An introduction to the Finite Volume Method for Conservation Laws

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Outline

Hyperbolic Scalar Conservation Laws

We consider the *Cauchy* problem for scalar hyperbolic conservations laws

$$(\text{CL}) \qquad \begin{aligned} \partial_t u + \partial_x f(u) &= 0, \quad x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) &= u_0 \end{aligned}$$

Example : Euler system for isentropic gas dynamics

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \qquad \text{conservation of mass} \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) &= 0 \qquad \text{conservation of momentum} \end{aligned}$$

ρ density , u velocity , p pressure

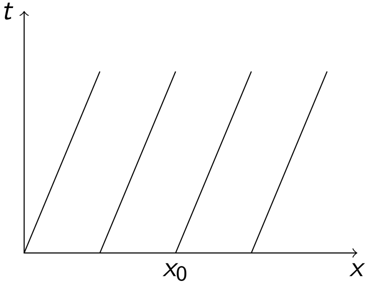
Hyperbolic : the Jacobian $A(u) = f'(u)$ has real eigenvalues and full set of eigenvectors.

Classical solution: u cont. differentiable w.r.t (x,t) and satisfies (CL)

Basic properties

- Characteristic Curves: $x(t)$ defined by

$$\frac{d}{dt}x(t) = f'(u(x(t),t)), \quad x(0) = x_0$$



Basic properties

- Characteristic Curves: $x(t)$ defined by

$$\frac{d}{dt}x(t) = f'(u(x(t),t)), \quad x(0) = x_0$$

- The solution is constant along the characteristic

$$\frac{d}{dt}u(x(t),t) = \partial_x u \; x'(t) + \partial_t u = \partial_t u + \partial_x u \; f'(u(x(t),t)) \overset{(CL)}{=} 0$$

$$\Longrightarrow u(x(t),t) = u(x(0),0) = u(x_0,0) = u_0(x_0)$$

$$\Longrightarrow x(t) = x_0 + f'(u_0(x_0)) \; t$$

$$\Longrightarrow u(x,t) = u_0(x - f'(u)t)$$

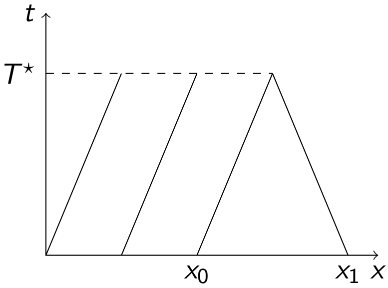
$$\Longrightarrow \partial_x u = \frac{u_0'}{1 + t \; u_0' \; f''(u_0(x_0))}$$

- So there is a time $t = T^\star$ where $\partial_x u$ becomes infinite!!
- Classical solutions cease to exist !

Basic properties

- Characteristic Curves: $x(t)$ defined by

$$\frac{d}{dt}x(t) = f'(u(x(t),t)), \quad x(0) = x_0$$



- At $t = T^\star$ the characteristics meet!! Formation of a shock

Weak solutions

To allow discontinuities weak solutions are introduced

Weak solution

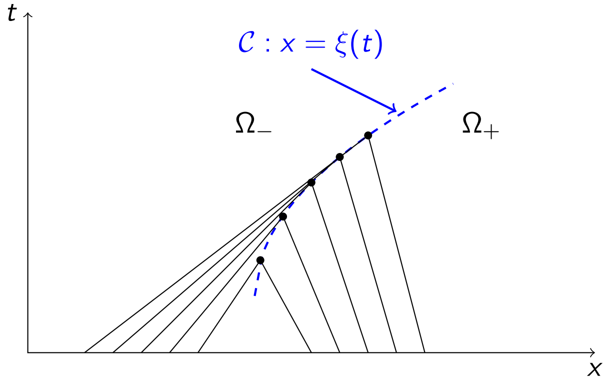
A bounded and measurable $u(x,t)$ is called a weak solution of (CL) if

$$\forall \phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^+) \int_0^T \int_{\mathbb{R}} (u\partial_t\phi + f(u)\partial_x\phi)\, dxdt + \int_{\mathbb{R}} \phi(x,0)u(x,0)dx = 0$$

