Understanding large-scale atmospheric and oceanic flows with layered rotating shallow water models

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Plan

Large-scale atmospheric and oceanic flows

Primitive equations

- Ocean
- Atmosphere

Vertical averaging and RSW models

Properties of the RSW model(s)

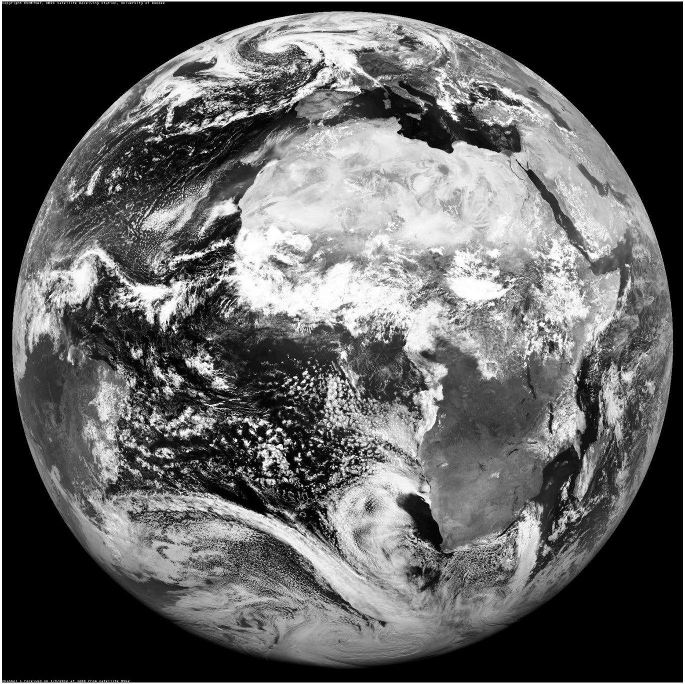
- Main properties
- Conservation laws

Well-balanced finite-volume schemes for 1-layer RSW

- Conceptual example
- Hydrostatic reconstruction
- Rotation as apparent topogrpahy
- Going second order
- Going two-dimensional

Efficient scheme for 2-layer RSW

Literature



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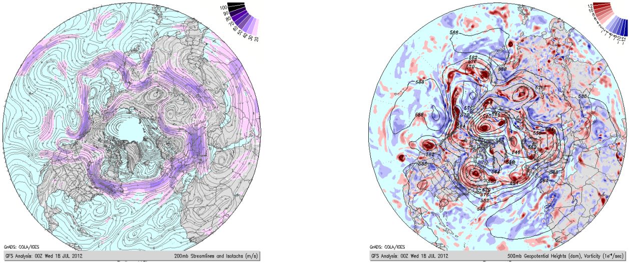
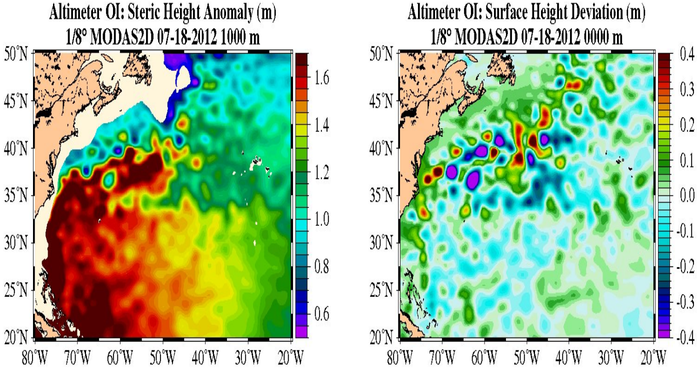
Midlatitude atmospheric jet



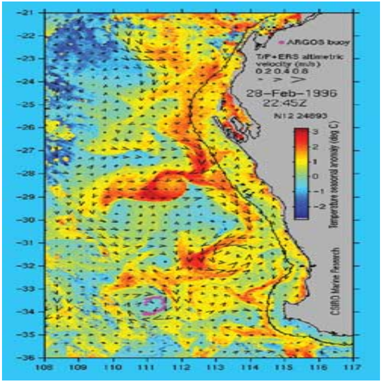
Midlatidude upper-tropospheric jet (left) and related synoptic systems (right).

Oceanic currents : Gulfstream



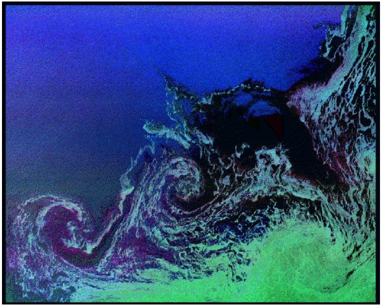
Gulfstream (left) and related vortices (right). Velocity follows isopleths of the height anomaly in the first approximation.

Coastal current and associated vortices



Velocity (arrows) and temperature anomaly (colors) of the Leeuwin curent near Australian coast.

