Lecture 3: definition and theory of T-splines for IGA

G. Sangalli

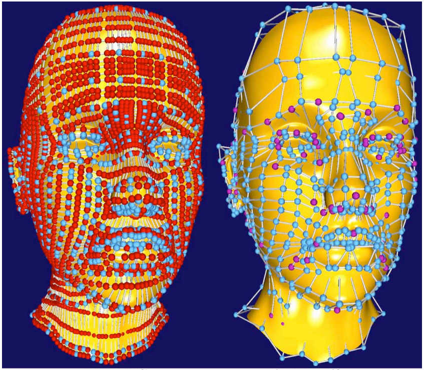
University of Pavia & IMATI-CNR “E. Magenes”

5th European Women in Mathematics Summer School

June 2013 - ICTP Trieste

Motivation

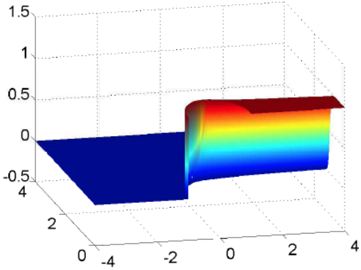
Tensor product B-splines do not allow local mesh refinement, as in



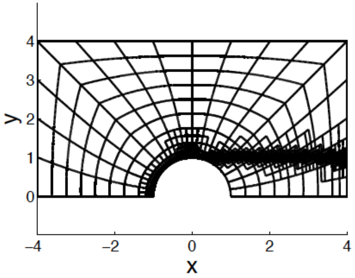
a. NURBS b. T-spline

(Courtesy of T. Sederberg)

Tensor product B-splines do not allow local mesh refinement, as in



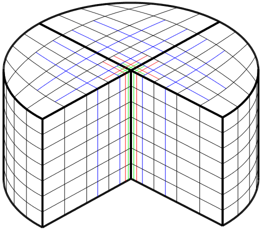
(a) Refined solution with 13129 DOF

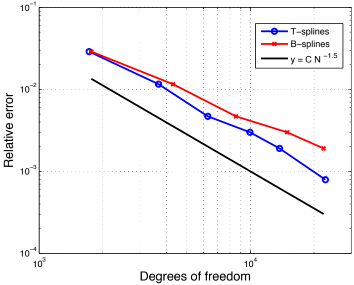


(b) T-mesh after the last step

Motivation

Tensor product B-splines do not allow local mesh refinement, as in





exact solution $\mathbf{u} \approx \mathbf{grad}(r^{2/3} \sin 2\theta/3 \sin(\pi z))$ (by R. Vazquez)

Motivation

Possible extensions:

- Hierarchical splines [A.-V. Vuong, C. Giannelli, B. Juettler, B. Simeon, CMAME 2011] , . . .
- PS-Bsplines [H. Speleers, C. Manni, F. Pelosi, M. L. Sampoli, CMAME, 2012] , . . .
- T-splines [Sederberg, Cardon, Finnigan, North, Zheng, Lyche, 2004] , . . .
- LR-splines [T. Dokken, T. Lyche, K. F. Pettersen, CAGD, 2013] , . . .

Outline

- Recalling again some definition on B-splines

- Intro on T-splines for isogeometric analysis
- Arbitrary degree T-splines
- AS class
- DC class and properties
- Equivalence of AS and DC
- Maths properties for IGA

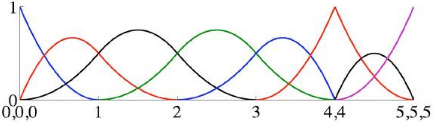
- T-spline based differential forms

Univariate B-splines: a little change in notation

Given a *non-uniform knot vector* $\Xi = \{\xi_1, ..., \xi_{n+p+1}\}$, B-spline are:

$$B^0_{i,\Xi}(s) = \begin{cases} 1 \text{ if } \xi_i \leq s < \xi_{i+1} \\ 0 \text{ otherwise.} \end{cases}$$

$$B^p_{i,\Xi}(s) = \frac{s - \xi_i}{\xi_{i+p} - \xi_i} B^{p-1}_{i,\Xi}(s) + \frac{\xi_{i+p+1} - s}{\xi_{i+p+1} - \xi_{i+1}} B^{p-1}_{i+1,\Xi}(s).$$



New notation: $B^2_{5,\Xi}(s) = B_2[2,3,4,4](s) = B_p[\overbrace{\xi}^{\text{l.k.v.}}](s)$

Univariate B-splines: a dual basis

A dual basis is a set of functionals that give the spline coefficients:

$$f(s) = \sum_{i=1}^{n} \lambda_i(f) B_i(s), \qquad \forall f \in S_{p,\Xi}$$

Various definitions are possible [deBoor, BOOK] , [L. L. Schumaker BOOK, 2007] , e.g. in "Standard and Non-Standard CAGD Tools for Isogeometric Analysis: a Tutorial" by Carla Manni and Hendrik Speleers.

