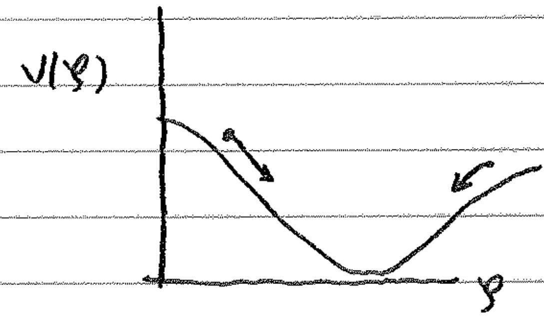
Inflation



An epoch of scalar field potential energy domination

$$S = \int d^4x \sqrt{g} \left[\frac{R}{16\pi G} - \frac{1}{2}(\partial\varphi)^2 - V(\varphi) \right]$$

$$\frac{3}{8\pi G} H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \qquad \dot{} = \frac{d}{dt}$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0$$

if V' is sufficiently small, whilst V is large then $\varphi \sim$ constant and $\dot{\varphi}^2 \ll V$ so that $H \sim$ constant

SLOW ROLL PARAMETERS

$$\epsilon \equiv -\dot{H}/H^2, \quad \delta \equiv \ddot{H}/2H\dot{H}$$

Require $\epsilon, \delta \ll 1$ for inflation

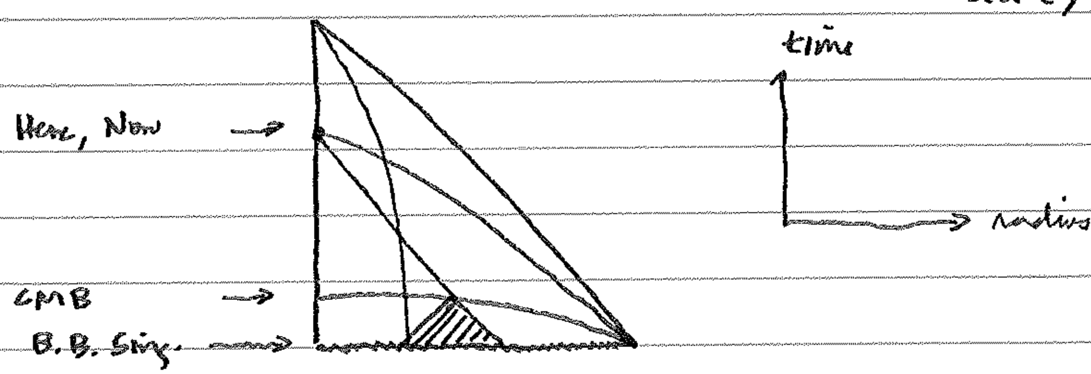
INFLATION

An epoch of exponential expansion
which solves horizon problem, etc
generates fluctuations of matter, metric.

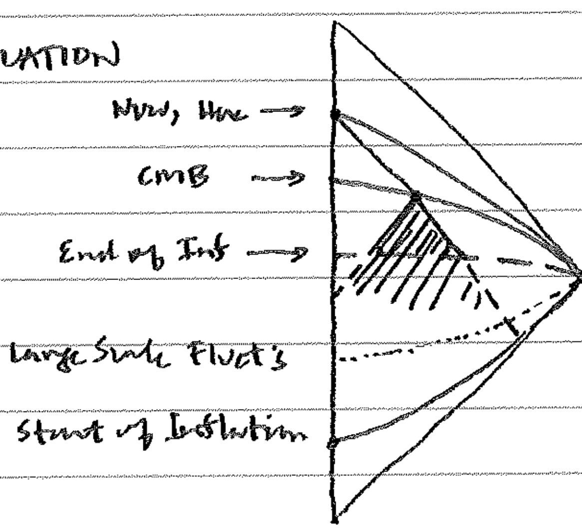
Sets the initial conditions for the HOT Big Bang

STD FW spacetime $\quad ds^2 = a(\tau)^2[-d\tau^2 + d\vec{x}^2]$, $\tau > 0$

$a \propto \tau, \tau^2$



+ INFLATION



No inflation RW: $ds^2 = a^2(\tau)\left[-d\tau^2 + d\vec{x}^2\right]$

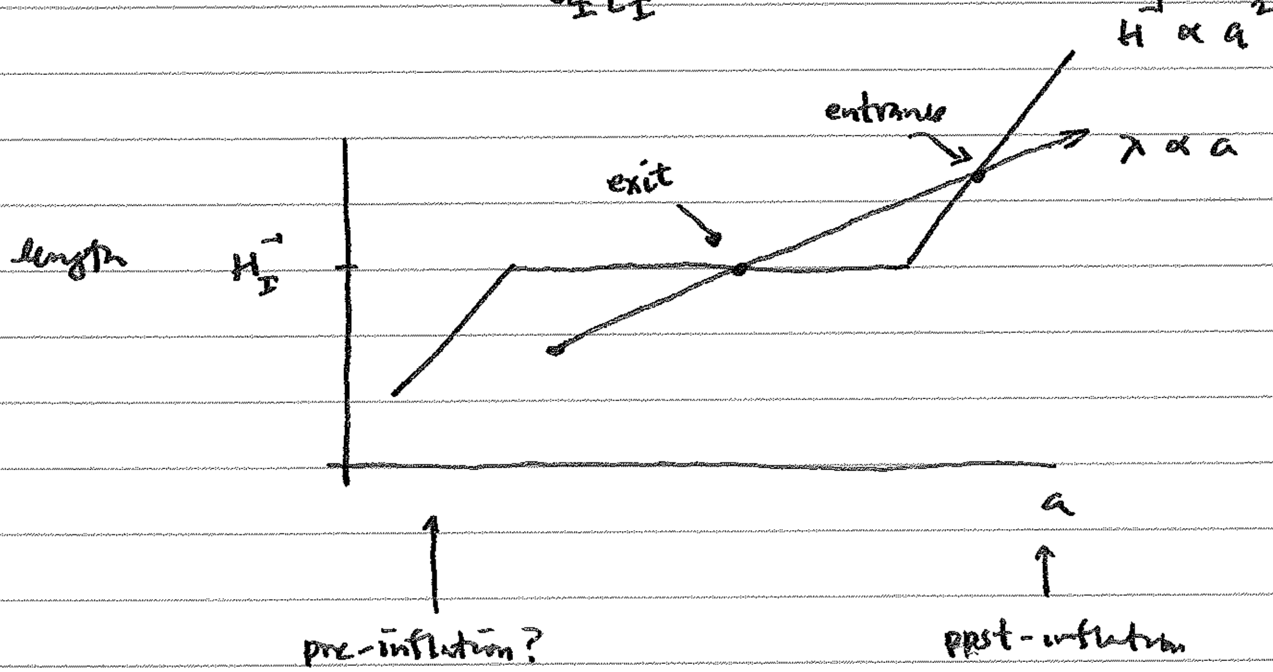
$a \propto \tau, \tau^2 \qquad 0 < \tau < \infty$