Shock curve

Let $\mathcal{C}$ be the curve separating the domain in Ω_- and Ω_+. Consider a function $U \in C^1$ in $\bar{\Omega}_-$ and $\bar{\Omega}_+$. Then U solves (CL) in distributional sense in Ω iff U is a classical solution in Ω_- and Ω_+ and the Rankine-Hugoniot jump condition is satisfied with $U_+ = U|_{\Omega_+}$ $U_- = U|_{\Omega_-}$

$$f(U_+) - f(U_-) = \xi'(U_+ - U_-) \quad \text{in } \mathcal{C} \cap \Omega$$



Entropy

Entropy

An entropy for (CL) is a pair of smooth functions $(\eta(U),\ G(U))$ satisfying $G'(u) = \eta'(u) f'(u)$, where *prime* denotes differentiation w.r.t u

For smooth solutions we have

$$\partial_t(\eta(U)) + \partial_x(G(U)) = 0$$

but in case of discontinuous functions U for η *convex* we have

$$\text{(EN)} \qquad \partial_t(\eta(U)) + \partial_x(G(U)) \le 0$$

A weak solution U of (CL) is entropy satisfying if (EN) holds. This is a criteria for selecting a unique solution to the (CL). For the Euler system we can take

$$\eta = \rho u^2/2 + \rho\ e(\rho),\ \ G = (\rho u^2/2 + \rho\ e(\rho) + p(\rho))u, \quad e'(\rho) = \frac{p(\rho)}{\rho^2}$$

Numerical methods for Conservation Laws

- Finite Differences
- Finite Elements, continuous or discontinuous
- Finite Volume

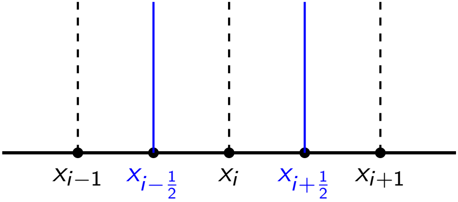
A scheme should respect the physical properties of the system. Desirable propreties

- Conservative : to be able to capture the discontinuities
- Stable : free of oscillations
- Accurate
- Convergent

Finite Volume method

Basic concepts:

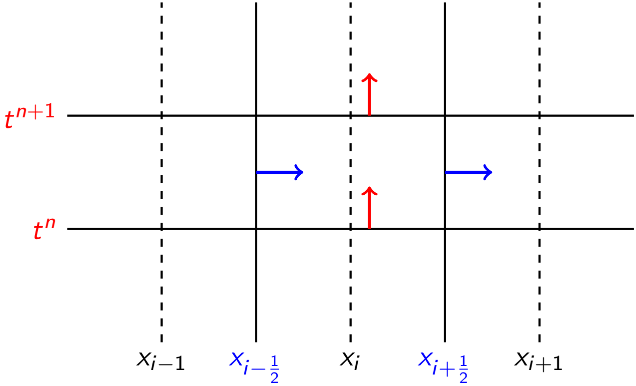
- Given an partition of the domain create a “dual” mesh consisting of control volumes



- Discretize the integral form of the conservation law over each control volume
- Evaluate the boundary values(integrals)
- Evolve the solution at the next time level

Finite Volume method - 1D

Consider a uniform partition of $\mathbb{R} \times \mathbb{R}^+$ in cells $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]$



$x_i = i\Delta x,\ \ i \in \mathbb{Z},\ x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2},\ t^n = n\Delta t$

Finite volume discretization I

Integrate the conservation law over the control volume using Gauss theorem $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]$

$$
\begin{aligned}
0 &= \int_{t^n}^{t^{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [u_t + f(u)_x] \, dx \, dt \\
&= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[u(x, t^{n+1}) - u(x, t^n) \right] \, dx \\
&+ \int_{t^n}^{t^{n+1}} \left[f(u(x_{i+\frac{1}{2}}, t)) - f(u(x_{i-\frac{1}{2}}, t)) \right] dt \\
&= \text{Change of Mass} + \text{Difference of Fluxes}
\end{aligned}
$$

The rate of change of mass in the control volume is equal to the net mass flux through its boundary

