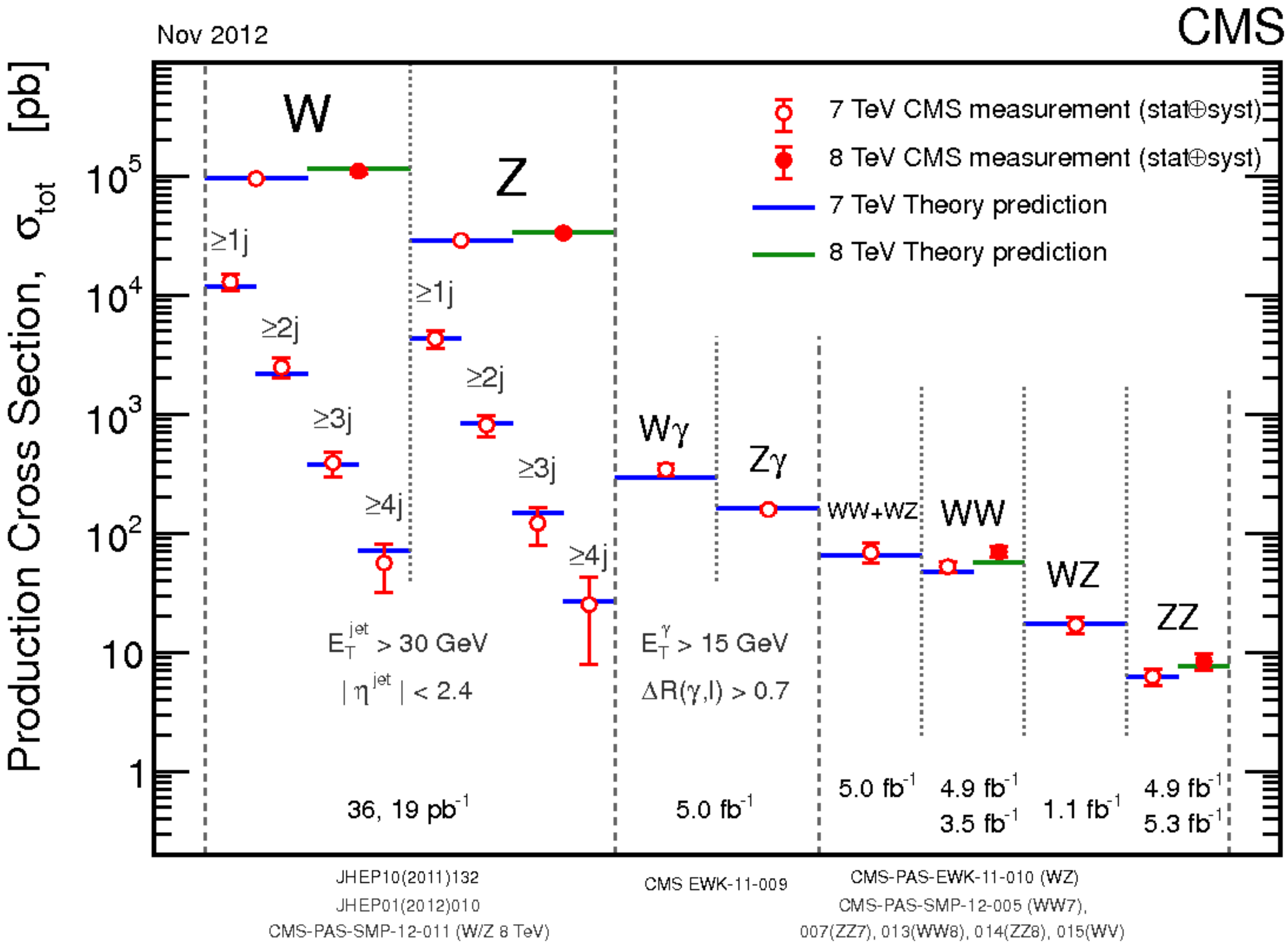
Overview II

- Many of the most interesting processes involve the production of (pairs of) vector bosons.
- These processes display the cancellations inherent in gauge theory and have an important role as a background to the Higgs.
- Plot includes branching ratio to one generation of charged fermions and cuts on the photon p_T where applicable.