Destabilizing coastal flow



Instability of a coastal current in the Weddell sea.

Main features of the large-scale flows

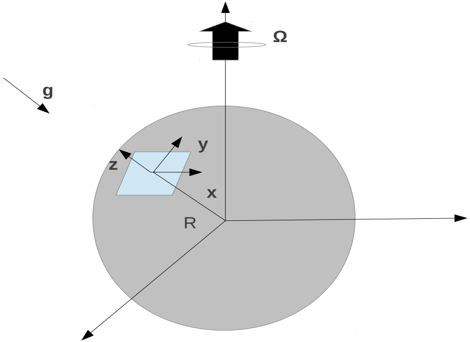
Large-scale atmospheric and oceanic flows are :

- rotating and stratified
- close to hydrostatic equilibrium
- having typical horizontal scale L and typical vertical scale H, $L \gg H$

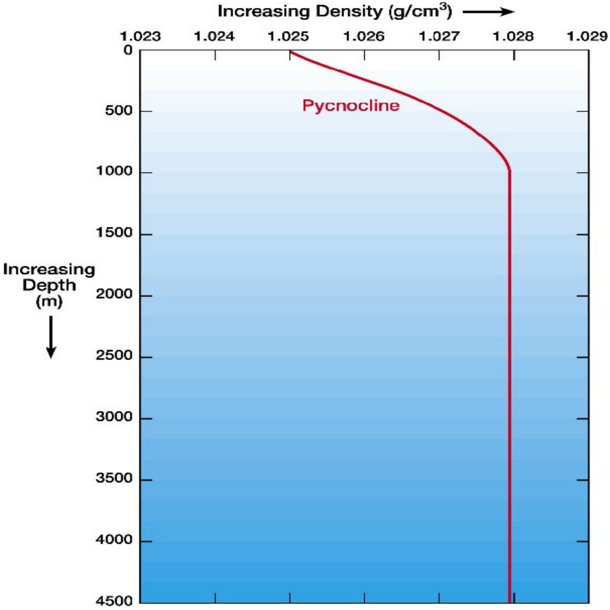
Yet,

$L \ll R$, the Earth's radius $\longrightarrow$ tangent plane approximation.

Tangent-plane approximation



Mean oceanic stratification



Primitive equations : ocean

Hydrostatics

$$g\rho + \partial_z P = 0, \tag{1}$$

$$\begin{aligned} P &= P_0 + P_s(z) + \pi(x, y, z; t), \\ \rho &= \rho_0 + \rho_s(z) + \sigma(x, y, z; t), \quad \rho_0 \gg \rho_s \gg \sigma \end{aligned}$$

Incompressibility

$$\vec{\nabla} \cdot \vec{v} = 0, \quad \vec{v} = \vec{v}_h + \hat{z} w. \tag{2}$$

Euler :

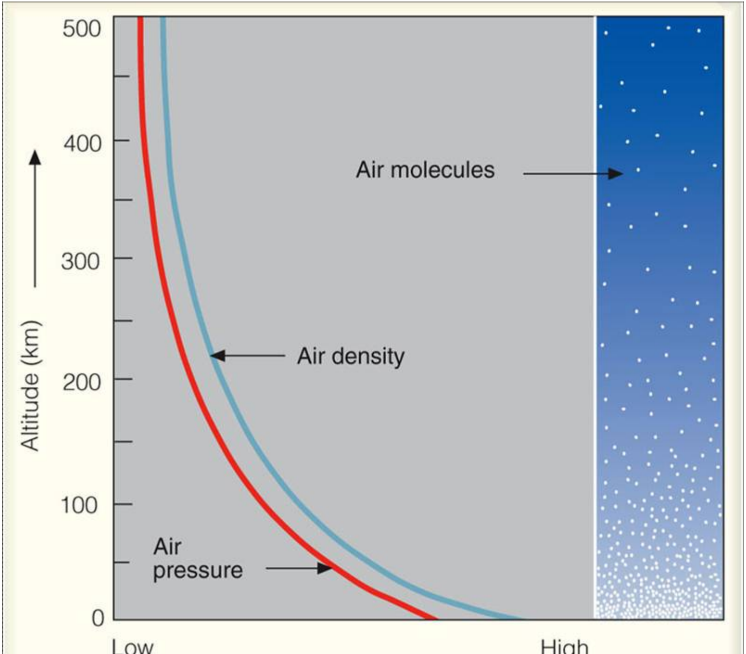
$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \phi. \tag{3}$$

$\phi = \frac{\pi}{\rho_0}$ - geopotential.

