An introduction to the Finite Element Method (FEM)

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Outline

Motivation

- Physical problem $\rightarrow$ Mathematical Model
- Mathematical Model $\rightarrow$ Set of Partial Differential Equations
- PDE's $\rightarrow$ Complex, impossible to solve analytically
- Rely on a Numerical Method to solve (approximate) the pde's

Motivation

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Example

- Fluid Flow : Conservation of Mass and Momentum
- Euler equations :

$$\rho_t + \operatorname{div}(\rho u) = 0$$
$$(\rho u)_t + \operatorname{div}(\rho u^2 + p(\rho)) = 0$$
$$\rho \text{ density }, \quad u \text{ velocity }, \quad p \text{ pressure}$$

- Can't be solved analytically,
- Approximate the solution by a Numerical Method

Numerical Methods

Numerical Methods

- Finite Difference Method(FD)
  - The first method used
  - Conceptually the simplest of all
  - Given a set of discrete points P_i finds U_i which approximate the values of the solution at these points : $U_i \approx u(P_i)$
  - Increasing the number of points P_i we get (hopefully) better approximation

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  - ...but it is like a FD method since it computes a set of discrete approximations to the solution

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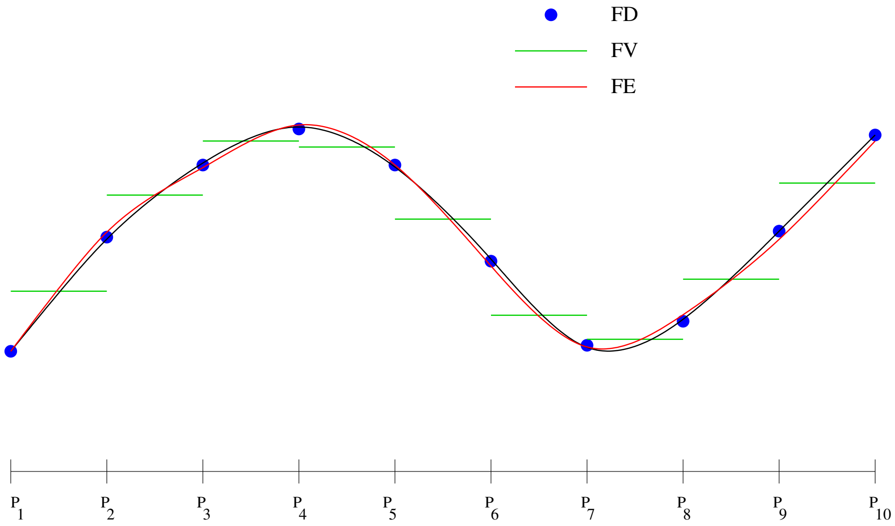
- Finite Element Method (FE)
  - Conceptually differs from FD and FV
  - ....

The Finite Element Method - A picture



The Finite Element Method - Basic Idea

For an unknown function u the FEM produces an approximating function U

$$u(x,t) \approx U(x,t) = \sum_{i=1}^{M} \alpha_i(t)\phi_i(x)$$

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FEM vs FD or FV

Advantages of the FEM over FD or FV

- easier handling of complex geometries
- boundary conditions are handled in a systematic way
- nonlinearities are treated more easily
- "Arbitrary" order of approximation
- Rigorous mathematical framework

FEM : Applications areas

- Engineering applications : Mechanical, Civil, Automotive, Aerospace
- Structural / Stress Analysis
- Fluid flow
- Heat Transfer
- Electromagnetism
- Biomechanics
- ...many more

FEM : basic steps

Physical problem

$\Downarrow$

Mathematical Model - Set of p.d.e's

$\Downarrow$

A Variational Formulation

$\Downarrow$

A Discrete Variational problem

$\Downarrow$

Solve an Algebraic System of equations

$\Downarrow$

Solution, Postprocessing

A model problem in 1D

Consider the O.D.E

$$-(p(x)u'(x))' + q(x)u(x) = f(x), \quad x \in (a,b)$$

$p(x),\ q(x),\ f(x)$ are known data of the problem, $u(x)$ is the solution to be determined