$$H = \frac{1}{a^2}\frac{da}{d\tau} = \frac{n}{a\tau} \qquad n = 1, 2$$

Yes inflation $\quad a = a_I \dfrac{\tau_I}{2\tau_I - \tau}, \quad -\infty < \tau < \tau_I$

inflation extends the past conformal history

$$H = \frac{1}{a_I \tau_I} = H_I$$



"Horizon" exit and entrance refers to $\lambda = H^{-1}$

Inflation

Scalar Field

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu}\left(\tfrac{1}{2}(\partial\varphi)^2 + V \right)$$

$$\nabla^\mu T_{\mu\nu} = 0 \quad \rightarrow \quad \Box\varphi - V' = 0$$

in conformal time: $\varphi'' + 2\mathcal{H}\varphi' + a^2 V' = 0$

in a weak gravitational field

$$ds^2 = a^2(\tau)\left[-(1+2\Psi)d\tau^2 + (1-2\Phi)d\vec{x}^2 \right]$$

$$\varphi \rightarrow \varphi + \delta\varphi(\tau, \vec{x})$$

$$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \nabla^2\delta\varphi + a^2 V''\delta\varphi = (3\Phi' + \Psi')\varphi' - 2a^2 V'\Psi$$

$$\delta T^0{}_0 = -\left(\frac{1}{a^2}\varphi'\delta\varphi' + V'\delta\varphi - \frac{1}{a^2}\varphi'^2\Psi \right)$$

$$\delta T^i{}_j = \left(\frac{1}{a^2}\varphi'\delta\varphi' - V'\delta\varphi - \frac{1}{a^2}\varphi'^2\Psi \right)\delta^i{}_j$$

$$\delta T^0{}_i = -\frac{1}{a^2}\varphi'\partial_i\delta\varphi$$

SCALAR FIELD HAS NO ANISOTROPIC STRESS : $\Psi = \Phi$.

Use Einstein Eq'ns to obtain

(A) $$\phi'' + 2\left(\mathcal{H} - \frac{\varphi''}{\varphi'}\right)\phi' + 2\left(\mathcal{H}' - \frac{\varphi''}{\varphi'}\mathcal{H}\right)\phi - \nabla^2\phi = 0$$

remark: in $k \to 0$ limit, the quantity

$$\zeta = \phi + \frac{\delta\varphi}{\varphi'}\mathcal{H}$$
$$= \phi + \frac{2}{3}\frac{\phi' + \mathcal{H}\phi}{\mathcal{H}(1+w)}$$

is constant, $\zeta' = 0$

(B) $$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \nabla^2\delta\varphi + a^2 V''\delta\varphi = 4\phi'\varphi' - 2a^2\phi V'$$

Define $R = \phi + \frac{\delta\varphi}{\varphi'}\mathcal{H}$ to get

(C) $$R'' + 2\frac{z'}{z}R' - \nabla^2 R = 0 \qquad z = a\varphi'/\mathcal{H}$$

COMMENTS: ON SMALL SCALES DURING INFLATION, $-k\tau \to \infty$,
R-eq'n looks like Minkowski wave eq'n.

ON LARGE SCALES DURING INFLATION, $k \ll \mathcal{H}$,
R has constant and decaying solutions.
Long duration of inflation ensures decaying sol'ns are negligible.

$\mathcal{R}$: $$\mathcal{R}'' + 2\frac{z'}{z}\mathcal{R}' - \nabla^2\mathcal{R} = 0$$

or $$\frac{d^2}{d\tau^2}\mathcal{R} + 2\mathcal{H}(1+\epsilon+\delta)\frac{d}{d\tau}\mathcal{R} - \nabla^2\mathcal{R} = 0$$

$\left\{\begin{array}{l}\text{Weinberg}\\ \text{Peter \& Uzan}\end{array}\right.$

QFT of $\mathcal{R}$

set $q = z\mathcal{R}$

$$S_q = \int d\tau\, d^3x \ \ \frac{1}{2}\left[q'^2 - \partial_i q\, \partial^i q + \frac{z''}{z} q^2 \right]$$

$$\pi = \frac{\partial \mathcal{L}}{\partial q'} = q' \qquad \text{s.t.} \qquad [q, \pi] = i\,\delta(\bar{x}-\bar{y})$$

$$\mathcal{R}(\tau,\bar{x}) = \int \frac{d^3k}{(2\pi)^3}\left[\mathcal{R}_k(\tau)\, e^{i\vec{k}\cdot\vec{x}}\, \hat{a}(\vec{k}) + \text{H.C.} \right]$$

$$\langle \mathcal{R}(\tau,\bar{x})\, \mathcal{R}(\tau,\vec{y}) \rangle = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\bar{x}-\bar{y})}\, P_{\mathcal{R}}(k)$$

$$P_{\mathcal{R}}(k) = |\mathcal{R}_k|^2, \quad \Delta^2_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} P_{\mathcal{R}} = A\left(\frac{k}{k_0}\right)^{n_s - 1}$$

ORIGIN OF INHOMOGENEITIES: AMPLIFICATION OF QUANTUM FLUCTUATIONS DURING INFLATION.