The dual basis also depends locally on the knot vector. Using a similar notation as in the previous slide we will denote it as $\lambda_p[\xi]$.

Univariate B-splines: a dual basis

A dual basis is a set of functionals that give the spline coefficients:

$$f(s) = \sum_{i=1}^{n} \lambda_i(f) B_i(s), \qquad \forall f \in S_{p,\Xi}$$

Various definitions are possible [deBoor, BOOK] , [L. L. Schumaker BOOK, 2007] , e.g. in "Standard and Non-Standard CAGD Tools for Isogeometric Analysis: a Tutorial" by Carla Manni and Hendrik Speleers.

The dual basis also depends locally on the knot vector. Using a similar notation as in the previous slide we will denote it as $\lambda_p[\xi]$.

Univariate B-splines: a dual basis

A dual basis is a set of functionals that give the spline coefficients:

$$f(s) = \sum_{i=1}^{n} \lambda_i(f) B_i(s), \qquad \forall f \in S_{p,\Xi}$$

Various definitions are possible [deBoor, BOOK] , [L. L. Schumaker BOOK, 2007] , e.g. in "Standard and Non-Standard CAGD Tools for Isogeometric Analysis: a Tutorial" by Carla Manni and Hendrik Speleers.

The dual basis also depends locally on the knot vector. Using a similar notation as in the previous slide we will denote it as $\lambda_p[\xi]$.

Univariate B-splines: a dual basis

A dual basis is a set of functionals that give the spline coefficients:

$$f(s) = \sum_{i=1}^{n} \lambda_i(f) B_i(s), \qquad \forall f \in S_{p,\Xi}$$

Various definitions are possible [deBoor, BOOK] , [L. L. Schumaker BOOK, 2007] , e.g. in "Standard and Non-Standard CAGD Tools for Isogeometric Analysis: a Tutorial" by Carla Manni and Hendrik Speleers.

The dual basis also depends locally on the knot vector. Using a similar notation as in the previous slide we will denote it as $\lambda_p[\xi]$.

I adopt here a "function representation":

$$\lambda_p[\xi](f) = \int_{\mathbb{R}} f(s) \lambda_p[\xi](s) ds$$

Univariate B-splines: a dual basis

A dual basis is a set of functionals that give the spline coefficients:

$$f(s) = \sum_{i=1}^{n} \lambda_i(f) B_i(s), \qquad \forall f \in S_{p,\Xi}$$

Various definitions are possible [deBoor, BOOK] , [L. L. Schumaker BOOK, 2007] , e.g. in "Standard and Non-Standard CAGD Tools for Isogeometric Analysis: a Tutorial" by Carla Manni and Hendrik Speleers.

The dual basis also depends locally on the knot vector. Using a similar notation as in the previous slide we will denote it as $\lambda_p[\xi]$.

This is a dual basis in the sense that

$$\int_{\mathbb{R}} B_p[\xi^A]\lambda_p[\xi^A] = 1,$$
$$A \neq B \Rightarrow \int_{\mathbb{R}} B_p[\xi^A]\lambda_p[\xi^B] = 0,$$

Bivariate B-splines: new notation

Bivariate B-splines

Tensor product B-splines are defined as

$$B^{\boldsymbol{A}}_{p,q}(s,t) = B_p[\boldsymbol{\xi}^{A_1}](s)\, B_q[\boldsymbol{\eta}^{A_2}](t)$$

and span the space $S_{p,q} := S_p \otimes S_q$

Tensor product dual functions are defined as

$$\lambda^{\boldsymbol{A}}_{p,q} = \lambda_p[\boldsymbol{\xi}^{A_1}]\, \lambda_q[\boldsymbol{\eta}^{A_2}].$$

we still have the duality poperty:

$$\int_{\mathbb{R}^2} B^{\boldsymbol{A}}_{p,q}\lambda^{\boldsymbol{A}}_{p,q} = \int_{\mathbb{R}} B_p[\boldsymbol{\xi}^{A_1}]\lambda_p[\boldsymbol{\xi}^{A_1}] \int_{\mathbb{R}} B_q[\boldsymbol{\eta}^{A_2}]\lambda_q[\boldsymbol{\eta}^{A_2}] = 1$$

$$\boldsymbol{A} \neq \boldsymbol{B} \Rightarrow \int_{\mathbb{R}^2} B^{\boldsymbol{A}}_{p,q}\lambda^{\boldsymbol{B}}_{p,q} = \int_{\mathbb{R}} B_p[\boldsymbol{\xi}^{A_1}]\lambda_p[\boldsymbol{\xi}^{B_1}] \int_{\mathbb{R}} B_q[\boldsymbol{\eta}^{A_2}]\lambda_q[\boldsymbol{\eta}^{B_2}] = 0.$$

T-splines, from www.tsplines.com

T-Splines

search...