Assumptions : $p \in C^1([a,b]),\ p(x) \geq p_0 > 0,\ q \in C([a,b]),\ q(x) \geq 0,$ $f \in C([a,b])$

We need Boundary Conditions :

Dirichlet $u(a) = A,\ u(b) = B$

Neumann $p(a)u'(a) = A,\ p(b)u'(b) = B$

Mixed $u(a) + p(a)u'(a) = A,\ u(b) + p(b)u'(b) = B$

Applications: heat transfer, elastic cables-bars, flow of viscous fluids, flow through porous medium, electrostatics

A model problem in 1D - Variational formulation

Idea : multiply the equation by a test function v and integrate over the domain

$$\int_a^b \left(-(p(x)u'(x))' + q(x)u(x)\right) v(x)\, dx = \int_a^b f(x)v(x)\, dx.$$

Integrate the first term by parts

$$\int_a^b \big(p(x)u'(x)v'(x) + q(x)u(x)v(x)\big)\, dx - \big[p(x)u'(x)v(x)\big]_a^b = \int_a^b f(x)v(x)dx.$$

Dirichlet(zero) boundary conditions : $u(a) = u(b) = 0$, then we choose $v(a) = v(b) = 0$

Variational Formulation and Dirichlet B.C's

Find u such that $u(a) = u(b) = 0$ and

$$\text{(WP)} \qquad \mathcal{B}(u, v) = \ell(v), \qquad \forall v, \quad v(a) = v(b) = 0$$

where

$$\mathcal{B}(u, v) := \int_a^b \big(p(x)u'(x)v'(x) + q(x)u(x)v(x)\big)\, dx$$
$$\ell(v) := \int_a^b f(x)v(x)dx.$$

$\mathcal{B}$ is a bilinear form and ℓ a linear functional.

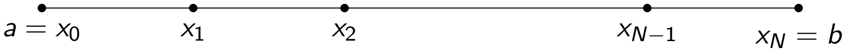
Under certain assumptions problem (WP) has a unique solution.

Also called the weak formulation of the problem

FE Solution of 1D model : setup

We look for an approximate solution to the weak formulation of the problem.

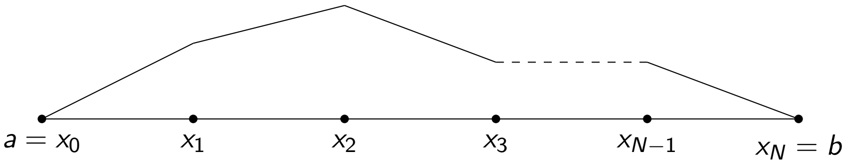
FEM : split $[a, b]$ into subintervals(elements) and approximate the solution in each interval by a polynomial



A uniform mesh of $N-$subintervals, $N + 1$ nodes :
$x_i = a + ih, \ h = (b - a)/N, i = 0, \dots, N$

FE Solution of 1D model : Linear Elements

We approximate the solution in each subinterval by a linear polynomial



Approximate solution $u_h \approx u$: Piecewise Linear polynomial

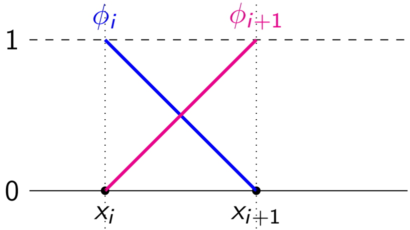
$$u_h(x) = \sum_{i=0}^{N} U_i \phi_i(x)$$

where $\phi_i(x)$ linear polynomials defined locally (compact support)

FE Solution of 1D model : Linear Basis functions

We define

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x_i \le x \le x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$



any linear polynomial in $[x_i, x_{i+1}]$ is a linear combination of $\phi_i,\ \phi_{i+1}$

$$u_h(x)|_{[x_i, x_{i+1}]} = U_i \phi_i(x) + U_{i+1} \phi_{i+1}(x)$$

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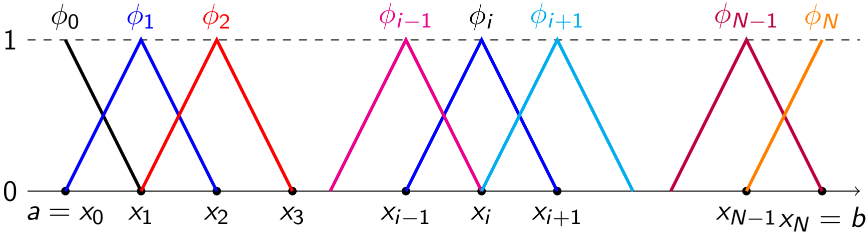