Continuity :

$$\partial_t \rho + \vec{v} \cdot \vec{\nabla} \rho = 0. \tag{4}$$

Mean atmospheric stratification



Primitive equations : atmosphere, pseudo-height vertical coordinate

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \phi, \tag{5}$$

$$-g \frac{\theta}{\theta_0} + \frac{\partial \phi}{\partial z} = 0, \tag{6}$$

$$\frac{\partial \theta}{\partial t} + \vec{v} \cdot \vec{\nabla} \theta = 0; \quad \vec{\nabla} \cdot \vec{v} = 0. \tag{7}$$

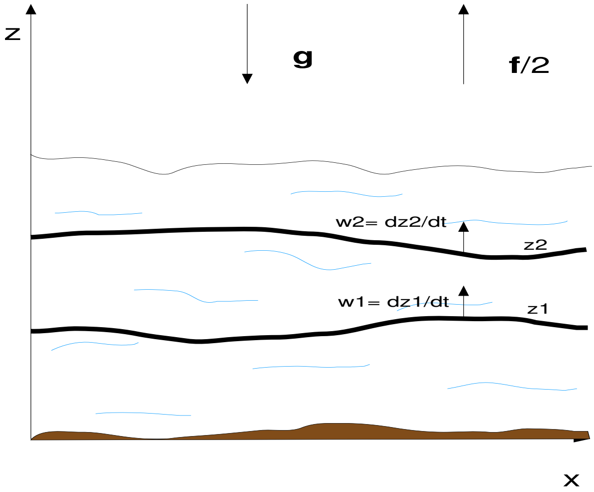
Identical to oceanic ones with $\sigma \to -\theta$, potential temperature.
Vertical coordinate : pseudo-height, P - pressure.

$$\bar{z} = z_0 \left(1 - \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}} \right) \tag{8}$$

Equivalence of the primitive equations

A key fact of geophysical Fluid Dynamics
:
Under clever changes of variables ocean and atmosphere primitive equations are equivalent : incompressible stratified fluid on the rotating plane in hydrostatic equilibrium.

Fluid layers beteen material surfaces



Vertical averaging and RSW models

- Take horizontal momentum equation in conservative form :

$$(\rho u)_t + (\rho u^2)_x + (\rho v u)_y + (\rho w u)_z - f\rho v = -p_x, \quad (9)$$

and integrate between a pair of material surfaces $z_{1,2}$:

$$w|_{z_i} = \frac{dz_i}{dt} = \partial_t z_i + u\partial_x z_i + v\partial_y z_i, \quad i = 1, 2. \quad (10)$$

- Use Leibnitz formula and get :

$$\begin{aligned} \partial_t \int_{z_1}^{z_2} dz \rho u \quad + \quad \partial_x \int_{z_1}^{z_2} dz \rho u^2 + \partial_y \int_{z_1}^{z_2} dz \rho u v - \\ f \int_{z_1}^{z_2} dz \rho v \quad = \quad -\partial_x \int_{z_1}^{z_2} dz p - \partial_x z_1 \; p|_{z_1} + \partial_x z_2 \; p|_{z_2} . \end{aligned}$$

(analogously for v).

- Use continuity equation and get

$$\partial_t \int_{z_1}^{z_2} dz\rho + \partial_x \int_{z_1}^{z_2} dz\rho u + \partial_y \int_{z_1}^{z_2} dz\rho v = 0. \quad (11)$$

- Introduce the mass- (entropy)- averages :

$$\langle F \rangle = \frac{1}{\mu} \int_{z_1}^{z_2} dz\rho F, \quad \mu = \int_{z_1}^{z_2} dz\rho. \quad (12)$$

and obtain averaged equations :

$$\begin{aligned} &\partial_t (\mu\langle u\rangle) + \partial_x \left(\mu\langle u^2\rangle\right) + \partial_y (\mu\langle uv\rangle) - f\mu\langle v\rangle \\ = \quad &-\partial_x \int_{z_1}^{z_2} dzp - \partial_x z_1 \, p|_{z_1} + \partial_x z_2 \, p|_{z_2}, \quad (13) \end{aligned}$$

$$\begin{aligned} &\partial_t (\mu\langle v\rangle) + \partial_x (\mu\langle uv\rangle) + \partial_y \left(\mu\langle v^2\rangle\right) + f\mu\langle u\rangle \\ = \quad &-\partial_y \int_{z_1}^{z_2} dzp - \partial_y z_1 \, p|_{z_1} + \partial_y z_2 \, p|_{z_2}, \quad (14) \end{aligned}$$

$$\partial_t\mu + \partial_x (\mu\langle u\rangle) + \partial_y (\mu\langle v\rangle) = 0. \quad (15)$$