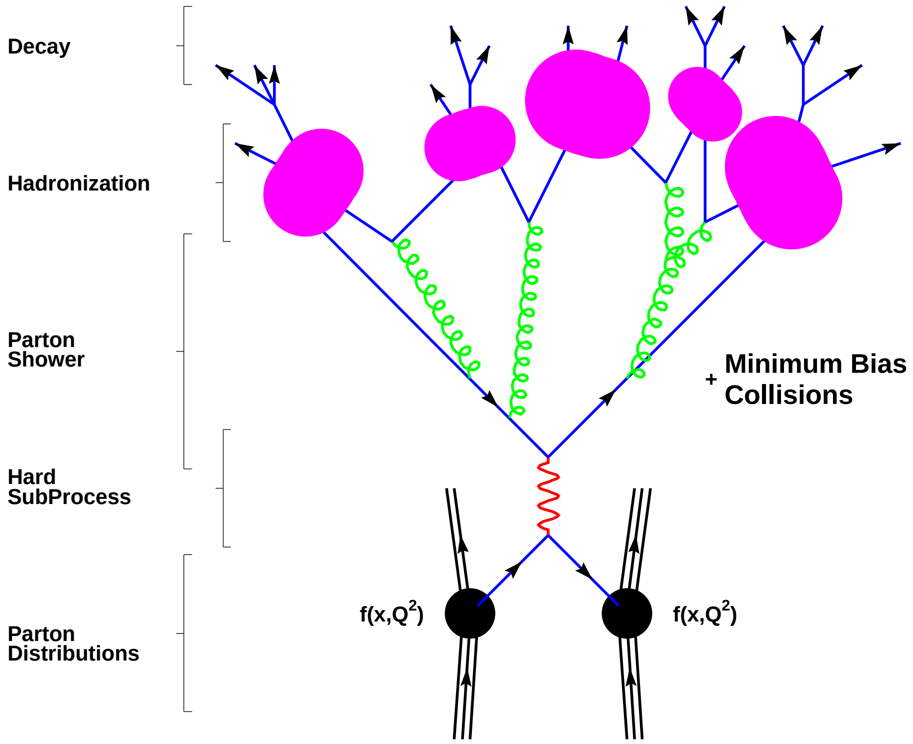
Overview III

- Many of these processes have been observed at the LHC $\sqrt{s} = 7, 8$ TeV.



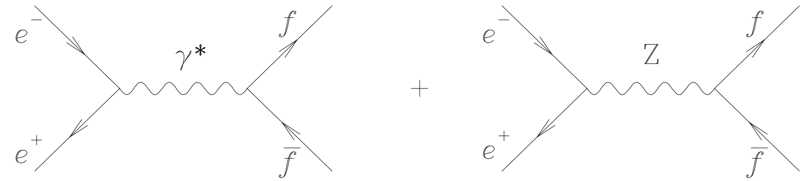
Overview IV

- Several different pieces required for a full description of the event.
- I shall concentrate on methods for the hard scattering cross section, parton distribution functions and the parton shower.
- All of these have aspects that can be derived from a perturbative treatment of the QCD Lagrangian.



Motivation for Colour SU(3)

- Consider the ratio R of the e^+e^- total hadronic cross section to the cross section for the production of a pair of point-like, charge-one objects such as muons.
- The virtual photon excites all electrically charged constituent-anticonstituent pairs from the vacuum.



- At low energy the virtual photon excites only the u, d and s quarks, each of which occurs in three colours.

$$R = N_c \sum_i Q_i^2 = 3\left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right] = 2\,.$$

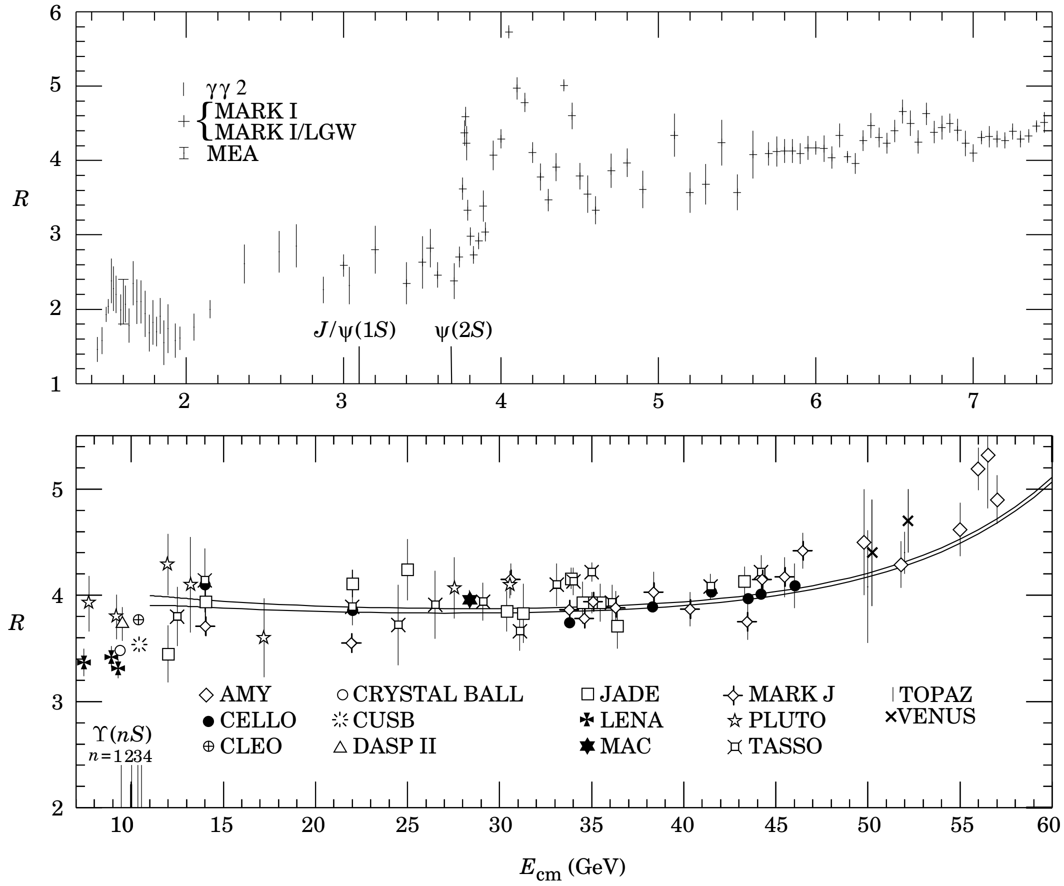
- For centre-of-mass energies $E_{\rm cm} \geq 10$ GeV, one is above the threshold for the production of pairs of c and b quarks, and so

$$R = 3\left[2 \times \left(\frac{2}{3}\right)^2 + 3 \times \left(-\frac{1}{3}\right)^2\right] = \frac{11}{3}\,.$$