FE Solution of 1D model : Discrete problem

Find u_h such that (Discrete variational problem)

(DWP) $$\mathcal{B}(u_h, v_h) = \ell(v_h), \quad \forall v_h$$

Recall

$$u_h(x) = \sum_{i=0}^{N} U_i \phi_i(x), \quad U_i \in \mathbb{R}$$

and taking $v_h = \phi_j$ we get

$$\mathcal{B}\left(\sum_{i=0}^{N} U_i \phi_i, \phi_j\right) = \ell(\phi_j) \Rightarrow \sum_{i=0}^{N} U_i \mathcal{B}(\phi_i, \phi_j) = \ell(\phi_j), \ j = 0, \ldots, N$$

In matrix form

$$\mathcal{K}\mathbf{U} = \mathbf{f}, \quad \mathbf{U} = \{U_i\}$$
$$\mathcal{K}_{ij} = \mathcal{B}(\phi_i, \phi_j), f_i = \ell(\phi_i)$$

FE Solution of 1D model : system matrix $\mathcal{K}$

$$\mathcal{K}_{ij} = \mathcal{B}(\phi_i, \phi_j) = \int_a^b \big(p(x)\phi_i'(x)\phi_j'(x) + q(x)\phi_i(x)\phi_j(x)\big)\,dx$$

$$\sum_{m=0}^{N-1} \int_{x_m}^{x_{m+1}} \big(p(x)\phi_i'(x)\phi_j'(x) + q(x)\phi_i(x)\phi_j(x)\big)\,dx$$

$$\mathcal{K}_{ij} \neq 0 \Longleftrightarrow i, j = m, m+1$$

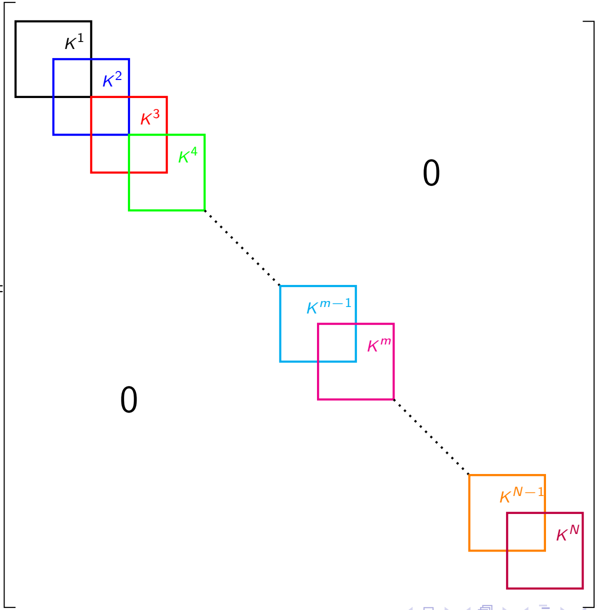
$$\mathcal{K} = \sum_{m=0}^{N-1} K^m, \quad K^m = \text{ element system matrix}$$

$$K^{m-1} = \begin{bmatrix} \mathcal{K}_{m-1,m-1} & \mathcal{K}_{m-1,m} \\ \mathcal{K}_{m,m-1} & \mathcal{K}_{m,m} \end{bmatrix} K^m = \begin{bmatrix} \mathcal{K}_{m,m} & \mathcal{K}_{m,m+1} \\ \mathcal{K}_{m+1,m} & \mathcal{K}_{m+1,m+1} \end{bmatrix}$$

i.e. $\mathcal{K}$ is a Tridiagonal Matrix ($supp\phi_i = 2$ intervals)