Home | Products | Applications | Community | Support | About us

Home > About us > Technology

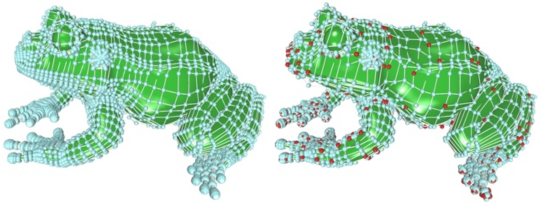
T-Splines technology

What is the magic underneath the covers that makes T-Splines such a unique and compelling new technology? How does T-Splines overcome limitations that are inherent to existing NURBS and subdivision based modeling approaches? Four key components of the patented T-Splines technology make it all possible:

T-Points

A non-uniform rational b-spline Surface or NURBS surface is defined by a set of control points which lie, topologically, in a rectangular grid. This means that, in practice, a large percentage of NURBS control points are superfluous in that they contain no significant geometric information, but merely are needed to satisfy the topological constraint.

In the frog model below, 55% of the NURBS control points are superfluous. In contrast, a T-Spline's control grid is allowed to have partial rows of control points. A partial row of control points terminates in a T-Point, hence the name T-Splines. In the T-Splines frog, the red control points are T-Points.



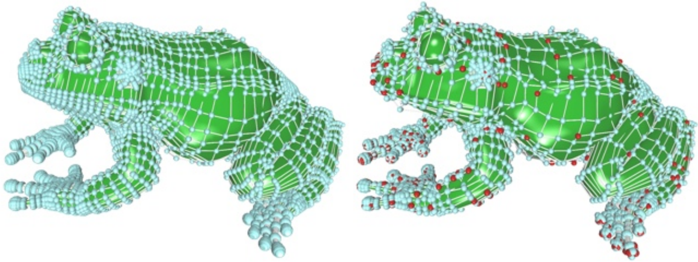
NURBS frog (11625 control points) Courtesy Zygote Media   T-Splines frog (5035 control points)

Minimizing control points makes it easier to create models, control surface smoothness, decrease file size, and speed up editing time.

T-Splines technology

- Overview
- Technical Papers
- Education Portal

T-splines, from `www.tsplines.com`

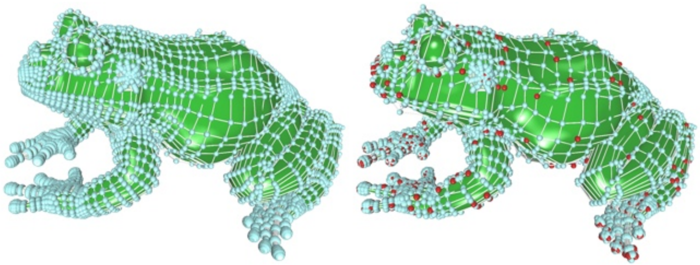


NURBS frog (11625 control points) Courtesy Zygote Media T-Splines frog (5035 control points)

T-splines are an extension of tensor-product NURBS:

- allow local refinement
- allow accurate patch union
- allow more flexibility
- from CAD [Sederberg et al., ACM SIGGRAPH 2003-04] ...
  ...to IGA [Bazilevs, Calo, Cottrell, Evans, Hughes, Lipton, Scott, Sederberg, CMAME, 2010]

T-splines, from www.tsplines.com

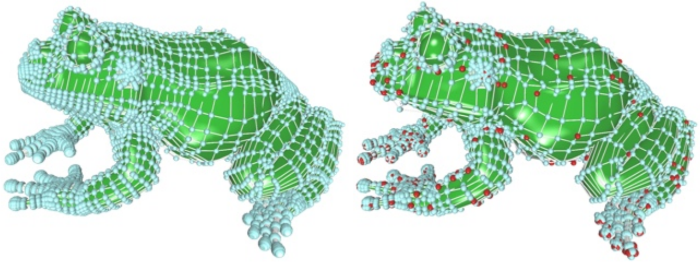


NURBS frog (11625 control points) Courtesy Zygote Media   T-Splines frog (5035 control points)

To use T-splines in IGA we need

- a restriction: Analysis-Suitable class [Li, Zheng, Sederberg, Scott, Hughes, CAGD, 2012] ,
- a generalization to arbitrary-degree [Finnigan, PhD , 2008] ,[Bazilevs, Calo, Cottrell, Evans, Hughes, Lipton, Scott, Sederberg, CMAME 2010] ,
- a generalization to 3D and unstructured meshes [Wang, Zhang, Liu, Hughes, CAD, 2012]