Inflationary Power Spectrum

$$P_R = \frac{2\pi}{k^3} \frac{H_*^2}{\epsilon M_p^2} \left(\frac{k}{a_* H_*} \right)^{n_s - 1}$$

"*" indicates values at horizon crossing, $k = aH$

WMAP 7 (Komatsu et al, ApJ Supp. 192 18 2011)

$K_0 = 0.002\ \mathrm{Mpc}^{-1}$

$A = 2.43(\pm 0.091) \times 10^{-9}$

$n_s = 0.967 \pm 0.014$

so $\epsilon = 0.0162 \pm 0.0067$

$H_* = 1.12(\pm 0.23) \times 10^{-5}\ M_p$

TENSOR PERTURBATIONS

$$H_{ij}: \qquad H_{ij}'' + 2\mathcal{H} H_{ij}' - \nabla^2 H_{ij} = 8\pi G a^2 p \underbrace{\pi^T_{ij}}_{\to 0}$$

$$H^i{}_i = 0, \quad \nabla^i H_{ij} = 0$$

QFT of H

$$S = \frac{M_p^2}{64\pi} \int d\tau\, d^3x \quad a^2 \left[(H_{ij}')^2 - (\nabla_k H_{ij})^2 \right]$$

Define $H_{ij} = \sum_\lambda h_\lambda \varepsilon^\lambda_{ij}$, $\quad q_\lambda = \sqrt{\frac{M_p^2}{32\pi}}\, a\, h_\lambda$

$$= \frac{1}{2} \sum_\lambda \int d\tau\, d^3x \left[(q_\lambda')^2 - (\nabla q_\lambda)^2 + \frac{a''}{a} (q_\lambda)^2 \right]$$

$$\pi_\lambda = \frac{\partial L}{\partial q_\lambda'} = q_\lambda' \quad \to \quad [q_\lambda(\bar{x},\tau), \pi_{\lambda'}(\bar{y},\tau)] = i\delta_{\lambda\lambda'}\delta(\bar{x}-\bar{y})$$

EOM: $\quad q_\lambda'' - \left(\frac{a''}{a} + \nabla^2 \right) q_\lambda = 0$

$$q_\lambda(\bar{x},\tau) = \int \frac{d^3k}{(2\pi)^3} \left[q_\lambda\, e^{i\vec{k}\cdot\vec{x}}\, \hat{a}(\vec{k}) + \text{H.C.} \right]$$

recipe: choose sol'n $q_\lambda(k) \propto e^{-ik\tau}$ as $-k\tau \to \infty$

$$P_T = 2 \times \frac{k^3}{2\pi^2} |h_\lambda|^2, \qquad n_T = \frac{\partial \ln P_T}{\partial \ln k}$$

POWER LAW INFLATION

$$n_T = 3 - 2\nu \simeq -2\epsilon$$

$$P_T = 16\epsilon P_R = -8 n_T P_R$$

ϵ CONTROLS RELATIONSHIP BETWEEN R & H

IN GENERAL, FOR SINGLE FIELD, SLOW ROLL INFLATION

$$\frac{P_T}{P_R} = 16\epsilon = -8 n_T$$

$$n_s = 1 + 2\delta - 4\epsilon$$
$$n_T = -2\epsilon$$

NOTE: INDICES MAY "RUN"

$$\alpha_s = \frac{d n_s}{d \ln k} , \quad \alpha_T = \frac{d n_T}{d \ln k}$$

Quantity of Inflationary Perturbations

Single field inflation produces adiabatic perturbations

$$\frac{\delta p}{\delta \rho} = \frac{\dot{p}}{\dot{\rho}}$$

Adiabatic pert'ns transfer to radiation, matter

radn: $\delta p = \frac{1}{3}\delta \rho$

matter: $\delta p = 0$

$$\frac{\delta_r}{1+w_r} = \frac{\delta_m}{1+w_m} \rightarrow \frac{3}{4}\delta_r = \delta_m$$

WHAT IS THE CMB TELLING US ABOUT INFLATION?

$n_s = 0.9603 \pm 0.0073$

$r < 0.11$ (95%)

$\frac{dn_s}{d\ln k} = -0.0134 \pm 0.0090$

$V = \frac{1}{2}m^2\phi^2$,

$M^4(1+\cos\phi/f)$

ON THE EDGE?

BUT RELAXED ASSUMPTIONS ABOUT COSMIC ENERGY CONTENT

N_ν, Y_p, Σm_ν, dark energy

ADJUST THESE BOUNDS A BIT.

SEE PLANCK XXII (2013).

RECALL: $\epsilon_V = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2$, $\eta_V = \frac{1}{8\pi G}\left(\frac{V''}{V}\right)$

$n_s - 1 \simeq 2\eta_V - 6\epsilon_V$

$n_t \simeq -2\epsilon_V$ $\qquad r = 16\epsilon_V$

$A_s = V/24\pi^2 M_p^4 \epsilon_V$, $A_T = 2V/3\pi^2 M_p^4$

& $N = -\sqrt{4\pi G}\int_\phi^{\phi_F} d\phi/\sqrt{\epsilon}$

PICK A MODEL OF INFLATION. DOES IT FIT?