FE Solution of 1D model : system matrix $\mathcal{K}$

$\mathcal{K} =$

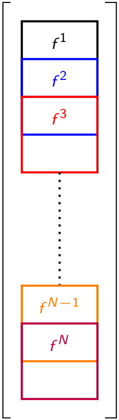


FE Solution of 1D model : load vector **f**

$$f_i = \ell(\phi_i) = \int_a^b f(x)\phi_i(x)dx = \sum_{m=0}^{N-1} \int_{x_m}^{x_{m+1}} f(x)\phi_i(x)dx$$

$$f_i \neq 0 \iff i = m, m+1$$

$\mathbf{f} =$



FE Solution of 1D model : Remarks

- FE solution is obtained by solving a linear system :
  - (Sparse) Tridiagonal(Banded) matrix. Size of the band is related to the support of basis functions (bw= 2supp-1)
  - Matrix is positive definite
  - Effective linear solvers
- Numerical quadrature for evaluating the integrals :
  - Matrix entries : if $p(x)$, $q(x)$ are constants or polynomials then using, e.g. Gauss quadrature rule the integrals are evaluated exactly. Otherwise there is quadrature error
  - Load vector entries : unless $f(x)$ is simple, there is always quadrature error.
  - Make sure to use accurate enough quadrature rule so that does not influence the accuracy of the method.
- Higher order elements : quadratic, cubic, quartic, ...
- Lagrange Basis, Splines, ...

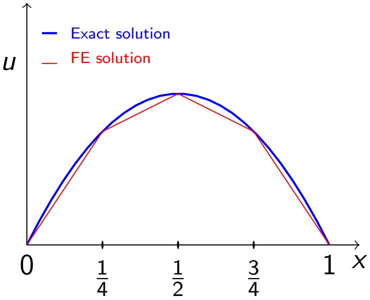
FE Solution of 1D model : Theory

Error estimates : It can be shown that

$$\|u - u_h\|_{L^2(a,b)} \leq Ch^2 \qquad \|u' - u_h'\|_{L^2(a,b)} \leq Ch$$

where C depends only on the solution and the data of the problem.

Example : $[a, b] = [0, 1],\ p(x) = 1,\ q(x) = 0,\ f(x) = 1$ so we are solving $-u'' = 1,\ u(0) = u(1) = 0 \Rightarrow u(x) = \frac{1}{2}x(1 - x)$



FEM for elliptic problems : setup

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$ be a bounded domain

$$(\text{GEP}) \quad -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x),\ x \in \Omega$$

$$u = 0,\ x \in \partial\Omega$$

$$a_{ij} \in C^1(\bar{\Omega}),\ b_i \in C(\bar{\Omega}),\ c \in C(\bar{\Omega}),\ f \in C(\bar{\Omega})$$

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \tilde{c}\sum_{i=1}^{d}\xi_i^2,\ \tilde{c} > 0 \quad \forall \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d,\ x \in \bar{\Omega}$$

$$c(x) - \frac{1}{2}\sum_{i=1}^{d}\frac{\partial b_i}{\partial x_i} \geq 0,\ x \in \Omega$$

Classical solution : $u \in C^2(\Omega) \cap C(\Omega)$ satisfying (GEP). Such smoothness is VERY restrictive for applications $\Longrightarrow$ Weak Solutions

FEM for elliptic problems : Weak solutions

Definition

Let $a_{ij},\ b_i,\ c \in L^\infty(\Omega),\ i,j = 1,\dots,d,\ f \in L^2(\Omega)$. A $u \in H_0^1(\Omega)$ satisfying

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{d} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i} v dx + \int_\Omega c(x) u v dx$$
$$= \int_\Omega f(x) v dx,\ \forall v \in H_0^1(\Omega)$$

is called Weak solution of (GEP). spaces

if u Classical solution $\Longrightarrow$ u Weak solution ... but not the contrary!!!