T-splines, from www.tsplines.com



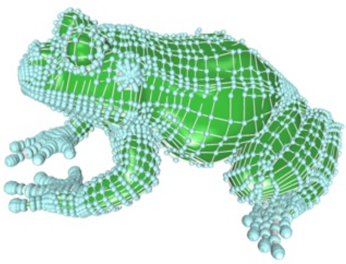
NURBS frog (11625 control points) Courtesy Zygote Media — T-Splines frog (5035 control points)

To use T-splines in IGA we need

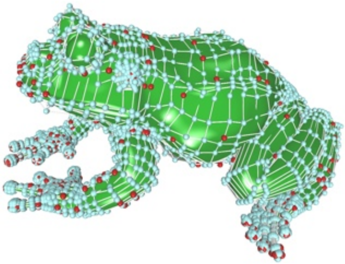
- a restriction: Analysis-Suitable class [Li, Zheng, Sederberg, Scott, Hughes, CAGD, 2012] ,
- a generalization to arbitrary-degree [Finnigan, PhD , 2008] ,[Bazilevs, Calo, Cottrell, Evans, Hughes, Lipton, Scott, Sederberg, CMAME 2010] ,
- a generalization to 3D and unstructured meshes [Wang, Zhang, Liu, Hughes, CAD, 2012]

$\longrightarrow$[Beirão da Veiga,Buffa, GS, Vázquez, M^3AS, 2013]

T-splines, from `www.tsplines.com`



NURBS frog (11625 control points) Courtesy Zygote Media



T-Splines frog (5035 control points)

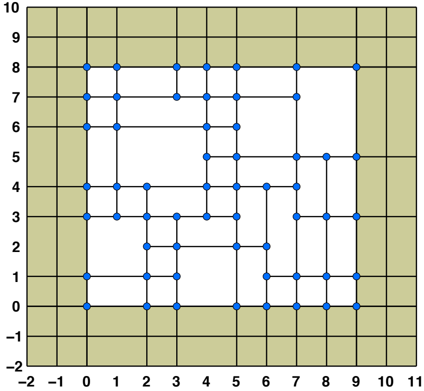
Two important features of IGA:

- (k, p)-refinement + local h-refinement
- smooth discrete “differential forms”+ local h-refinement

  [Buffa, GS, Vázquez, JCP, submitted]

From here the interest for arbitrary degree AS T-splines

Cubic T-spline meshes

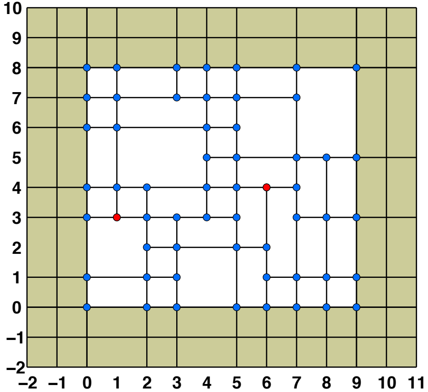


A T-mesh $\mathcal{M}$ in the index space. Indices are associated to knots:

$$\xi_{-2} = \xi_{-1} = \xi_0 < \xi_1 \leq \xi_2 \leq \ldots \leq \xi_8 < \xi_9 = \xi_{10} = \xi_{11}$$

$$\eta_{-2} = \eta_{-1} = \eta_0 < \eta_1 \leq \eta_2 \leq \ldots \leq \eta_7 < \eta_8 = \eta_9 = \eta_{10}$$

Cubic T-spline meshes

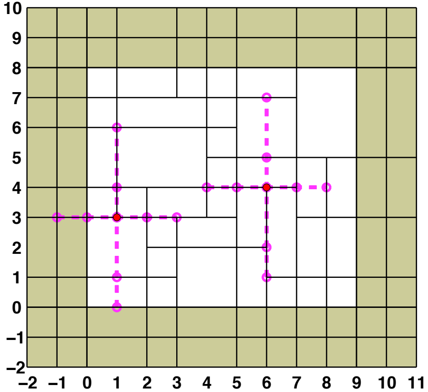


To each vertex (anchor) of $\mathcal{M}$, we associate a cubic bivariate B-spline, defined by its horizontal and vertical local knot vectors

$$B^A_{3,3}(s,t) = B_3[\xi_{-1}, \xi_0, \xi_1, \xi_2, \xi_3](s)\, B_3[\eta_0, \eta_1, \eta_3, \eta_4, \eta_6](t)$$

$$B^B_{3,3}(s,t) = B_3[\xi_4, \xi_5, \xi_6, \xi_7, \xi_8](s)\, B_3[\eta_1, \eta_2, \eta_4, \eta_5, \eta_7](t)$$