- Use hydrostatics and get, introducing mean constant density $\bar{\rho}$:

$$p(x,y,z,t) \approx -g\bar{\rho}(z - z_1) + p|_{z_1} \, . \tag{16}$$

- Use the mean-field (= columnar motion) approximation :

$$\langle uv\rangle \approx \langle u\rangle\langle v\rangle, \; \langle u^2\rangle \approx \langle u\rangle\langle u\rangle, \; \langle v^2\rangle \approx \langle v\rangle\langle v\rangle. \tag{17}$$

and get master equation for the layer :

$$\begin{aligned} & \bar{\rho}(\mathbf{z}_2 - \mathbf{z}_1)(\partial_\mathbf{t}\mathbf{v_h} + \mathbf{v}\cdot\nabla\mathbf{v_h} + \mathbf{f}\hat{\mathbf{z}}\wedge\mathbf{v_h}) = \\ - \quad & \nabla_\mathbf{h}\left(-\mathbf{g}\bar{\rho}\frac{(\mathbf{z}_2 - \mathbf{z}_1)^2}{2} + (\mathbf{z}_2 - \mathbf{z}_1)\,\mathbf{p}|_{\mathbf{z}_1}\right) \\ - \quad & \nabla_\mathbf{h}\mathbf{z}_1\,\mathbf{p}|_{\mathbf{z}_1} + \nabla_\mathbf{h}\mathbf{z}_2\,\mathbf{p}|_{\mathbf{z}_2} \, . \end{aligned} \tag{18}$$

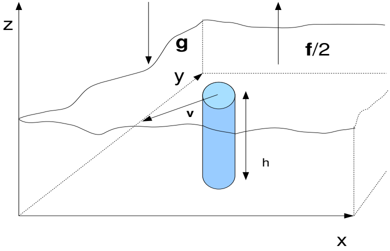
- Pile up layers, with lowermost boundary fixed by topography, and uppermost free or fixed.

1-layer RSW, $z_1 = 0$, $z_2 = h$

$$\partial_t v + v \cdot \nabla v + f\hat{z} \wedge v + g\nabla h = 0\,, \tag{19}$$

$$\partial_t h + \nabla \cdot (vh) = 0\,. \tag{20}$$

⇒ 2d barotropic gas dynamics + Coriolis force.
In the presence of nontrivial topography $b(x, y)$:
$h \rightarrow h - b$ in the second equation.



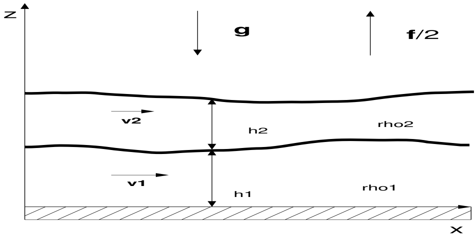
2-layer rotating shallow water model with a free surface : $z_1 = 0$, $z_2 = h_1$, $z_3 = h_1 + h_2$

$$\partial_t v_2 + v_2 \cdot \nabla v_2 + f\hat{z} \wedge v_2 = -\nabla(h_1 + h_2) \tag{21}$$

$$\partial_t v_1 + v_1 \cdot \nabla v_1 + f\hat{z} \wedge v_1 = -\nabla(rh_1 + h_2), \tag{22}$$

$$\partial_t h_{1,2} + \nabla \cdot \left(v_{1,2} h_{1,2}\right) = 0\,, \tag{23}$$

where $r = \frac{\rho_1}{\rho_2} \leq 1$ - density ratio, and $h_{1,2}$ - thicknesses of the layers.



Useful notions

Balanced vs unbalanced motions

Geostrophic balance : balance between the Coriolis force and the pressure force. In shallow-water model :

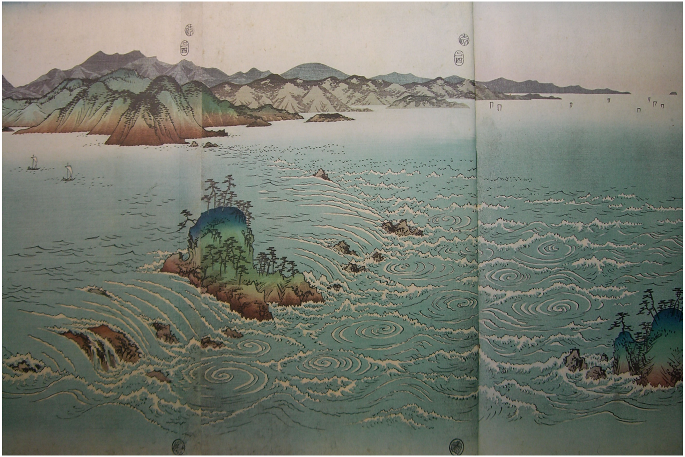
$$f\hat{z} = -g\nabla h \tag{24}$$

Valid at small Rossby numbers : $Ro = U/fL$, where $U,\ L$ - characteristic velocity and horizontal scale.
Balanced motions at small Ro : vortices.
Unbalanced motions : inertia-gravity waves.

Relative, absolute and potential vorticity

Relative vorticity in layered models : $\zeta = \hat{z}\cdot\nabla\wedge v$.
Absolute vorticity : $\zeta + f$. Potential vorticity (PV) : $q_{12} = \frac{\zeta+f}{z_2-z_1}$ for the fluid layer between z_2 and z_1.