FEM for elliptic problems : Weak formulation

$$a(u,v) := \sum_{i,j=1}^{d} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{d} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i} v dx + \int_\Omega c(x) uv dx$$

$$\ell(v) := \int_\Omega f(x) v dx$$

Weak problem :

Find $u \in H_0^1(\Omega)$ such that $a(u,v) = \ell(v), \quad \forall v \in H_0^1(\Omega)$

Existence and uniqueness of such u is provided by the Lax-Milgram theorem LMTheorem with $V = H_0^1$

FEM for elliptic problems : Discrete variational problem

Recall for $V = H_0^1(\Omega)$

$$\text{Find } u \in V \text{ such that } a(u,v) = \ell(v), \quad \forall v \in V$$

replace V by a finite-dimensional subspace $V_h \subset V$ of continuous piecewise polynomials associated with a subdivision of length h of Ω

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = \ell(v_h), \quad \forall v_h \in V_h$$

Let

$$\dim\ V_h = N_h \quad V_h = \text{ span } \{\phi_1, \ldots, \phi_{N_h}\}$$

where the ϕ_i's form a basis of V_h and have "small" support.

FEM for elliptic problems : Discrete variational problem

We write

$$u_h(x) = \sum_{i=1}^{N_h} U_i \phi_i(x), \quad U_i \text{ to be determined}$$

Then the Discrete problem is

$$\text{Find } (U_1, \ldots, U_{N_h}) \in \mathbb{R}^{N_h} \text{ s.t } \sum_{i=1}^{N_h} a(\phi_i, \phi_j) U_i = \ell(\phi_j),\ j = 1, \ldots, N_h$$

so U_i's are the solution of a linear system.

FEM for elliptic problems : Approximation result

Cea's Lemma

Theorem

The FE approximation u_h to u is the near-best fit to u in H^1-norm

$$\|u - u_h\|_{H^1(\Omega)} \le \frac{c1}{c0} \min_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}$$

Approximation result

Theorem

For the "usual" finite element spaces we have

$$\min_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} \le C(u) h^s$$

- $C(u) > 0$ *positive constant depending on the smoothness of* u
- $s > 0$ *depending on the smoothness of* u *and the degree of the polynomials comprising the space* V_h

FEM for elliptic problems : Error estimate

Combining the previous results we get the following error estimate

$$\|u - u_h\|_{H^1(\Omega)} \leq C(u)\frac{c1}{c0}h^s$$

- An *a priori* bound
- As $h \to 0$ the sequence of FE solutions $\{u_h\}_h \to u \in H^1(\Omega)$
- Difficult to quantify the error since it depends on the solution itself.
- A posteriori estimates

$$\|u - u_h\|_{H^1(\Omega)} \leq \eta(u_h)h^s, \quad \eta(u_h) \text{ computable quantity}$$

FE - Bibliography

- P. Ciarlet, *The finite element method for elliptic problems*, SIAM 2002
- V. Thomée *Galerkin Finite Element Methods for Parabolic Problems* , Springer 2006

Thank You

Lax-Milgram Theorem

Theorem

Suppose that V is a real Hilbert space equipped with norm $\|\cdot\|_V$. Let $a(\cdot,\cdot)$ be a bilinear functional on $V \times V$ such that :

1. $\exists c_0 > 0 \ \forall v \in V \ a(v,v) \geq c_0 \|v\|_V^2$ *(coercivity)*
2. $\exists c_1 > 0 \ \forall v, w \in V \ |a(w,v)| \leq c_1 \|w\|_V \|v\|_V$ *(continuity)*
3. $\ell(\cdot)$ *is linear functional on V such that* $\exists c_2 > 0 \ \forall v \in V \ |\ell(v)| \leq c_2 \|v\|_V$ *(continuity)*

Then there exists a unique $u \in V$ such that

$$a(u,v) = \ell(v) \quad \forall v \in V$$

back

Function Spaces

$$L^p(\Omega) := \{u : \|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx\right)^{1/p} < \infty\}$$

$$L^\infty(\Omega) := \{u : \|u\|_{L^\infty(\Omega)} := \text{ ess sup}_{x\in\Omega}|u(x)| < \infty\}$$

$$W_p^k(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \leq k\}$$

$$H^k(\Omega) := W_2^k \text{ (Hilbert space)}$$

$$H_0^1 := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}$$

back