Finite volume discretization II

Notation

$$U_i^n \sim \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^n)dx,$$

$$F_{i+\frac{1}{2}}^n := F(U_i^{R,n}, U_{i+1}^{L,n}) \sim \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}},t))dt$$

- u is assumed to be discontinuous across the control volume boundaries (typically constant within each volume)
- U_i^n approximates the average of u in $C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ at time t^n
- $F_{i+\frac{1}{2}}^n$ approximates the average in $[t^n, t^{n+1}]$ at $x = x_{i+\frac{1}{2}}$
- F is a **numerical flux** function
- $U_i^{R,n}$ some approximation of $u(x_{i+\frac{1}{2}} - 0, t^n)$
- $U_{i+1}^{L,n}$ some approximation of $u(x_{i+\frac{1}{2}} + 0, t^n)$

Basic FV scheme

Advance in time with an Euler step

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right)$$

- F numerical flux function:
  1. Consistency : $F(u,u) = f(u)$
  2. Conservation : $F(u,v) = -F(v,u)$
  3. Monotonicity : $\partial_u F > 0,\ \partial_v F < 0$
- CFL condition : $\sup_u |f'(u)| \frac{\Delta t}{\Delta x} \leq 1$ Stability condition

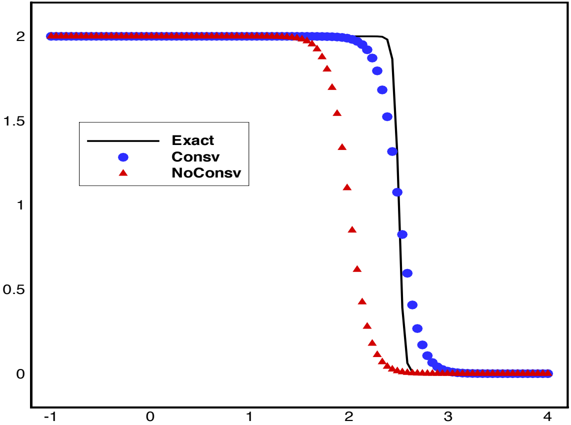
Convergence : Lax,Wendroff, Crandall, Majda

Assume the FV scheme is conservative, consistent, monotone and satisfies the CFL-condition. Then as Δt and Δx tend to zero with $\Delta t/\Delta x =$constant U_i^n converges boundedly almost everywhere to some function $u(x,t)$ which is an entropy satisfying weak solution of (CL)

Conservation really maters!!

Solution of Burgers eqn.

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$



Numerical Fluxes

- Central Differences

$$F_{i+\frac{1}{2}} = \frac{1}{2}\left(F(U_i^R) + F(U_{i+1}^L)\right), \quad \text{or} \quad F_{i+\frac{1}{2}} = F\left(\frac{U_i^R + U_{i+1}^L}{2}\right)$$

- Lax-Friedrichs flux

$$F_{i+\frac{1}{2}} = \frac{1}{2}\left(F(U_i^R) + F(U_{i+1}^L)\right) - \frac{\Delta x}{2\Delta t}\left(U_{i+1}^L - U_i^R\right)$$

- Upwind flux, $\lambda_{i+1/2} = f'\left(\frac{U_i^R + U_{i+1}^L}{2}\right)$

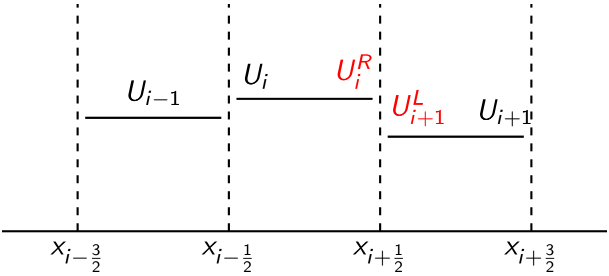
$$F_{i+\frac{1}{2}} = \begin{cases} F(U_i^R) & \text{if } \lambda_{i+1/2} \geq 0 \\ F(U_{i+1}^L) & \text{if } \lambda_{i+1/2} < 0 \end{cases}$$

- Lax-Wendroff

$$F_{i+\frac{1}{2}} = \frac{1}{2}\left(F(U_i^R) + F(U_{i+1}^L)\right) - \frac{\Delta t \lambda_{i+1/2}}{2\Delta x}\left(F(U_{i+1}^L) - F(U_i^R)\right)$$

Reconstruction process : the values U_i^R, U_{i+1}^L

The simplest choice : Constant values



We take $U_i^R = U_i, \; U_{i+1}^L = U_{i+1}$.