Dynamical actors in RSW : vortices & waves



Conservation laws in RSW (no topography)

Equations in conservative form (momentum & mass)

$$
\begin{aligned}
\partial_t(hu) + \partial_x(hu^2) + \partial_y(huv) - fhv + g\partial_x \frac{h^2}{2} &= 0, \\
\partial_t(hv) + \partial_x(huv) + \partial_y(hv^2) + fhu + g\partial_y \frac{h^2}{2} &= 0, \\
\partial_t h + \partial_x(hu) + \partial_y(hv) &= 0.
\end{aligned}
$$

Energy is locally conserved :

$$E = \int dxdy\, e = \int dxdy\, \left(h\frac{u^2+v^2}{2} + g\frac{h^2}{2} \right),$$

$\partial_t e + \nabla \cdot f_e = 0.$

Lagrangian conservation of potential vorticity :

$(\partial_t + u\partial_x + v\partial_y)\, q = 0.$

Reminder

RSW system :

- Equivalent to 2d barotropic gas dynamics (if no topography and rotation).
- Hyperbolic (except at resonant points (crossing of eigenvalues of the characteristic matrix).
- Rotation - stiff source.
- Weak solutions $\leftrightarrow$ Rankine-Hugoniot conditions. Selection : energy decrease across shocks (equivalent to entropy increase in gas dynamics).
- Natural numerical method : finite-volume, shock-capturing

1-dimensional SW with topography :

Equations in conservative form, where $Z(x)/g$ - topography :

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + gh^2/2)_x + hZ_x = 0, \end{cases} \tag{25}$$

Convex entropy (energy) :

$$e = hu^2/2 + gh^2/2 + ghZ$$

with entropy flux $\left(e + gh^2/2\right) u$.

Numerical difficulties :

- keeping $h \geq 0$,
- maintaining steady states at rest ("well-balanced" property $u = 0$, $gh + Z = \text{const}$
- treatment of drying $h \to 0$,
- satisfying a discrete entropy inequality.

First-order three-point finite-volume schemes

Discretization :

Grid $x_{i+1/2}$, $i \in \mathbb{Z}$, cells (finite volumes)
$C_i = (x_{i-1/2}, x_{i+1/2})$, centers $x_i = (x_{i-1/2} + x_{i+1/2})/2$,
lengths $\Delta x_i = x_{i+1/2} - x_{i-1/2}$.
Discrete data (U_i^n, Z_i), U_i^n – approximation of $U = (h, hu)$.

Evolution :

$$U_i^{n+1} - U_i^n + \frac{\Delta t}{\Delta x_i}(F_{i+1/2-} - F_{i-1/2+}) = 0, \tag{26}$$

Z_i does not evolve,

$$F_{i+1/2-} = \mathcal{F}_l(U_i, U_{i+1}, \Delta Z_{i+1/2}),\ F_{i+1/2+} = \mathcal{F}_r(U_i, U_{i+1}, \Delta Z_{i+1/2}), \tag{27}$$

with $\Delta Z_{i+1/2} = Z_{i+1} - Z_i$.

Consistency

Numerical fluxes $\mathcal{F}_l$ and $\mathcal{F}_r$ must satisfy two consistency properties.

- consistency with the conservative term :

  $$\mathcal{F}_l(U,U,0)=\mathcal{F}_r(U,U,0)=F(U)\equiv(hu,hu^2+gh^2/2), \tag{28}$$

- consistency with the source :

  $$\mathcal{F}_r(U_l,U_r,\Delta Z)-\mathcal{F}_l(U_l,U_r,\Delta Z)=(0,-h\Delta Z)+o(\Delta Z), \tag{29}$$

  as U_l, $U_r \to U$ and $\Delta Z \to 0$.

Well-balancing and mass conservation

A required global property is the conservation of mass,

$$\mathcal{F}_l^h(U_l, U_r, \Delta Z) = \mathcal{F}_r^h(U_l, U_r, \Delta Z) \equiv F^h(U_l, U_r, \Delta Z). \tag{30}$$

The property for the scheme to be well-balanced is that

$$F_{i+1/2-} = F(U_i) \text{ and } F_{i+1/2+} = F(U_{i+1})$$
$$\text{whenever } u_i = u_{i+1} = 0 \text{ and } gh_{i+1} - gh_i + \Delta Z_{i+1/2} = 0. \tag{31}$$

Hydrostatic reconstruction scheme (Audusse *et al*, 2003)

$$
\begin{aligned}
F_l(U_l, U_r, \Delta Z) = \mathcal{F}(U_l^*, U_r^*) + \begin{pmatrix} 0 \\ \frac{g}{2}h_l^2 - \frac{g}{2}h_{l*}^2 \end{pmatrix}, \\
F_r(U_l, U_r, \Delta Z) = \mathcal{F}(U_l^*, U_r^*) + \begin{pmatrix} 0 \\ \frac{g}{2}h_r^2 - \frac{g}{2}h_{r*}^2 \end{pmatrix},
\end{aligned}
\qquad (32)
$$

where $U_l^* = (h_{l*}, h_{l*}u_l)$, $U_r^* = (h_{r*}, h_{r*}u_r)$, and

$$
\begin{aligned}
h_{l*} &= \max(0, h_l - \max(0, \Delta Z/g)), \\
h_{r*} &= \max(0, h_r - \max(0, -\Delta Z/g)).
\end{aligned}
$$

$\mathcal{F}$ is any entropy satisfying consistent numerical flux for the problem with $Z = cst$. Multiple choices for $\mathcal{F}$ in the literature - approximate Riemann solvers (Roe, HLL, HLLC,...).