EXAMPLE $V = \lambda\phi^4$

$$\epsilon_V = 1/\pi G\phi^2 \qquad \eta_V = 3/4\pi G\phi^2$$

SO $n_s - 1 = -9/2\pi G\phi^2$

$$r = 16/\pi G\phi^2$$

THESE GIVE LIMITS ON ϕ AT HORIZON EXIT FOR $k > 0.002\ h$-MVMPC MODES.

USE $N = -\sqrt{4\pi G}\int_{\phi}^{\phi_F} d\phi/\sqrt{\epsilon}$

$$= -2\pi G(\phi_F^2 - \phi^2)$$

BUT $\epsilon_V \approx 1$ AT END OF INFLATION : $\phi_F^2 = \frac{1}{\pi G}$

SO $N = 2\pi G\phi^2 - 2$

IN ORDER THAT $N \simeq 60$, $\pi G\phi^2 \approx 31$

SO $r = \frac{16}{31}$ TOO BIG!

$n_s - 1 = \frac{-9}{62}$ n_s TOO SMALL!

RULE OUT MODEL, OR FIND A WAY TO RAISE N!

HIGGS INFLATION

$$S = \int d^4x \sqrt{g} \left[\tfrac{1}{2} M_P^2 R - (\partial H^\dagger)(\partial H) - V(H^\dagger H) - \xi H^\dagger H R + \mathcal{L}_{SM} \right]$$

$$\mathcal{L}_{SM} \ni -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} g^2 A_\mu^2 H^\dagger H \ \dots$$

FOR SIMPLICITY $H \to \varphi/\sqrt{2}$

$$\mathcal{L} = \tfrac{1}{2} M_P^2 R \left(1 - \xi \frac{\varphi^2}{M_P^2}\right) - \tfrac{1}{2} (\partial\varphi)^2 - V(\varphi) + \mathcal{L}_{SM}$$

$$V(\varphi) = \frac{\lambda}{4} (\varphi^2 - \sigma^2)^2$$

SHOW THAT THIS SYSTEM IS CAPABLE OF INFLATION
[Bezrukov + Shaposhnikov, PLB 659 703 (2008)]

Step 1. Factor $\times$ R: $\Omega^2 \equiv (1 - \xi \varphi^2 / M_p^2)$

THIS IS CONFORMAL FACTOR TO RELATE EINSTEIN & JORDAN-B-D... FRAMES.

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \hat{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}$$

$$\sqrt{-\det(\hat{g})} = \Omega^4 \sqrt{-\det(g)}$$

$$\hat{R} = \Omega^{-2} R - 6 \Omega^{-3} \Box_g \Omega$$

WE WILL ASSUME SM IS CONFORMALLY INVARIANT...

$$S = \int d^4x \sqrt{-\hat{g}} \Big[\tfrac{1}{2} M_p^2 \hat{R} + 3 M_p^2 \Omega^{-3} \Box_g \Omega - \tfrac{1}{2} \Omega^{-2} \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$$
$$- \Omega^{-4} V(\varphi) + \hat{\mathcal{L}}_{SM} \Big]$$

$$= \int d^4x \sqrt{-\hat{g}} \Big[\tfrac{1}{2} M_p^2 \hat{R} - 3 M_p^2 \Omega^{-2} \hat{g}^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega$$
$$- \tfrac{1}{2} \Omega^{-2} \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$$
$$- \Omega^{-4} V(\varphi) + \hat{\mathcal{L}}_{SM} \Big]$$

NOW IN EINSTEIN FRAME W/ UNUSUAL SCALAR SYSTEM.

STEP 2. SIMPLIFY KINETIC ENERGY

$$\Omega^{-2}\left[3M_p^2(\partial\Omega)^2 + \frac{1}{2}(\partial\varphi)^2\right] = \frac{1}{2}(\partial\chi)^2$$

$$\Omega = \sqrt{1-\xi\varphi^2/M_p^2}$$

$$\partial\Omega = -\xi\,\Omega^{-1}\frac{\varphi}{M_p^2}\partial\varphi$$

$$\Omega^{-2}\,\hat{g}^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi\left(\frac{1}{2} + 3M_p^2\left(\frac{\xi\varphi}{\Omega M_p^2}\right)^2\right)$$

$$\partial\chi = \partial\varphi\,\Omega^{-1}\sqrt{1+6\xi^2\varphi^2/\Omega^2 M_p^2}$$

$$\frac{d\chi}{d\varphi} = \sqrt{\frac{\Omega^2+6\xi^2\varphi^2/M_p^2}{\Omega^4}}$$

SO $\varphi(\chi)$ ALLOWS TO WRITE

$$S = \int d^4x\sqrt{-\hat{g}}\left[\frac{1}{2}M_p^2\hat{R} - \frac{1}{2}(\hat{\partial}\chi)^2 - U(\chi) + \hat{\mathcal{L}}_{SM}\right]$$

$$\& \quad U(\chi) = \Omega^{-4}\frac{\lambda}{4}\left(\varphi(\chi)^2-\sigma^2\right)^2$$

STEP 3. MAP OUT NEW POTENTIAL, U

INTEGRATE $\frac{d\chi}{d\varphi}$, TAKE $|\xi| \gg 1$, $|\xi \frac{\varphi^2}{M_p^2}| \gg 1$ LIMIT

$$\varphi^2 \simeq -\frac{M_p^2}{\xi} \exp\left(\sqrt{\tfrac{2}{3}} \frac{\chi}{M_p}\right) \qquad (\xi < 0)$$

SO THAT $U \simeq \frac{\lambda}{4} \frac{M_p^2}{\xi^2} \left(1 + e^{-\sqrt{\frac{2}{3}}\chi/M_p}\right)^{-2}$

STEP 4. STUDY BEHAVIOR WHEN

$$|\xi \frac{\varphi^2}{M_p^2}| \gg 1, \quad \frac{\chi}{M_p} \gg 1$$

FLAT POTENTIAL!