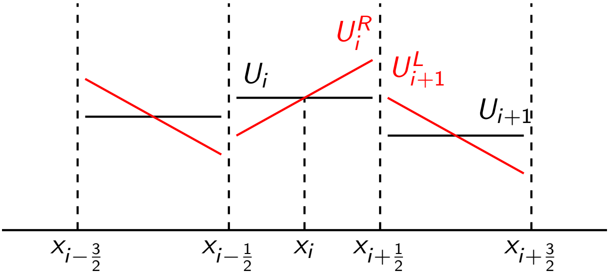
The resulting scheme is 1st order accurate in space

Reconstruction process : the values U_i^R, U_{i+1}^L

Linear in $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ with cell average U_i

$$p_1^i(x) = U_i + \frac{x - x_i}{\Delta x} S_i, \quad S_i \approx \Delta x \; u'(x_{i+\frac{1}{2}}), \; x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$$

Then $U_i^R = p_1^i(x_{i+\frac{1}{2}}), \; U_{i+1}^L = p_1^{i+1}(x_{i+\frac{1}{2}})$



$$S_i = U_i - U_{i-1} \;\text{ or }\; S_i = U_{i+1} - U_i \;\text{ or }\; S_i = \Psi\left(U_i - U_{i-1}, U_{i+1} - U_i\right)$$

MUSCL schemes

- Godunov showed that it is not possible for a linear scheme to be both higher than first order accurate and free of spurious oscillations
- ...so if we replace the constants by linear polynomials spurious oscillations will be generated around discontinuities
  - Add artificial diffusion/viscosity
  - appropriate weighted combination of a low order, non-oscillatory flux and one higher order oscillatory flux
- Van Leer, 1979 : Introduced MUSCL schemes: Monotone Upstream-centered Scheme for Conservation Laws
  - Extension to 2nd order by introducing Slope Limiters
  - Limiters : Nonlinear functions ϕ to control the gradient of the solution at discontinuities
  - Prevention of appearance of unphysical spurious oscillations around discontinuities
  - $\Psi\left(U_i - U_{i-1}, U_{i+1} - U_i\right) = \phi(r_{i+\frac{1}{2}}),\ r_{i+\frac{1}{2}} = \frac{U_{i+1}-U_i}{U_i-U_{i-1}}$

Remarks on Limiting

- A way to introduce artificial diffusion
- ϕ should be smooth for $r = 1$ for 2nd order accuracy
- ϕ should be symmetric

$$\phi\left(\frac{1}{r}\right) = \frac{\phi(r)}{r}$$

- poor choices of ϕ can generated unwanted oscillations
- good choices suppress oscillations and forces the method to be TVD
- Total Variation(TV)

$$TV(U^n) := \sum_i |U^n_{i+1} - U^n_i|$$

TVD scheme

A numerical scheme is called Total Variation Diminishing if

$$TV(U^{n+1}) \leq TV(U^n)$$

Limiter functions

- Van Leer's limiter

$$\phi(r) = \frac{r + |r|}{1 + |r|}$$

- Minmod limiter

$$\phi(r) = \max(0, \min(1, r))$$

- Superbee limiter

$$\phi(r) = \max(0, \min(1, 2r), \min(2, r))$$

- Monotonized Central limiter

$$\phi(r) = \max(0, \min(2r, (1 + r)/2, 2))$$

- Sweby's limiter

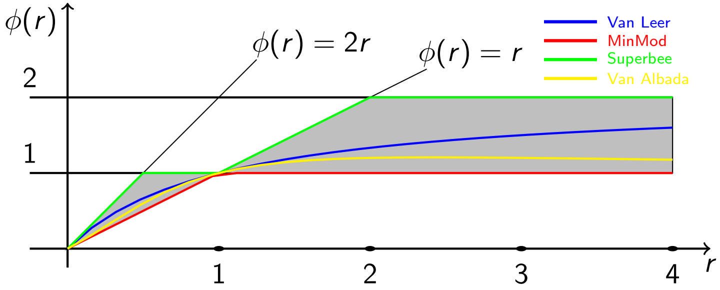
$$\phi(r) = \max(0, \min(\beta r, 1), \min(r, \beta)), \quad 1 \leq \beta \leq 2$$