Cubic T-spline meshes



To each vertex (anchor) of $\mathcal{M}$, we associate a cubic bivariate B-spline, defined by its horizontal and vertical local knot vectors

$$B^A_{3,3}(s,t) = B_3[\xi_{-1},\xi_0,\xi_1,\xi_2,\xi_3](s)\; B_3[\eta_0,\eta_1,\eta_3,\eta_4,\eta_6](t)$$

$$B^B_{3,3}(s,t) = B_3[\xi_4,\xi_5,\xi_6,\xi_7,\xi_8](s)\; B_3[\eta_1,\eta_2,\eta_4,\eta_5,\eta_7](t)$$

Definition of arbitrary-degree T-splines

T-splines of arbitrary degree (p,q)

A bivariate T-spline of degree (p, q)

$$B_{p,q}[\xi, \eta](s, t) = B_p[\xi](s)\, B_q[\eta](t)$$

is defined by $(p+2)$ horizontal knot values ξ and $(q+2)$ vertical knot values η for the $t-$coordinate.

Anchors indicate the center of the local knot vectors, then:

p	q	anchors
odd	odd	vertexes
even	odd	horizontal edges
odd	even	vertical edges
even	even	elements

Similar to: [Finnigan, PhD , 2008] ,[Bazilevs, Calo, Cottrell, Evans, Hughes, Lipton, Scott, Sederberg, CMAME 2010]

T-splines of arbitrary degree (p,q)

A bivariate T-spline of degree (p, q)

$$B_{p,q}[\xi, \eta](s, t) = B_p[\xi](s)\, B_q[\eta](t)$$

is defined by $(p + 2)$ horizontal knot values ξ and $(q + 2)$ vertical knot values η for the $t-$coordinate.

Anchors indicate the center of the local knot vectors, then:

p	q	anchors
odd	odd	vertexes
even	odd	horizontal edges
odd	even	vertical edges
even	even	elements

Similar to: [Finnigan, PhD , 2008] ,[Bazilevs, Calo, Cottrell, Evans, Hughes, Lipton, Scott, Sederberg, CMAME 2010]

T-splines of arbitrary degree (p,q)

A bivariate T-spline of degree (p, q)

$$B_{p,q}[\xi, \eta](s, t) = B_p[\xi](s)\, B_q[\eta](t)$$

is defined by $(p+2)$ horizontal knot values ξ and $(q+2)$ vertical knot values η for the $t-$coordinate.

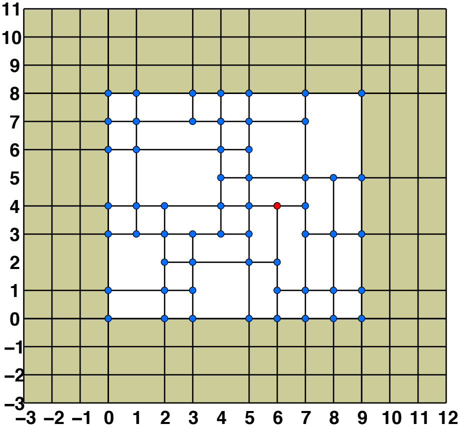
Anchors indicate the center of the local knot vectors, then:

p	q	anchors
odd	odd	vertexes
even	odd	horizontal edges
odd	even	vertical edges
even	even	elements

Similar to: [Finnigan, PhD , 2008] ,[Bazilevs, Calo, Cottrell, Evans, Hughes, Lipton, Scott, Sederberg, CMAME 2010]

The odd-odd case (the simplest to generalize)

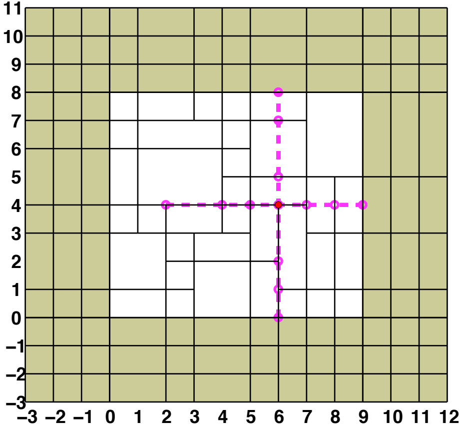
Let for example $p = q = 5$.



$$B^A_{5,5}(s,t) = B_5[\xi_2,\xi_4,\xi_5,\xi_6,\xi_7,\xi_8,\xi_9](s)\; B_5[\eta_0,\eta_1,\eta_2,\eta_4,\eta_5,\eta_7,\eta_8](t)$$

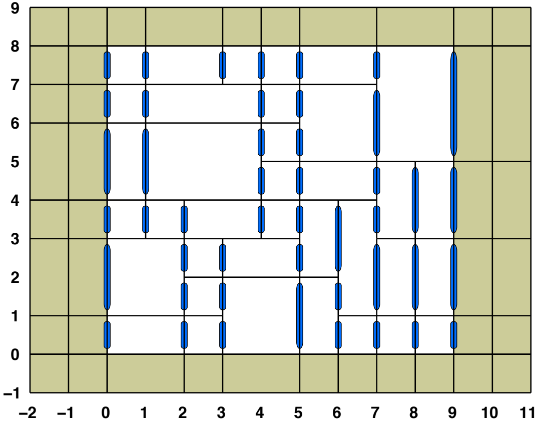
The odd-odd case (the simplest to generalize)