Data

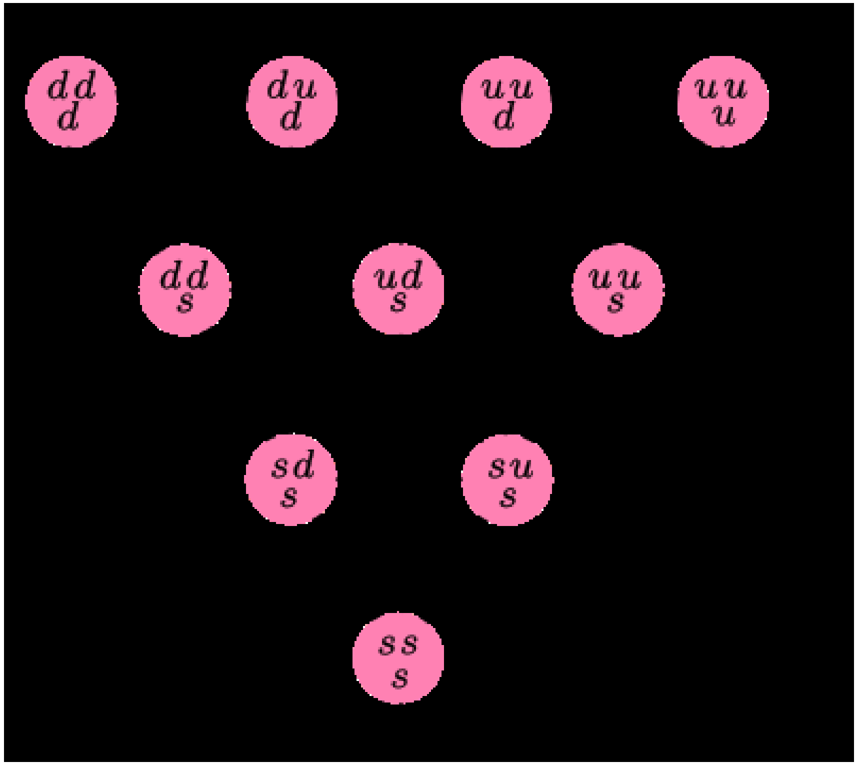
The data on R are in reasonable agreement with the prediction of the three colour model.

$$R_{e^+e^-} = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



Colour SU(3) and spectroscopy

- The observed baryons are interpreted as three-quark states.
- The quark constituents of the baryons are forced to have half-integral spin in order to account for the spins of the low-mass baryons.
- The quarks in the spin-$\frac{3}{2}$ baryons are then in a symmetrical state of space, spin and SU(3)$_f$ degrees of freedom.
- However the requirements of Fermi-Dirac statistics imply the total antisymmetry of the wave function.



- We introduce the colour degree of freedom: a colour index a with three possible values (usually called red, green, blue for $a = 1, 2, 3$) is carried by each quark.
- The baryon wave functions are totally antisymmetric in this new index.

Quark	Charge	Mass	Baryon Number	Isospin
u	$+\frac{2}{3}$	~ 4 MeV	$\frac{1}{3}$	$+\frac{1}{2}$
d	$-\frac{1}{3}$	~ 7 MeV	$\frac{1}{3}$	$-\frac{1}{2}$
c	$+\frac{2}{3}$	$\sim 1.5\ \mathrm{GeV}$	$\frac{1}{3}$	0
s	$-\frac{1}{3}$	~ 135 MeV	$\frac{1}{3}$	0
t	$+\frac{2}{3}$	$\sim 173\ \mathrm{GeV}$	$\frac{1}{3}$	0
b	$-\frac{1}{3}$	$\sim 5\ \mathrm{GeV}$	$\frac{1}{3}$	0

- The group of colour transformations is SU(3), with the quarks q_a transforming according to the fundamental representation
- Why does this new degree of freedom not lead to a proliferation of states?
- We hypothesize that only colour singlet states can exist in nature.
- For a baryon the colour singlet state is totally antisymmetric $(|a\rangle|b\rangle|c\rangle + |b\rangle|c\rangle|a\rangle + |c\rangle|a\rangle|b\rangle - |b\rangle|a\rangle|c\rangle - |a\rangle|c\rangle|b\rangle - |c\rangle|b\rangle|a\rangle)/\sqrt{6}$

Lagrangian of QCD

- Feynman rules for perturbative QCD follow from Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^A_{\alpha\beta} F^{\alpha\beta}_A + \sum_{\text{flavours}} \bar{q}_a (i\not{D} - m)_{ab} q_b + \mathcal{L}_{\text{gauge-fixing}}$$

$F^A_{\alpha\beta}$ is field strength tensor for spin-1 gluon field $\mathcal{A}^A_\alpha$,

$$F^A_{\alpha\beta} = \partial_\alpha \mathcal{A}^A_\beta - \partial_\beta \mathcal{A}^A_\alpha - g f^{ABC} \mathcal{A}^B_\alpha \mathcal{A}^C_\beta$$

Capital indices A, B, C run over 8 colour degrees of freedom of the gluon field. Third 'non-Abelian' term distinguishes QCD from QED, giving rise to triplet and quartic gluon self-interactions and ultimately to asymptotic freedom.