SLOW ROLL PARAMETERS

$$\epsilon = \frac{1}{2} M_p^2 \left(\frac{U'}{U}\right)^2 = \frac{4}{3} e^{-2\sqrt{\frac{2}{3}}\frac{\chi}{M_p}} = \frac{4}{3} \frac{M_p^4}{\xi^2 \varphi^4}$$

$$\eta = \frac{4}{3} e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_p}} = \frac{4}{3} \frac{M_p^2}{(-\xi)\varphi^2}$$

SLOW ROLL ENDS @ $\epsilon = 1$: $\varphi_{END}^4 = \frac{4}{3} M_p^4 / \xi^2$

e-FOLDINGS vs φ?

$$N = \int H\,dt = -\sqrt{4\pi G}\int \frac{dx}{\sqrt{\epsilon}}$$

$$\vdots$$

$$= \frac{3}{2}\frac{\xi}{M_p^2}\int_{\varphi}^{\varphi_{END}} \varphi\, d\varphi = \frac{3}{4}\frac{\xi}{M_p^2}\left(\varphi_{END}^2 - \varphi^2\right)$$

USE φ_{END} TO GET

$$N + \frac{\sqrt{3}}{2} = -\frac{3}{4}\frac{\xi}{M_p^2}\varphi^2$$

$$60 \simeq -\frac{3}{4}\frac{\xi}{M_p^2}\varphi^2$$

SO $\epsilon \simeq \frac{3}{4}\frac{1}{(60)^2}$, $\eta \simeq \frac{1}{60}$

WHICH MEANS $n_s - 1 = 2\eta - 6\epsilon = 0.032$ OR $n_s = 0.967$

$$r = 16\epsilon = \frac{1}{300} = 0.0033$$

SINCE $A_s = \dfrac{U}{24\pi^2 M_p^4 \epsilon} \simeq \dfrac{50\lambda}{\pi^2 \xi^2}$

$\lambda = m_H^2 / 2\sigma^2$ $\quad m_H = 125$ GeV!

$A_s = 2.196^{+0.051}_{-0.060} \times 10^{-9}$. (PLANCK XVI)

$- \xi = 17{,}300$ ✓

WHAT IS THE TEMPERATURE PATTERN ON THE SKY
...TELLING US ?

THERE IS A POWER ASYMMETRY !

$$\frac{\delta T}{T}(\hat{n}) = \frac{\delta T}{T}(\hat{n})\Big|_{\mathrm{iso}} \times (1 + A\,\hat{n}\cdot\hat{b})$$

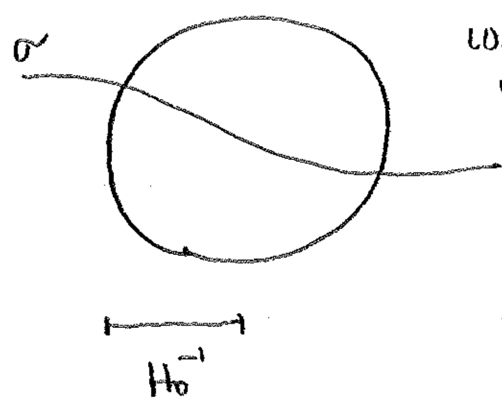
$A = 0.072 \pm 0.022$

HOFTUFT et al, ApJ 699 985 (2009)

extends to high ℓ...

$n_s \simeq 0.93$ in one direction, 0.99 in another.

AXELSSON et al, 1303.5371

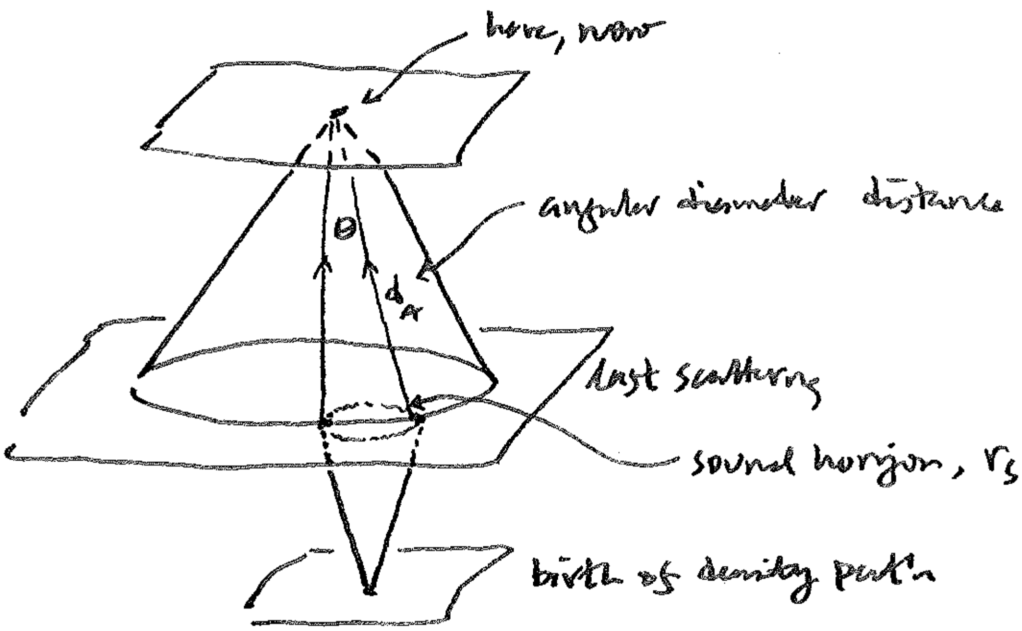


LONG WAVELENGTH PERTURBATION IN A SPECTATOR FIELD ?