Let for example $p = q = 5$.



$$B^{A}_{5,5}(s,t) = B_5[\xi_2, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8, \xi_9](s)\; B_5[\eta_0, \eta_1, \eta_2, \eta_4, \eta_5, \eta_7, \eta_8](t)$$

We consider the case $p = 3, q = 2$.

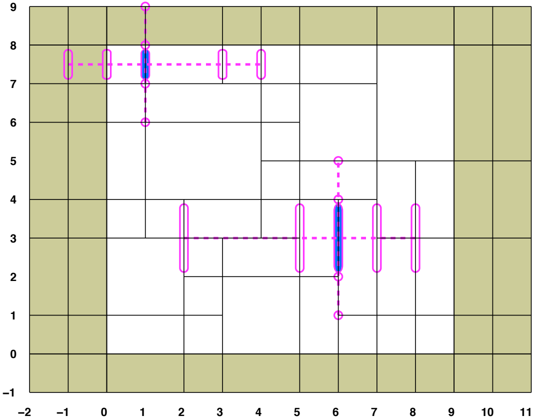


$$B^A_{3,2}(s,t) = B_3[\xi_{-1}, \xi_0, \xi_1, \xi_3, \xi_4](s)\, B_3[\eta_6, \eta_7, \eta_8, \eta_9](t)$$

$$B^B_{3,2}(s,t) = B_3[\xi_2, \xi_5, \xi_6, \xi_7, \xi_8](s)\, B_3[\eta_1, \eta_2, \eta_4, \eta_5](t)$$

The even-odd ad odd-even cases

We consider the case $p = 3, q = 2$.

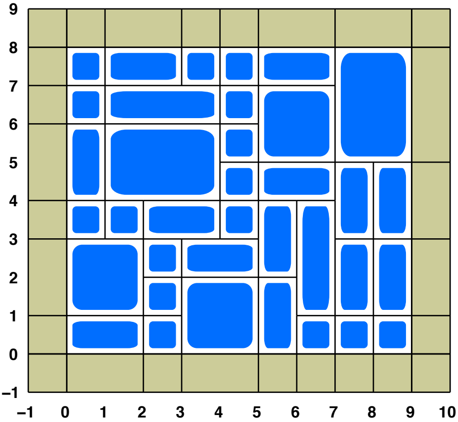


$$B^A_{3,2}(s,t) = B_3[\xi_{-1}, \xi_0, \xi_1, \xi_3, \xi_4](s)\, B_3[\eta_6, \eta_7, \eta_8, \eta_9](t)$$

$$B^B_{3,2}(s,t) = B_3[\xi_2, \xi_5, \xi_6, \xi_7, \xi_8](s)\, B_3[\eta_1, \eta_2, \eta_4, \eta_5](t)$$

The even-even case

Let for example $p = 2, q = 2$.

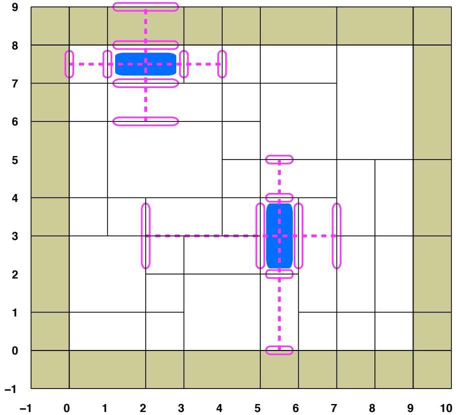


$$B^A_{2,2}(s,t) = B_3[\xi_0,\xi_1,\xi_3,\xi_4](s)\; B_3[\eta_6,\eta_7,\eta_8,\eta_9](t)$$

$$B^B_{2,2}(s,t) = B_3[\xi_2,\xi_5,\xi_6,\xi_7](s)\; B_3[\eta_0,\eta_2,\eta_4,\eta_5](t)$$

The even-even case

Let for example $p = 2, q = 2$.



$$B^A_{2,2}(s,t) = B_3[\xi_0,\xi_1,\xi_3,\xi_4](s)\, B_3[\eta_6,\eta_7,\eta_8,\eta_9](t)$$

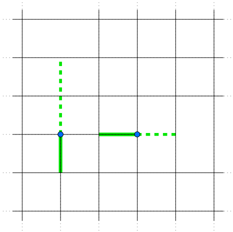
$$B^B_{2,2}(s,t) = B_3[\xi_2,\xi_5,\xi_6,\xi_7](s)\, B_3[\eta_0,\eta_2,\eta_4,\eta_5](t)$$

Definition of AS T-splines

Analysis Suitable T-splines of p, q degree

Extensions: every horizontal ($\vdash$ and $\dashv$) (resp. vertical ($\perp$ and $\top$)) T-junction is extended by $\lceil p/2 \rceil$-bays (resp., $\lceil q/2 \rceil$-bays) forward (called face extension) and by $\lfloor p/2 \rfloor$-bays (resp., $\lfloor q/2 \rfloor$-bays) backward (called edge extension). All extensions are closed lines.