- QCD coupling strength is $\alpha_S \equiv g^2/4\pi$. Numbers f^{ABC} ($A, B, C = 1, ..., 8$) are structure constants of the SU(3) colour group. Quark fields q_a ($a = 1, 2, 3$) are in triplet colour representation. D is covariant derivative:

$$\begin{aligned} (D_\alpha)_{ab} &= \partial_\alpha \delta_{ab} + ig \left(t^C \mathcal{A}^C_\alpha \right)_{ab} \\ (D_\alpha)_{AB} &= \partial_\alpha \delta_{AB} + ig (T^C \mathcal{A}^C_\alpha)_{AB} \end{aligned}$$

- t and T are matrices in the fundamental and adjoint representations of SU(3), respectively:

$$t^A = \frac{1}{2}\lambda^A, \quad [t^A, t^B] = i f^{ABC} t^C, \quad [T^A, T^B] = i f^{ABC} T^C$$

where $(T^A)_{BC} = -i f^{ABC}$. We use the metric $g^{\alpha\beta} =$ diag(1,–1,–1,–1) and set $\hbar = c = 1$. $\not{D}$ is symbolic notation for $\gamma^\alpha D_\alpha$. Normalisation of the t matrices is

$$\text{Tr}\; t^A t^B = T_R\, \delta^{AB}, \;\; T_R = \frac{1}{2}.$$

- Colour matrices obey the relations:

$$\begin{aligned} \sum_A t^A_{ab} t^A_{bc} &= C_F\, \delta_{ac}\,, \quad C_F = \frac{N^2-1}{2N} \\ \text{Tr}\; T^C T^D &= \sum_{A,B} f^{ABC} f^{ABD} = C_A\, \delta^{CD}\,, \quad C_A = N \end{aligned}$$

Thus $C_F = \frac{4}{3}$ and $C_A = 3$ for SU(3).

SU(3) generators

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The totally antisymmetric structure constants are given by, ($[\lambda^A, \lambda^B] = 2if^{ABC}\lambda^C$)

$$\begin{aligned} f^{123} &= 1 \\ f^{147} &= -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2} \\ f^{458} &= f^{678} = \frac{\sqrt{3}}{2}, \end{aligned}$$

and all other f^{ABC} not related to these by permutation are zero.

Gauge invariance

- QCD Lagrangian is invariant under local gauge transformations. That is, one can redefine quark fields independently at every point in space-time,

$$q_a(x) \to q'_a(x) = \exp(it\cdot\theta(x))_{ab}q_b(x) \equiv \Omega(x)_{ab}q_b(x)$$

  without changing physical content.

- Covariant derivative is so called because it transforms in same way as field itself:

$$D_\alpha q(x) \to D'_\alpha q'(x) \equiv \Omega(x) D_\alpha q(x) \ .$$

  (omitting the colour labels of quark fields from now on). Use this to derive transformation property of gluon field $\mathcal{A}$

$$\begin{aligned} D'_\alpha q'(x) &= \big(\partial_\alpha + igt\cdot\mathcal{A}'_\alpha\big)\Omega(x)q(x) \\ &\equiv (\partial_\alpha\Omega(x))q(x) + \Omega(x)\partial_\alpha q(x) + igt\cdot\mathcal{A}'_\alpha\Omega(x)q(x) \end{aligned}$$

  where $t\cdot\mathcal{A}_\alpha \equiv \sum_A t^A\mathcal{A}^A_\alpha$. Hence

$$t\cdot\mathcal{A}'_\alpha = \Omega(x)t\cdot\mathcal{A}_\alpha\Omega^{-1}(x) + \frac{i}{g}\big(\partial_\alpha\Omega(x)\big)\Omega^{-1}(x) \ .$$

- Transformation property of gluon field strength $F_{\alpha\beta}$ is

$$t \cdot F_{\alpha\beta}(x) \rightarrow t \cdot F'_{\alpha\beta}(x) = \Omega(x) t \cdot F_{\alpha\beta}(x) \Omega^{-1}(x) \ .$$

  Contrast this with gauge-invariance of QED field strength. QCD field strength is not gauge invariant because of self-interaction of gluons. Carriers of the colour force are themselves coloured, unlike the electrically neutral photon.

- Note there is no gauge-invariant way of including a gluon mass. A term such as

$$m^2 \mathcal{A}^\alpha \mathcal{A}_\alpha$$

  is not gauge invariant. This is similar to QED result for mass of the photon. On the other hand the quark mass term is gauge invariant.

Feynman rules

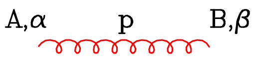
- Use free piece of QCD Lagrangian to obtain inverse quark and gluon propagators.
  - ⋆ Inverse quark propagator in momentum space obtained by setting $\partial^\alpha = -ip^\alpha$ for an incoming field. Result is on the next slide. The $i\varepsilon$ prescription for pole of propagator is determined by causality, as in QED.
  - ⋆ Gluon propagator impossible to define without a choice of gauge. The choice

$$\mathcal{L}_{\mathrm{gauge-fixing}} = -\frac{1}{2\,\lambda}\left(\partial^\alpha \mathcal{A}_\alpha^A\right)^2$$

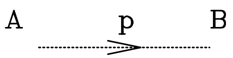
defines *covariant gauges* with gauge parameter λ. Inverse gluon propagator is then

$$\Gamma^{(2)}_{\{AB,\ \alpha\beta\}}(p) = i\delta_{AB}\ \left[p^2 g_{\alpha\beta} - (1 - \frac{1}{\lambda})p_\alpha p_\beta\right].$$

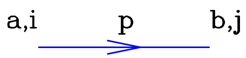
(Check that without gauge-fixing term this function would have no inverse.) Resulting propagator is in the table. $\lambda = 1$ (0) is *Feynman* (*Landau*) gauge.