Rotation as an apparent topography

1.5d shallow water with topography and Coriolis force

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + gh^2/2)_x + hZ_x - fhv = 0, \\ (hv)_t + (huv)_x + fhu = 0, \end{cases} \tag{33}$$

where $Z = Z(x)$, $f = f(x)$. Solutions at rest are given by $u = 0$, $fv = (gh + Z)_x$. The trick is to identify the two first equations in (33) as (25) with a new topography $Z + B$, where $B_x = -fv$. As v depends on time while B should be time-independent, so take $B_x^n = -fv^n$ and solve (25) on the time interval (t_n, t_{n+1}) with topography $Z + B^n$.

Discretized 1.5d RSW

$$
\begin{aligned}
h_i^{n+1} - h_i^n + \frac{\Delta t}{\Delta x_i}(F^h_{i+1/2} - F^h_{i-1/2}) &= 0,\\
h_i^{n+1}u_i^{n+1} - h_i^n u_i^n + \frac{\Delta t}{\Delta x_i}(F^{hu}_{i+1/2-} - F^{hu}_{i-1/2+}) &= 0, \qquad (34)\\
h_i^{n+1}v_i^{n+1} - h_i^n v_i^n + \frac{\Delta t}{\Delta x_i}(F^{hv}_{i+1/2-} - F^{hv}_{i-1/2+}) &= 0,
\end{aligned}
$$

with

$$
\begin{aligned}
(F^h_{i+1/2}, F^{hu}_{i+1/2-}) &= \mathcal{F}_l^{1d}(h_i, u_i, h_{i+1}, u_{i+1}, \Delta z_{i+1/2} + \Delta b^n_{i+1/2}),\\
(F^h_{i+1/2}, F^{hu}_{i+1/2+}) &= \mathcal{F}_r^{1d}(h_i, u_i, h_{i+1}, u_{i+1}, \Delta z_{i+1/2} + \Delta b^n_{i+1/2}),
\end{aligned}
\qquad (35)
$$

$$
\Delta b^n_{i+1/2} = -f_{i+1/2}\frac{v_i^n + v_{i+1}^n}{2}\Delta x_{i+1/2}/g. \qquad (36)
$$

and $\mathcal{F}_l^{1d}$ and $\mathcal{F}_r^{1d}$ - numerical hydrostatic reconstruction fluxes of the 1d shallow water.

Transverse momentum fluxes

A natural discretization associated to the equivalent conservation law (geostrophic momentum) $(h(v+\Omega))_t + (hu(v+\Omega))_x = 0$, with $\Omega_x = f$, which is strongly related to the potential vorticity :

$$F^{hv}_{i+1/2-} = \begin{cases} F^h_{i+1/2} v_i & \text{if } F^h_{i+1/2} \geq 0, \\ F^h_{i+1/2}(v_{i+1} + \Delta\Omega_{i+1/2}) & \text{if } F^h_{i+1/2} \leq 0, \end{cases} \tag{37}$$

$$F^{hv}_{i+1/2+} = \begin{cases} F^h_{i+1/2}(v_i - \Delta\Omega_{i+1/2}) & \text{if } F^h_{i+1/2} \geq 0, \\ F^h_{i+1/2} v_{i+1} & \text{if } F^h_{i+1/2} \leq 0, \end{cases} \tag{38}$$

with

$$\Delta\Omega_{i+1/2} = f_{i+1/2}\Delta x_{i+1/2}. \tag{39}$$

Second-order reconstruction - general

Reconstruction operator : $(U_i, Z_i) \to U_{i+1/2-}, Z_{i+1/2-}, U_{i+1/2+}, Z_{i+1/2+}$ for $i \in \mathbb{Z}$, in a way that it is

- conservative in U :

$$\frac{U_{i-1/2+} + U_{i+1/2-}}{2} = U_i, \tag{40}$$

- second-order, i.e. that whenever for all i,

$$U_i = \frac{1}{\Delta x_i} \int_{C_i} U(x)\, dx, \qquad Z_i = \frac{1}{\Delta x_i} \int_{C_i} Z(x)\, dx, \tag{41}$$

for smooth $U(x)$, $Z(x)$, then, for $\delta = \sup_i \Delta x_i$

$$\begin{aligned} U_{i+1/2-} &= U(x_{i+1/2}) + O(\delta^2), & U_{i+1/2+} &= U(x_{i+1/2}) + O(\delta^2), \\ Z_{i+1/2-} &= Z(x_{i+1/2}) + O(\delta^2), & Z_{i+1/2+} &= Z(x_{i+1/2}) + O(\delta^2). \end{aligned} \tag{42}$$

Possible reconstructions : *minmod* (respects max principle), *ENO* (non-oscillatory), ...