MODULATION OF ϵ ?

see LIANG et al, 1303.6949

CMB GEOMETRY



Size of CMB feature, on the sky $\theta = \dfrac{r_s}{d_A}$

COSMOLOGIES WITH SAME:

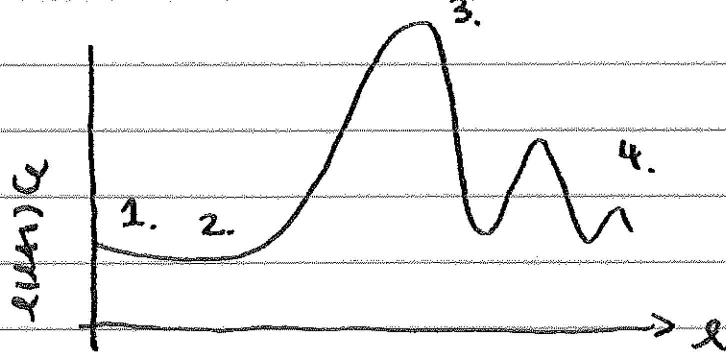
$\theta, \Omega_b h^2, \Omega_m h^2, r_s \ldots$

HAVE (NEARLY) THE SAME CMB POWER SPECTRA
OUT TO SMALL ANGULAR SCALES ($\ell \lesssim 1000$)

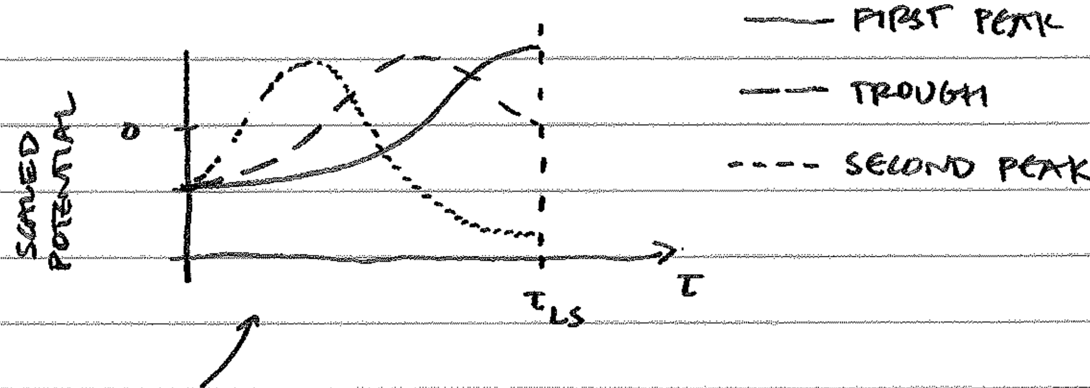
MORE CMB EFFECTS

SACHS-WOLFE, INTEGRATED SACHS-WOLFE,
PHOTON DENSITY PERTURBATIONS,
DOPPLER SHIFT,
POLARIZATION,
LENSING, ...

CMB Features



1. INTEGRATED SACHS-WOLFE
2. SACHS-WOLFE
3. ACOUSTIC PEAKS:



POTENTIALS "ENTER HORIZON" AND GROW UNTIL RAD. PRESSURE BECOMES IMPORTANT

4. DAMPING OF SMALL SCALE ANISOTROPIES:

γ NOT PERFECTLY COUPLED TO e, p

MEAN FREE PATH WASHES OUT SMALLER SCALE FEATURES $k > \frac{2\pi}{\lambda_{MFP}}$

LARGE SCALE CMB in "SUDDEN" LIMIT

SACHS-WOLFE EFFECT

Temperature of CMB $\propto$ Photon Energy

$$\frac{T_{OBS}}{T_{EMIT}} = \frac{\sqrt{-g_{00}(\text{emit})}}{\sqrt{-g_{00}(\text{obs})}} = \frac{a(t_{EMIT})}{a(t_{OBS})}\sqrt{\frac{(1+2\phi_e)}{(1+2\phi_0)}}$$

$$\simeq \frac{a_e}{a_0}\left(1+\phi_e(t_e,\vec{x}_e)-\phi_0(t_0,\vec{x}_0)\right)$$

$$T_{OBS} = \bar{T}_0 + \Delta T$$

But T_{emit} depends on direction

"EMIT" WHEN ρ_γ = particular value

$$\bar{\rho}_0\left(\frac{a_0}{a_e}\right)^4 = \bar{\rho}_0\left(\frac{T_e}{\bar{T}_0}\right)^4$$

$$\left(\frac{t_0}{t_e+\delta t}\right)^{8/3} = \left(\frac{T_e}{\bar{T}_0}\right)^4$$

$$t_e+\delta t = t_e(1+\phi_e)$$

$$\left(\frac{t_0}{t_e}\right)^{2/3}\left(1-\frac{2}{3}\phi_e\right) = \frac{T_e}{\bar{T}_0}$$

$$T_{OBS} = \bar{T}_0 + \Delta T = T_e\frac{a_e}{a_0}(1+\phi_e) = \bar{T}_0\left(1-\frac{2}{3}\phi_e\right)(1+\phi_e)$$

$$\simeq \bar{T}_0\left(1+\frac{1}{3}\phi_e\right)$$

CMB

$$\frac{\Delta T}{T_0} = \frac{1}{3}\Phi_e = \frac{1}{3}\Phi(\tau=\tau_{LS},\ \vec{x}=(\tau_0-\tau_{LS})\hat{n})$$

INFLATION INFORMS US OF THE STATISTICS OF CMB $\frac{\Delta T}{T}$.

$$\frac{\Delta T}{T}(\hat{n}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{n})$$

$$\left\langle \frac{\Delta T}{T}(\hat{n}) \frac{\Delta T}{T}(\hat{n}')\right\rangle = C(\theta)$$

$$= \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} C_\ell P_\ell(\cos\theta)$$

$$C_\ell = \frac{1}{2\ell+1}\sum_m |a_{\ell m}|^2$$

or $\langle a_{\ell m} a^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'} C_\ell$