Example for p=2, q=3:



Definition of Analysis Suitable (AS)

A T-mesh is $AS_{p,q}$ if no horizontal T-junction extension intersects a vertical T-junction extension.

Definition of DC T-splines

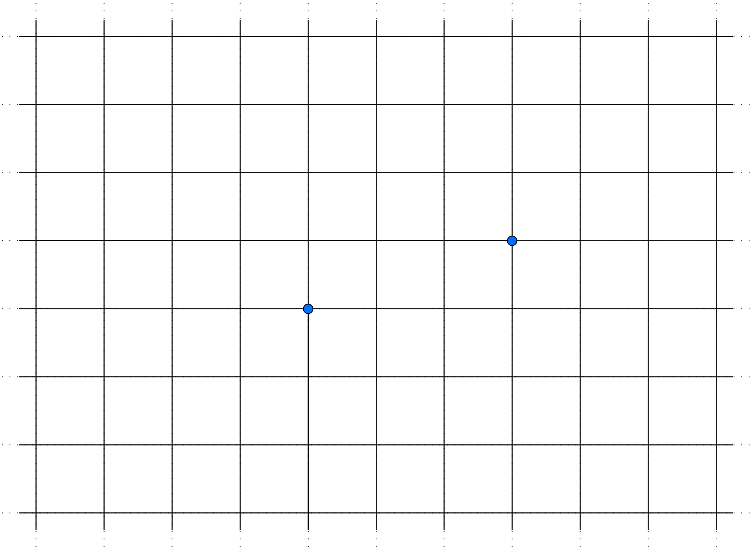
Overlapping index vectors

Two local index vector may be coincident, different and overlapping or different and non-overlapping (staggered):

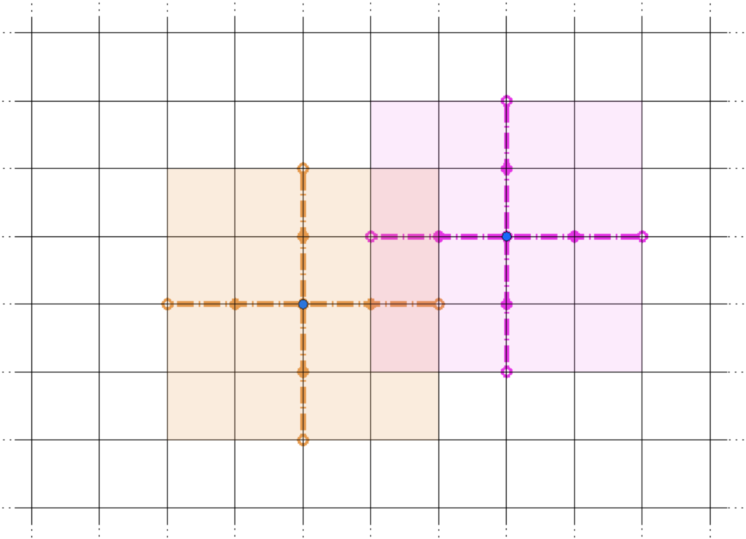
- overlapping

- non-overlapping

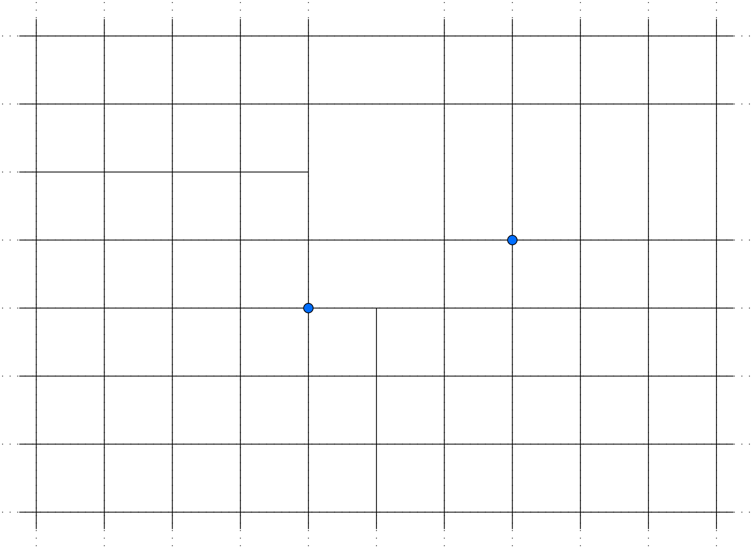
Overlapping B-splines ($p = q = 3$)