Second-order scheme :

$$U_i^{n+1} - U_i^n + \frac{\Delta t}{\Delta x_i}(F_{i+1/2-} - F_{i-1/2+} - F_i) = 0, \qquad (43)$$

with

$$\begin{aligned} F_{i+1/2-} &= \mathcal{F}_l\left(U^n_{i+1/2-}, U^n_{i+1/2+}, Z^n_{i+1/2-}, Z^n_{i+1/2+}\right), \\ F_{i+1/2+} &= \mathcal{F}_r\left(U^n_{i+1/2-}, U^n_{i+1/2+}, Z^n_{i+1/2-}, Z^n_{i+1/2+}\right), \\ F_i &= \mathcal{F}_c\left(U^n_{i-1/2+}, U^n_{i+1/2-}, Z^n_{i-1/2+}, Z^n_{i+1/2-}\right), \end{aligned} \qquad (44)$$

where the arguments are obtained from the reconstruction operator applied to (U_i^n, Z_i^n), and the centered flux function $\mathcal{F}_c$ to be chosen.

2nd-order reconstruction for shallow water

$(U_i, z_i) = (h_i, h_i u_i, z_i)$
$U_{i+1/2\pm} \equiv (h_{i+1/2\pm}, h_{i+1/2\pm} u_{i+1/2\pm})$ -reconstructed values. Then

$$\frac{h_{i-1/2+} + h_{i+1/2-}}{2} = h_i, \qquad (45)$$
$$\frac{h_{i-1/2+}u_{i-1/2+} + h_{i+1/2-}u_{i+1/2-}}{2} = h_i u_i.$$

Equivalent to

$$h_{i-1/2+} = h_i - \frac{\Delta x_i}{2} Dh_i, \qquad h_{i+1/2-} = h_i + \frac{\Delta x_i}{2} Dh_i,$$
$$u_{i-1/2+} = u_i - \frac{h_{i+1/2-}}{h_i}\frac{\Delta x_i}{2} Du_i,$$
$$u_{i+1/2-} = u_i + \frac{h_{i-1/2+}}{h_i}\frac{\Delta x_i}{2} Du_i, \qquad (46)$$

for some slopes Dh_i, Du_i. *Minmod*, *ENO*, ENO_m for them. ENO_m for $h + z$ variable.
Centered flux : $\mathcal{F}_c(U_l, U_r, \Delta z) = \left(0, -\frac{h_l+h_r}{2} g\Delta z\right)$.

Second-order accuracy in time

The second-order accuracy in time can be obtained by the Heun method. The second-order scheme in x can be written as

$$U^{n+1} = U^n + \Delta t\, \Phi(U^n), \tag{47}$$

where $U = (U_i)_{i\in\mathbb{Z}}$, and Φ is a nonlinear operator depending on the mesh. Then the second-order scheme in time is

$$\begin{aligned} \widetilde{U}^{n+1} &= U^n + \Delta t\, \Phi(U^n), \\ \widetilde{U}^{n+2} &= \widetilde{U}^{n+1} + \Delta t\, \Phi(\widetilde{U}^{n+1}), \\ U^{n+1} &= \frac{U^n + \widetilde{U}^{n+2}}{2}. \end{aligned} \tag{48}$$

If the operator Φ does not depend on Δt, this procedure gives a fully second-order scheme in space and time. The convex average in (48) enables to ensure the stability without any further limitation on the CFL.

From 1- to 2-dimensions :

Our interest - systems of the form :

$$\partial_t U+\partial_x(F_1(U,Z))+\partial_y(F_2(U,Z))+B_1(U,Z)\partial_x Z+B_2(U,Z)\partial_y Z=0. \tag{49}$$

2d quasilinear system :

$$\partial_t U + A_1 U \partial_x U + A_2 U \partial_y U = 0. \tag{50}$$

Consider planar solutions of the form $U(t,x,y) = U(t,\zeta)$ with $\zeta = xn^1 + yn^2$ and (n^1, n^2) is a unit vector, which leads to

$$\partial_t U + A_n(U)\partial_\zeta U = 0, \tag{51}$$

with

$$A_n U = n^1 A_1(U) + n^2 A_2(U). \tag{52}$$

The notions introduced for one-dimensional systems can be applied to (51), and one defines hyperbolicity, entropies, and other notions for (50) by defining them for all directions n.