R is Gaussian distributed w/ zero mean, variance $\langle R^2\rangle$

so is Φ, so is $\frac{\Delta T}{T}$, $a_{\ell m}$

$$\left\langle \frac{\Delta T}{T}(\hat{n}) \frac{\Delta T}{T}(\hat{n}') \right\rangle = \frac{1}{9} \langle \phi(\tau_{ls}, \lambda \hat{n}) \, \phi(\tau_{ls}, \lambda \hat{n}') \rangle$$

$\lambda = \tau_0 - \tau_{ls}$

recall $\phi = \frac{3}{5} R$ in matter era, for long wavelengths

$$= \frac{1}{25} \langle R(\tau_{ls}, \lambda \hat{n}) \, R(\tau_{ls}, \lambda \hat{n}') \rangle$$

$$= \frac{1}{25} \int \frac{d^3k}{(2\pi)^3} \, e^{i\vec{k} \cdot \lambda (\hat{n} - \hat{n}')} \, P_R(k)$$

identity $e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_\ell i^\ell j_\ell(kr) \sum_m Y^*_{\ell m}(\hat{n}) \, Y_{\ell m}(\hat{k})$

$=$ after a few steps

$$= \frac{1}{25} \int \frac{k^2 dk}{(2\pi)^3} (4\pi)^2 P_R \sum_\ell \frac{2\ell+1}{4\pi} |j_\ell(k\lambda)|^2 P_\ell(\cos\theta)$$

$$= \sum_\ell \frac{2\ell+1}{4\pi} C_\ell P_\ell(\cos\theta)$$

so that $C_\ell = \frac{1}{25} \int_0^\infty \frac{k^2 dk}{(2\pi)^3} P_R \, (4\pi)^2 \, j_\ell(k\lambda)^2$

$P_R = \frac{2\pi^2}{k^3} A \left(\frac{k}{k_0} \right)^{n_s - 1}$

$$= 4\pi \frac{A}{25} \int_0^\infty \frac{dk}{k} \left(\frac{k}{k_0} \right)^{n_s - 1} j_\ell(k\lambda)^2$$

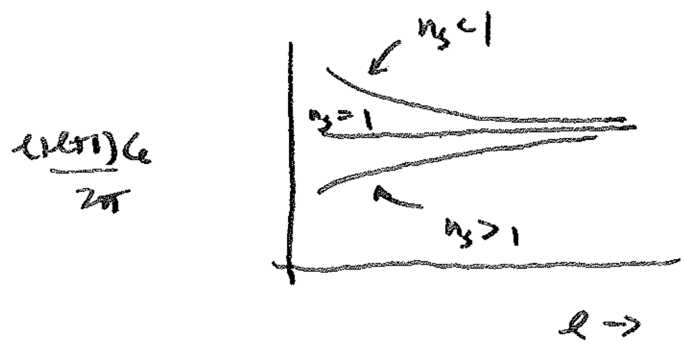
$$C_\ell = \frac{A}{25}\,\frac{2\pi}{\ell(\ell+1)} \qquad \text{for } n_s = 1$$

$$\ell(\ell+1)C_\ell/2\pi = \frac{A}{25}$$

or more generally

$$\ell(\ell+1)C_\ell/2\pi = \frac{A}{25} \times \frac{\sqrt{\pi}}{2}\,(H_0\lambda)^{1-n_s}\,\frac{\Gamma\left(\frac{3-n_s}{2}\right)\Gamma\left(\frac{2\ell-1+n_s}{2}\right)}{\Gamma\left(\frac{4-n_s}{2}\right)\Gamma\left(\frac{2\ell+5-n_s}{2}\right)}$$

$$\times\ \ell(\ell+1)$$



"Sachs-Wolfe plateau"

Q: Can one distinguish model universes based on such behavior?

A: Beware of COSMIC VARIANCE

$\langle C_\ell \rangle$ is predicted, with variance

$$\langle (C_\ell - \langle C_\ell \rangle)^2 \rangle = \frac{2}{2\ell+1}\langle C_\ell \rangle^2$$

← WIDTH DIMINISHES AT INCREASING ℓ.

DARK RADIATION ?

PLANCK + $\quad N_{eff} = 3.83 \pm 0.54 \;(2\sigma)$

WMAP + $\quad 3.62 \pm 0.5 \;(2\sigma)$

A HINT OF EXTRA RELATIVISTIC ENERGY AT DECOUPLING

$$\rho_R = \rho_\gamma \left(1 + \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3} N_{eff}\right)$$

$$N_{eff} = 3.046 + \Delta N$$

$\Delta N \sim 0.5$? $\quad$ (BBN: $\Delta N \lesssim 1$ @ 95%)

(See Valentino et al, 1304.5981)

HALF A NEUTRINO?

IF WE HAD $\mathcal{N}$ extra REL. ν's THAT DECOUPLED MUCH EARLIER, THEN

$$\rho_R = \rho_\gamma \left(1 + \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3} \times 3.046 + \frac{7}{8} \times \left(\frac{T_{\nu\mathcal{N}}}{T_\gamma}\right)^4 \mathcal{N}\right)$$

$$\frac{T_{\nu\mathcal{N}}}{T_\gamma} = \left(\frac{2}{g_{*D}}\right)^{1/3}$$

$g_{*D} = 2 + \frac{7}{8} \times 2 \times 2$ IN STD CASE

106.75 FOR SM?

CAN EASILY ARRANGE $\mathcal{N} \gg 1$

OBSERVE $\qquad \dfrac{\rho_\nu(0.5)}{\rho_\gamma + \rho_\nu(3.046)} \simeq 0.06$

~6% extra radiation

MAIN EFFECT IS TO INCREASE H

- DAMPS GROWTH OF B, DM δ's
- ENHANCES ANGULAR SCALE OF DIFFUSION DAMPING

• METRIC PERT'N

$$\ddot{h} + \mathcal{H}\dot{h} = -8\pi G a^2 (3\delta p - \delta\rho)$$

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j\right]$$

rad'n has $\delta p = \frac{1}{3}\delta\rho$, but boosts $\mathcal{H}$ term.

since $\dot{\delta}_m = -\frac{1}{2}\dot{h}$, δ_m is SLOWED.