- Van Albada's limiter

$$\phi(r) = \frac{r^2 + r}{r^2 + 1}$$

Admissible Limiters

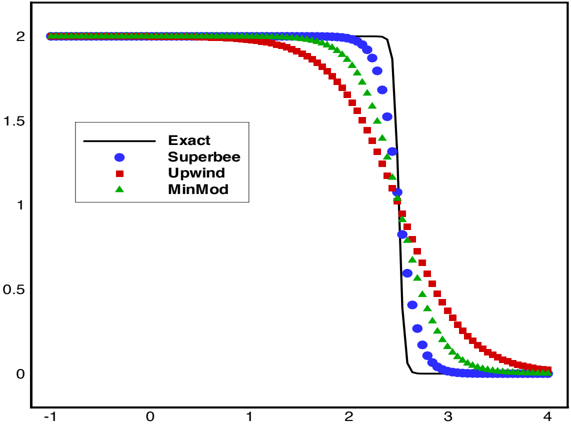
- All are second order TVD limiters
- all satisfy
  1. $\phi(1) = 1$
  2. for $0 \leq r \leq 1,\ r \leq \phi(r) \leq 2r$
  3. for $1 \leq r \leq 2,\ 1 \leq \phi(r) \leq r$
  4. for $r > 2,\ 1 \leq \phi(r) \leq 2$
- Sweby's diagram of admissible limiters



Effect of Limiters

Solution of Burgers eqn.

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$



Higher Order Reconstructions

- TVD is relaxed to TVB

TVB scheme

A numerical scheme is called Total Variation Bounded if

$$TV(U^n) \leq C(TV(U^0)), \ \forall n, \ n\Delta t \leq T$$

- Essentially Non Oscillatory(ENO) schemes (Harten,Osher, et al.)
- Uniformly Non Oscillatory(UNO) schemes (Harten, Osher)
- Weighted ENO(WENO) schemes (Liu, et al.)

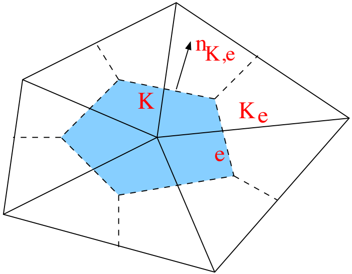
The 2D case

Using the Gauss divergence theorem we have

$$\int_{t^n}^{t^{n+1}} \int_K u_t \, dx + \int_{t^n}^{t^{n+1}} \int_K \nabla \cdot f(u) \, dx = 0$$

$$\int_K u(x, t^{n+1}) \, dx - \int_K u(x, t^n) \, dx + \int_{t^n}^{t^{n+1}} \int_{\partial K} f(u) \cdot n_K \, ds = 0$$

$$\int_K u(x, t^{n+1}) \, dx - \int_K u(x, t^n) \, dx + \sum_{e \in \partial K} \int_e \int_{t^n}^{t^{n+1}} f(u) \cdot n_{K,e} \, ds = 0$$



The FV scheme in 2D

The basic FV scheme reads

$$U_K^{n+1} = U_K^n - \frac{\Delta t}{|K|} \sum_{e \in \partial K} F(U_K^n, U_{K_e}^n, n_{K,e}), \quad U_K^n \approx \frac{1}{|K|} \int_K u(x, t^n)\, dx$$

- F is a numerical flux :
  1. Consistent : $F(u, u, n) = \int_e f(u) \cdot n$
  2. Conservative : $F(U_K, U_{K_e}, n_{K,e}) = -F(U_{K_e}, U_K, n_{K_e,e})$
  3. Monotone : $\partial_u F > 0, \ \partial_v F < 0$
- Convergence : as before but more technical
- Extension to 2nd order possible but technical. Higher order ... very technical
- For Cartesian grids things are much simpler!
- Actually this is also the multi-D version of the FV method

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