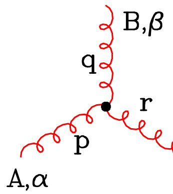
$$\delta^{AB}\ \left[-g^{\alpha\beta}+(1-\lambda)\,\frac{p^{\alpha}p^{\beta}}{p^{2}+i\epsilon}\right]\frac{i}{p^{2}+i\epsilon}$$



$$\delta^{AB}\ \frac{i}{(p^{2}+i\epsilon)}$$

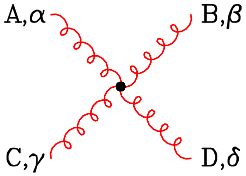


$$\delta^{ab}\ \frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

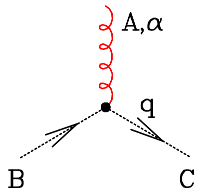


$$-g\ f^{ABC}[(p-q)^{\gamma}g^{\alpha\beta}+(q-r)^{\alpha}g^{\beta\gamma}+(r-p)^{\beta}g^{\gamma\alpha}]$$

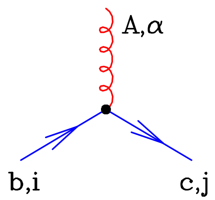
(all momenta incoming)



$$-ig^{2}\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^{2}\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^{2}\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$



$$g\ f^{ABC}q^{\alpha}$$



$$-ig\ (t^{A})_{cb}\ (\gamma^{\alpha})_{ji}$$

- Gauge fixing explicitly breaks gauge invariance. However, in the end physical results will be independent of gauge. For convenience, we usually use the Feynman gauge.
- In non-Abelian theories like QCD, covariant gauge-fixing term must be supplemented by a *ghost term* which we do not discuss here. The ghost field, shown by dashed lines in the above table, cancels unphysical degrees of freedom of gluon which would otherwise propagate in covariant gauges.
- Propagators determined from $-S$, interactions from S.

Feynman rules – recipe

- Consider a theory which contains only a complex scalar field ϕ and an action which contains only bilinear terms, $S = \phi^* (K + K') \phi$.
- MOE: both K and K' are included in the free Lagrangian, $S_0 = \phi^* (K + K') \phi$. Using the above rule the propagator Δ for the ϕ field is given by

$$\Delta = \frac{-1}{K + K'}.$$

- JOE: K is regarded as the free Lagrangian, $S_0 = \phi^* K \phi$, and K' as the interaction Lagrangian, $S_I = \phi^* K' \phi$. Now S_I is included to all orders in perturbation theory by inserting the interaction term an infinite number of times:

$$\begin{aligned} \Delta &= \frac{-1}{K} + \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) + \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) + \cdots \\ &= \frac{-1}{K + K'} \end{aligned}$$

Running coupling

- Consider dimensionless physical observable R which depends on a single large energy scale, $Q \gg m$ where m is any mass. Then we can set $m \to 0$ (assuming this limit exists), and dimensional analysis suggests that R should be independent of Q.
- This is not true in quantum field theory. Calculation of R as a perturbation series in the coupling $\alpha_S = g^2/4\pi$ requires renormalization to remove ultraviolet divergences. This introduces a second mass scale μ — point at which subtractions which remove divergences are performed. Then R depends on the ratio Q/μ and is not constant. The renormalized coupling α_S also depends on μ.
- But μ is arbitrary! Therefore, if we hold bare coupling fixed, R cannot depend on μ. Since R is dimensionless, it can only depend on Q^2/μ^2 and the renormalized coupling α_S. Hence

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_S\right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \frac{\partial}{\partial \alpha_S}\right] R = 0 \; .$$

- This is the simplest expression of the renormalization group. At fixed bare coupling, physical predictions cannot depend on an arbtitrary choice of renormalization scale.

- Introducing

$$\tau = \ln\left(\frac{Q^2}{\mu^2}\right), \quad \beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} ,$$

we have

$$\left[-\frac{\partial}{\partial \tau} + \beta(\alpha_S)\frac{\partial}{\partial \alpha_S}\right] R = 0.$$

This renormalization group equation is solved by defining running coupling $\alpha_S(Q)$:

$$\tau = \int_{\alpha_S}^{\alpha_S(Q)} \frac{dx}{\beta(x)} , \quad \alpha_S(\mu) \equiv \alpha_S .$$

Then

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = \beta(\alpha_S(Q)) , \quad \frac{\partial \alpha_S(Q)}{\partial \alpha_S} = \frac{\beta(\alpha_S(Q))}{\beta(\alpha_S)} .$$

and hence $R(Q^2/\mu^2, \alpha_S) = R(1, \alpha_S(Q))$. Thus all scale dependence in R comes from running of $\alpha_S(Q)$.

- We shall see QCD is asymptotically free: $\alpha_S(Q) \to 0$ as $Q \to \infty$. Thus for large Q we can safely use perturbation theory. Then knowledge of $R(1, \alpha_S)$ to fixed order allows us to predict variation of R with Q.