2-dimensional mesh :

Rectangles

$$C_{ij} = (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}), \qquad i \in \mathbb{Z},\ j \in \mathbb{Z}, \tag{53}$$

with sides :

$$\Delta x_i = x_{i+1/2} - x_{i-1/2} > 0, \qquad \Delta y_j = y_{j+1/2} - y_{j-1/2} > 0. \tag{54}$$

The centers of the cells : $x_{ij} = (x_i, y_j)$, with

$$x_i = \frac{x_{i-1/2} + x_{i+1/2}}{2}, \qquad y_j = \frac{y_{j-1/2} + y_{j+1/2}}{2}. \tag{55}$$

Finite-volumes in 2 dimensions

Goal : to approximate solution $U(t, x, y)$ to (49) by discrete values U_{ij}^n that are approximations of the mean value of U over the cell C_{ij} at time $t_n = n\Delta t$,

$$U_{ij}^n \simeq \frac{1}{\Delta x_i \Delta y_j} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} U(t_n, x, y)\, dxdy. \qquad (56)$$

A finite volume method for solving (49) takes the form

$$U_{ij}^{n+1} - U_{ij}^n + \frac{\Delta t}{\Delta x_i}(F_{i+1/2-,j} - F_{i-1/2+,j}) + \frac{\Delta t}{\Delta y_j}(F_{i,j+1/2-} - F_{i,j-1/2+}) = 0, \qquad (57)$$

Exchange terms :

$$\begin{aligned} F_{i+1/2\mp,j} &= \mathcal{F}^1_{l/r}(U_{ij}, U_{i+1,j}, Z_{ij}, Z_{i+1,j}), \\ F_{i,j+1/2\mp} &= \mathcal{F}^2_{l/r}(U_{ij}, U_{i,j+1}, Z_{ij}, Z_{i,j+1}), \end{aligned} \qquad (58)$$

for some numerical fluxes $\mathcal{F}^1_l$, $\mathcal{F}^1_r$, $\mathcal{F}^2_l$, $\mathcal{F}^2_r$.

Multi-layer 1-dimensional shallow-water system with topography in conservative form

$$\partial_t h_j + \partial_x \left(h_j u_j\right) = 0, \tag{59}$$

$$\partial_t \left(h_j u_j\right) + \partial_x \left(h_j u_j^2 + g h_j^2/2\right) + g h_j \left(z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k\right) = 0, \tag{60}$$

where $h_j \geq 0$, $j = 1, 3, ...m$ - layer depths, u_j - layer velocities, $z(x)$ - topography, and

$$0 < \rho_1 \leq ... \leq \rho_m$$

layer densities. Convex entropy $\equiv$ energy.

Equivalent 1-layer systems

Shallow-water systems for $U^j = (h_j, h_j u_j)$ with effective topography :

$$z^j = z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k$$

Finite volume scheme with numerical fluxes $\mathcal{F}_{l/r}$:

$$U_i^{j,n+1} - U_i^j + \frac{\Delta t}{\Delta x_i}\left(\mathcal{F}_l(U_i^j, U_{i+1}^j, z_i^j, z_{i+1}^j) - \mathcal{F}_r(U_{i-1}^j, U_i^j, z_{i-1}^j, z_i^j)\right) = 0. \quad (61)$$

For each j - effective shallow water $\Rightarrow$ well-balancing, hydrostatic reconstruction, apparent topography etc will be applied.

Splitting vs sum methods, an example

Consider an ODE

$$\frac{dU}{dt} + A(U) + B(U) = 0. \tag{62}$$

Solving $dU/dt + A(U) = 0$, and $dU/dt + B(U) = 0$, resp. :

$$\begin{aligned} U^{n+1} - U^n + \Delta t\, A(U^n) &= 0, \\ U^{n+1} - U^n + \Delta t\, B(U^n) &= 0. \end{aligned}$$

Splitting method :

$$\begin{aligned} U^{n+1/2} - U^n + \Delta t\, A(U^n) &= 0, \\ U^{n+1} - U^{n+1/2} + \Delta t\, B(U^{n+1/2}) &= 0. \end{aligned}$$

- solving (61) successively.

Sum method

$$U^{n+1} - U^n + \Delta t\, (A(U^n) + B(U^n)) = 0,$$

- solving (61) simultaneously

References, general + finite-volume wall-balanced numerical schemes

General

V. Zeitlin, ed. *"Nonlinear dynamics of rotating shallow water : methods and advances"*, Springer, NY, 2007, 391p.

Numerical schemes : 1-layer

F. Bouchut "Efficient numerical finite-volume schemes for shallow water models", Ch. 4 in V. Zeitlin, ed. *"Nonlinear dynamics of rotating shallow water : methods and advances"*, Springer, NY, 2007, 391p,
and references therein

Numerical scheme : Multi-layer

F. Bouchut and V. Zeitlin "A robust well-balanced scheme for multi-layer shallow water equations", *Disc. Cont. Dyn. Sys. B*, **13**, pp 739-758.