[see Ma + Bertschinger ApJ 1995]

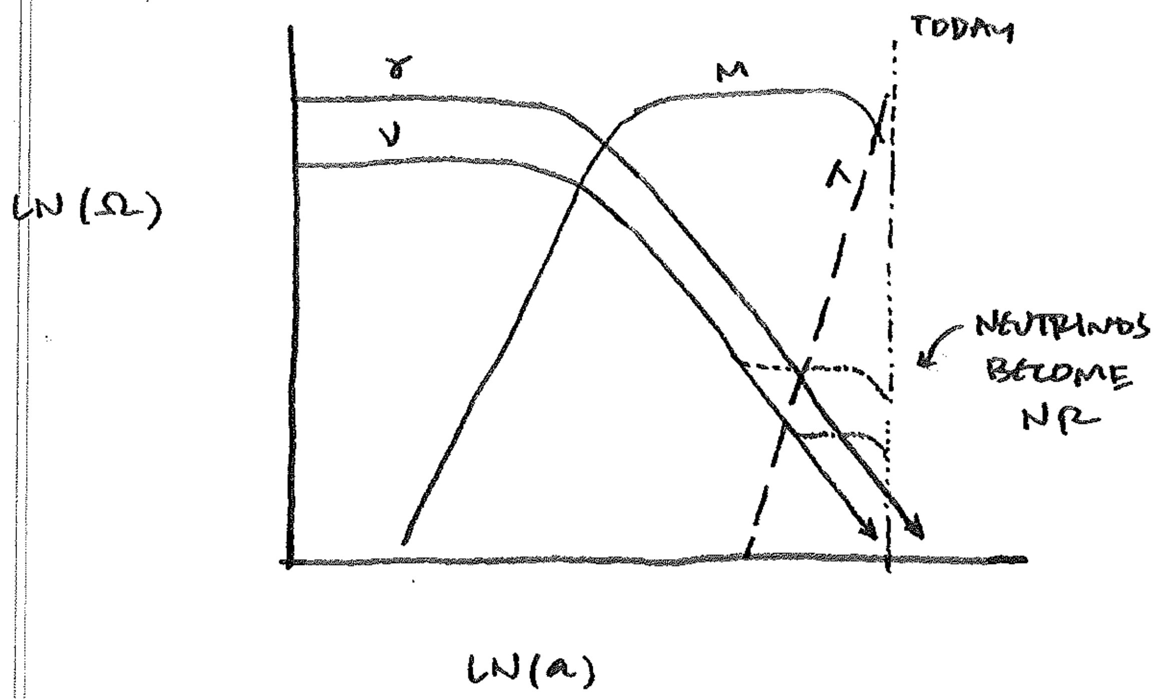
• DIFFUSION DAMPING (SILK)

$$\theta_d / \theta_s \propto H^{1/2}$$

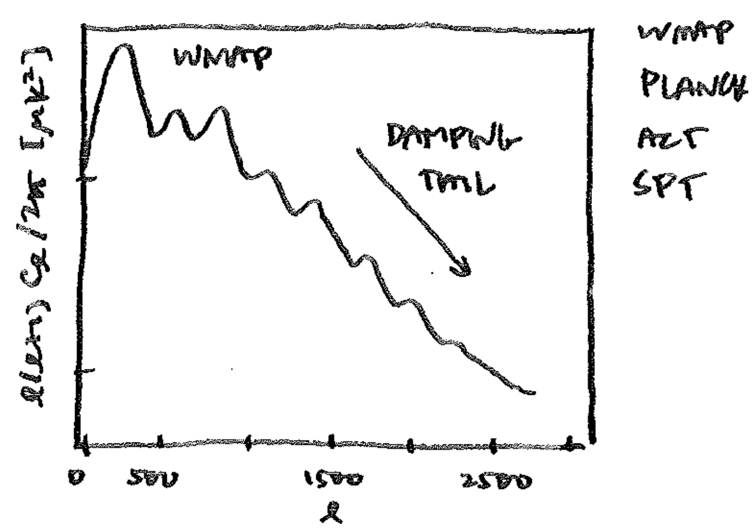
FOR FIXED θ_s (SOUND HORIZON), $N_{eff}\uparrow$ raises H, WHICH RAISES θ_d (DIFFUSION SCALE)

[see Hou et al, PRD 87, 083008 (2013)]

MASSIVE NEUTRINOS



[Lesgourgues & Pastor, PHYS REP 2006]



[Keisler et al, ApJ 743 28 (2011)]

NEUTRINO MASSES ?

JEANS STABILITY : $k_J = \sqrt{4\pi G \rho a^2} / v_{TH}$

$$v_{TH} = 3 \frac{T_\nu^0}{m_\nu} (1+z)$$

SO $k_J = 0.8 \frac{h}{Mpc} \left(\frac{m}{eV}\right) \frac{\sqrt{\Omega_m (1+z)^3 + 1 - \Omega_m}}{(1+z)^2}$

ON SMALL SCALES, $k > k_J$, NEUTRINOS FREE STREAM OR SMOOTH OUT

WHICH HELPS SMOOTH OTHER CLUMPY MATTER

USE MEASUREMENTS OF SMALL SCALE CMB TO LIMIT m_ν.

$$M_\nu = \Sigma m_\nu$$

? PLANCK: $M_\nu < 0.85$ eV @ 95% CL

WHEN LENSING INCLUDED

$M_\nu < 0.66$ eV @ 95% CL

FROM TT, WP

TENSION? SEE PLANCK XVII.