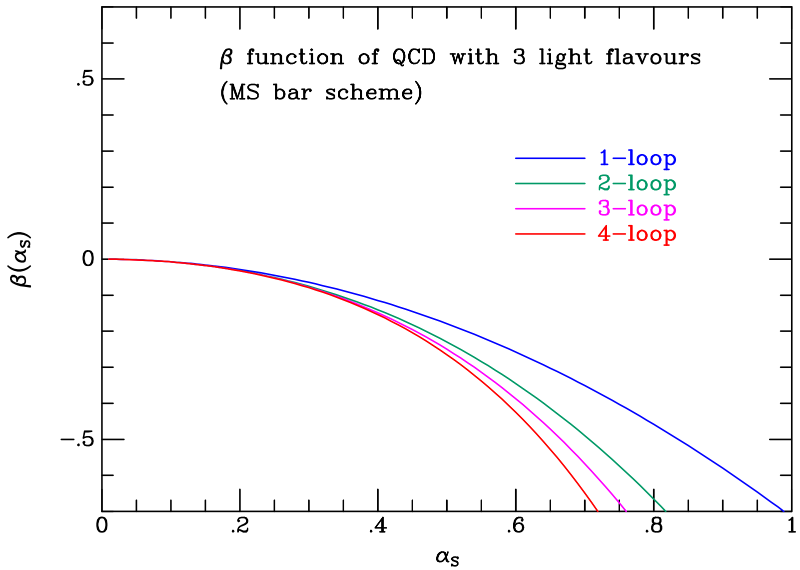
Beta function

- Running of the QCD coupling α_S is determined by the β function, which has the expansion

$$\beta(\alpha_S) = -b\alpha_S^2(1 + b'\alpha_S) + \mathcal{O}(\alpha_S^4)$$

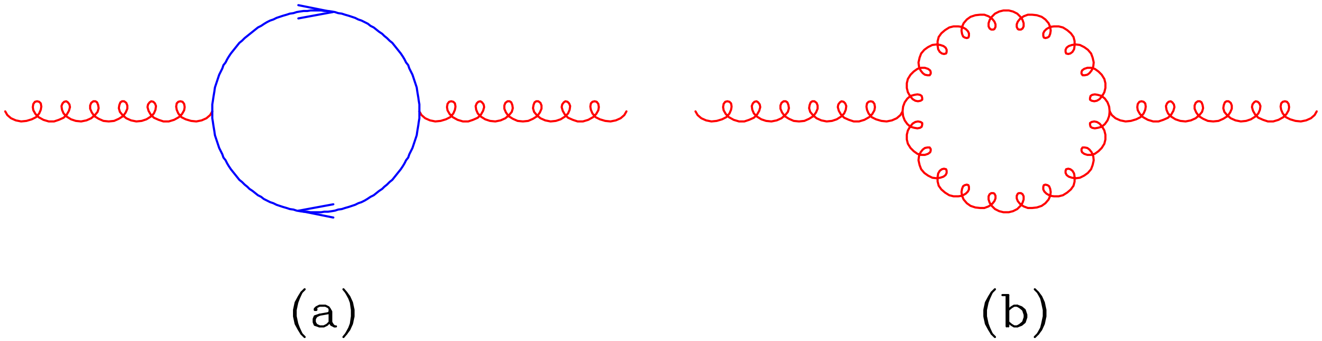
$$b = \frac{(11C_A - 2N_f)}{12\pi}, \quad b' = \frac{(17C_A^2 - 5C_A N_f - 3C_F N_f)}{2\pi(11C_A - 2N_f)},$$

where N_f is number of "active" light flavours. Terms up to $\mathcal{O}(\alpha_S^5)$ are known.



- ⋆ if $\frac{d\alpha_S}{d\tau} = -b\alpha_S^2(1 + b'\alpha_S)$ and $\alpha_S \to \bar{\alpha}_S(1 + c\bar{\alpha}_S)$, it follows that $\frac{d\bar{\alpha}_S}{d\tau} = -b\bar{\alpha}_S^2(1 + b'\bar{\alpha}_S) + O(\bar{\alpha}_S^4)$
- ⋆ first two coefficients b, b' are thus invariant under scheme change.

Asymptotic freedom



- Roughly speaking, quark loop diagram (a) contributes negative N_f term in b, while gluon loop (b) gives positive C_A contribution, which makes β function negative overall. (This statement is literally correct in the background field method).
- QED β function is $\beta_{QED}(\alpha) = \frac{1}{3\pi}\alpha^2 + \dots$
  Thus b coefficients in QED and QCD have opposite signs.
- From earlier slides,

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = -b\alpha_S^2(Q)\Big[1 + b'\alpha_S(Q)\Big] + \mathcal{O}(\alpha_S^4).$$

Neglecting b' and higher coefficients gives $\alpha_S(Q) = \frac{\alpha_S(\mu)}{1+\alpha_S(\mu)b\tau}\ , \quad \tau = \ln\left(\frac{Q^2}{\mu^2}\right).$

Asymptotic freedom

- As Q becomes large, $\alpha_S(Q)$ decreases to zero: this is asymptotic freedom. Notice that sign of b is crucial. In QED, $b < 0$ and coupling *increases* at large Q.
- Including next coefficient b' gives implicit equation for $\alpha_S(Q)$:

$$b\tau = \frac{1}{\alpha_S(Q)} - \frac{1}{\alpha_S(\mu)} + b' \ln\left(\frac{\alpha_S(Q)}{1 + b'\alpha_S(Q)}\right) - b' \ln\left(\frac{\alpha_S(\mu)}{1 + b'\alpha_S(\mu)}\right)$$

- What type of terms does the solution of the renormalization group equation take into account in the physical quantity R?
  Assume that R has perturbative expansion

$$R = \alpha_S + \mathcal{O}(\alpha_S^2)$$

Solution $R(1, \alpha_S(Q))$ can be re-expressed in terms of $\alpha_S(\mu)$:

$$\begin{aligned} R(1, \alpha_S(Q)) &= \alpha_S(\mu) \sum_{j=0}^{\infty} (-1)^j (\alpha_S(\mu) b\tau)^j \\ &= \alpha_S(\mu) \left[1 - \alpha_S(\mu) b\tau + \alpha_S^2(\mu)(b\tau)^2 + \ldots\right] \end{aligned}$$

Thus there are logarithms of Q^2/μ^2 which are automatically resummed by using the running coupling. Neglected terms have fewer logarithms per power of α_S.

Lambda parameter

- Perturbative QCD tells us how $\alpha_S(Q)$ varies with Q, but its absolute value has to be obtained from experiment. Nowadays we usually choose as the fundamental parameter the value of the coupling at $Q = M_Z$, which is simply a convenient reference scale large enough to be in the perturbative domain.
- Also useful to express $\alpha_S(Q)$ directly in terms of a dimensionful parameter (constant of integration) Λ:

$$\ln \frac{Q^2}{\Lambda^2} = -\int_{\alpha_S(Q)}^{\infty} \frac{dx}{\beta(x)} = \int_{\alpha_S(Q)}^{\infty} \frac{dx}{bx^2(1 + b'x + \ldots)}.$$

  Then (if perturbation theory were the whole story) $\alpha_S(Q) \to \infty$ as $Q \to \Lambda$. More generally, Λ sets the scale at which $\alpha_S(Q)$ becomes large.
- In leading order (LO) keep only first β-function b:

$$\alpha_S(Q) = \frac{1}{b \ln(Q^2/\Lambda^2)} \qquad \text{(LO).}$$

- In next-to-leading order (NLO) include also b':

$$\frac{1}{\alpha_S(Q)} + b' \ln\left(\frac{b'\alpha_S(Q)}{1+b'\alpha_S(Q)}\right) = b\ln\left(\frac{Q^2}{\Lambda^2}\right).$$

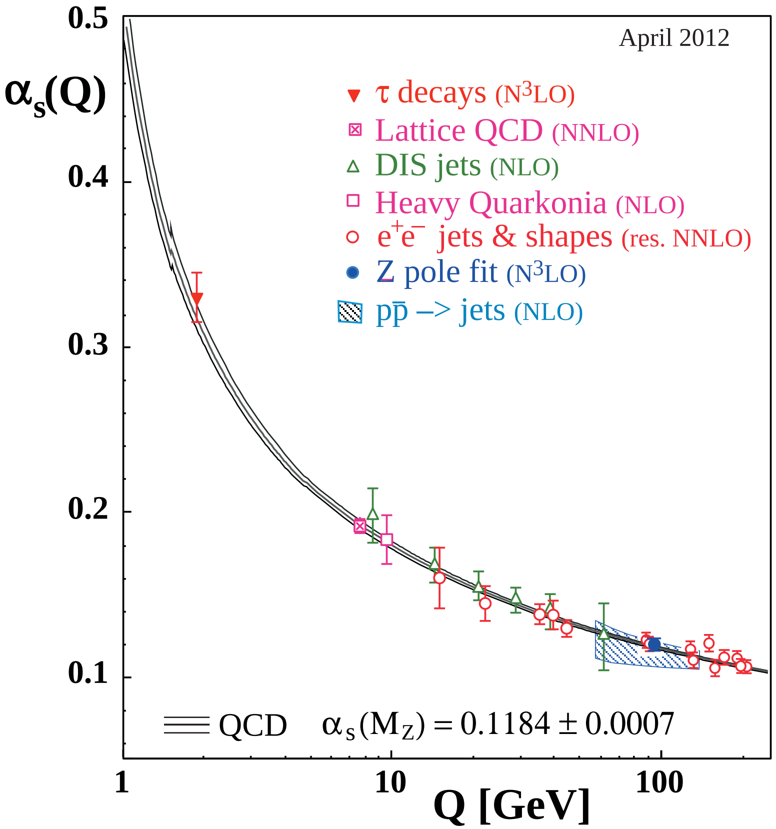
  This can be solved numerically, or we can obtain an approximate solution to second order in $1/\log(Q^2/\Lambda^2)$:

$$\alpha_S(Q) = \frac{1}{b\ln(Q^2/\Lambda^2)}\left[1 - \frac{b'}{b}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)}\right] \qquad \text{(NLO)}.$$

  This is Particle Data Group (PDG) definition.

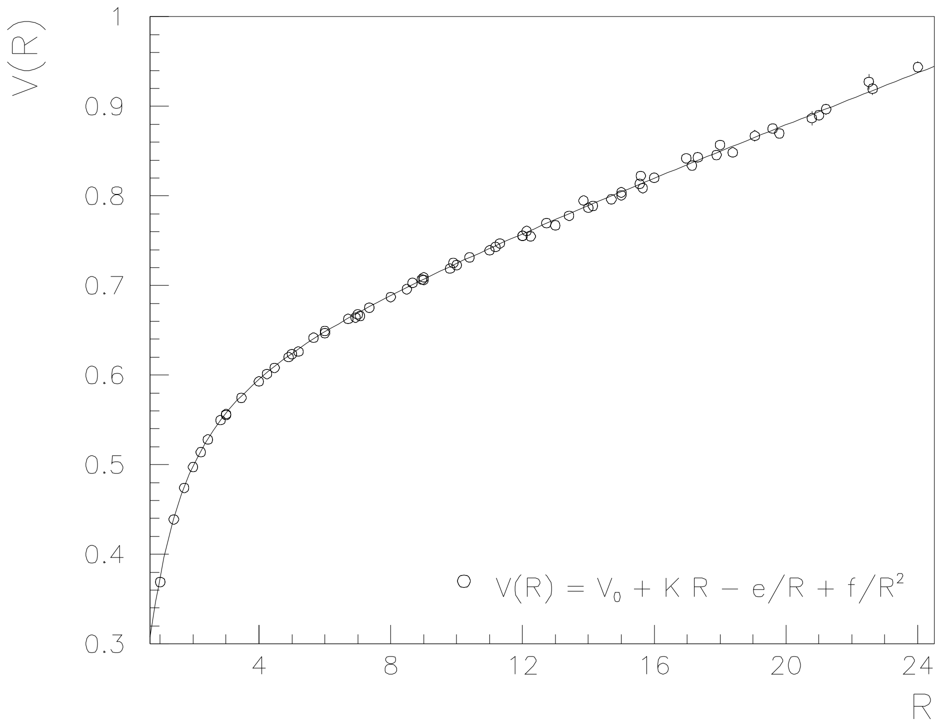
- Note that Λ depends on number of active flavours N_f. 'Active' means $m_q < Q$. Thus for $5 < Q < 175$ GeV we should use $N_f = 5$. See ESW for relation between Λ's for different values of N_f.

Measurements of α_S



- The most recent compilation of measurements of α_S (Bethke, arXiv:1210.0325) is shown above. Evidence that $\alpha_S(Q)$ has a logarithmic fall-off with Q is persuasive.
- The value of α_S is normally expressed as the value of the coupling constant in the $\overline{MS}$ scheme at the mass of the Z-boson.

The static force between two quarks



- Lattice calculation of the static $q\bar{q}$ potential, performed in the quenched approximation.
- At small distances $V(R) = V_0 - \frac{\alpha_S(aR)}{R}$, logarithmic modification of Coulomb potential.
- At large distances $V(R) = V_0 + KR$, linearly rising potential. This is infra-red slavery.

Renormalization schemes

- Λ also depends on renormalization scheme. Consider two calculations of the renormalized coupling which start from the same bare coupling α_S^0:

$$\alpha_S^A = Z^A \alpha_S^0 \; , \qquad \alpha_S^B = Z^B \alpha_S^0$$

Infinite parts of renormalization constants Z^A and Z^B must be same in all orders of perturbation theory. Therefore two renormalized couplings must be related by a finite renormalization:

$$\alpha_S^B = \alpha_S^A (1 + c_1 \alpha_S^A + \ldots).$$

- Note that first two β-function, coefficients, b and b', are unchanged by such a transformation: they are therefore renormalization-scheme independent.
- Two values of Λ are related by

$$\log \frac{\Lambda^B}{\Lambda^A} = \frac{1}{2} \int_{\alpha_S^A(Q)}^{\alpha_S^B(Q)} \frac{dx}{\beta(x)}$$

This must be true for all Q, so take $Q \to \infty$, to obtain

$$\Lambda^B = \Lambda^A \exp \frac{c_1}{2b}$$

Thus relations between different definitions of Λ are given by the one-loop calculation that fixes c_1.

- Nowadays, most calculations are performed in *modified minimal subtraction* ($\overline{\text{MS}}$) renormalization scheme. Ultraviolet divergences are 'dimensionally regularized' by reducing number of space-time dimensions to $D < 4$:

$$\frac{d^4 k}{(2\pi)^4} \longrightarrow (\mu)^{2\epsilon} \frac{d^{4-2\epsilon} k}{(2\pi)^{4-2\epsilon}}$$

  where $\epsilon = 2 - \frac{D}{2}$. Note that renormalization scale μ still has to be introduced to preserve dimensions of couplings and fields.

- Loop integrals of form

$$\int \frac{d^D k}{(k^2 + m^2)^2}$$

  lead to poles at $\epsilon = 0$. The *minimal subtraction* prescription is to subtract poles and replace bare coupling by renormalized coupling $\alpha_S(\mu)$. In practice poles always appear in combination

$$\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E,$$

  (Euler's constant $\gamma_E = 0.5772\ldots$). In *modified* minimal subtraction scheme $\ln(4\pi) - \gamma_E$ is subtracted as well. From argument above, it follows that

$$\Lambda_{\overline{\text{MS}}} = \Lambda_{\text{MS}} e^{[\ln(4\pi) - \gamma_E]/2} = 2.66\,\Lambda_{\text{MS}}$$

Recap

- QCD is an SU(3) gauge theory of quarks (3 colours) and gluons (8 colours,self-interacting).
- The Lagrangian of QCD has the property of local gauge invariance.
- Renormalization of dimensionless observables depending on a single large scale implies that the scale dependence enters through the running coupling.
- Asymptotic freedom implies the running coupling becomes small at high energies. The energy at which this occurs is determined from experiment. ($\Lambda \sim 300$ MeV).
- $\alpha(M_Z) \simeq 0.118$ in five flavour $\overline{MS}$-renormalization scheme.

Bibliography

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