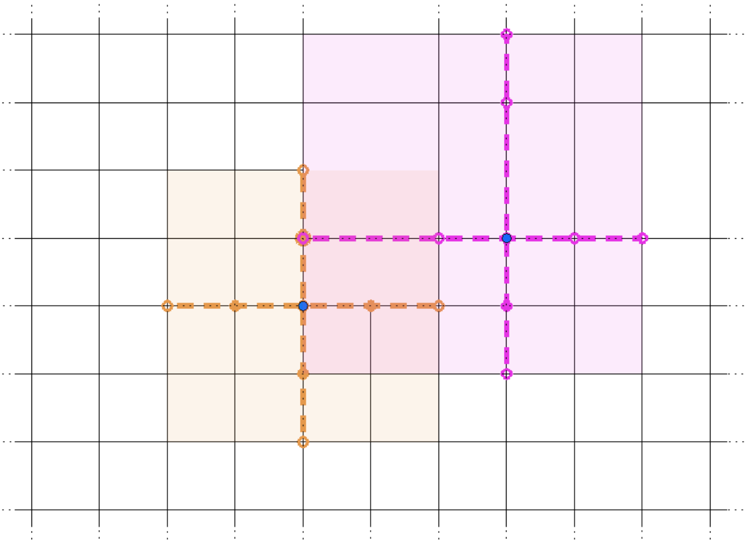
Overlapping B-splines ($p = q = 3$)



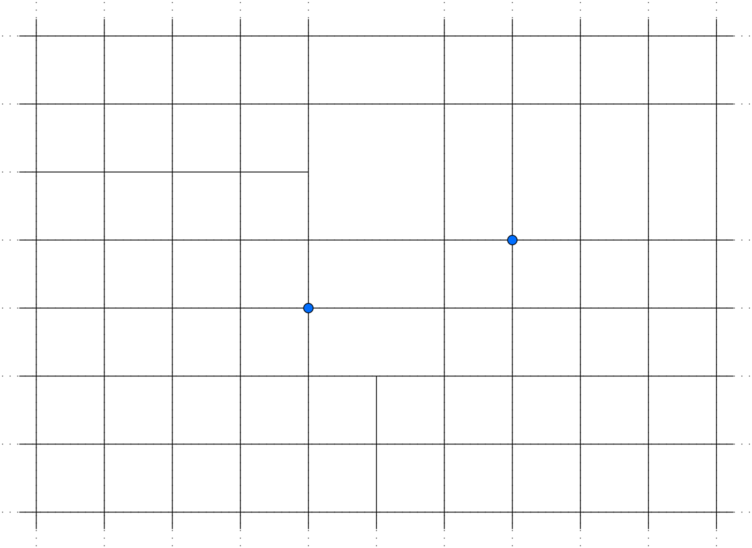
Non overlapping T-splines ($p = q = 3$)



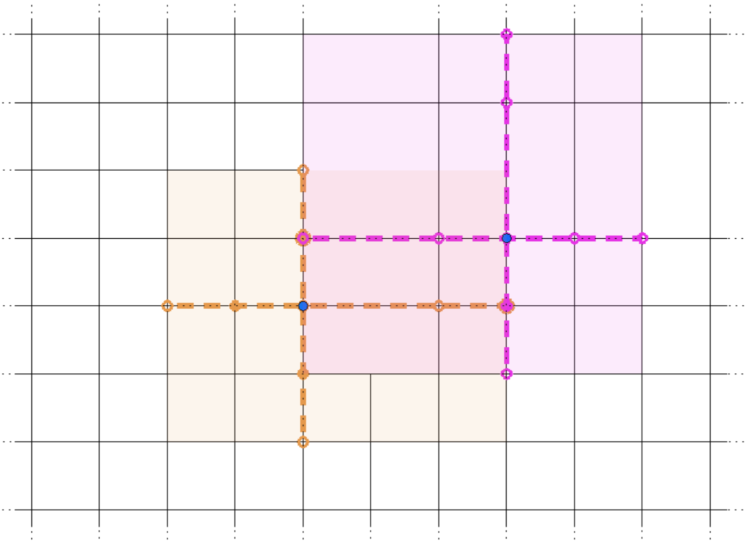
Non overlapping T-splines ($p = q = 3$)



Overlapping (in one direction) T-splines ($p = q = 3$)



Overlapping (in one direction) T-splines ($p = q = 3$)



Dual Compatible T-splines

Definition of Dual Compatible (DC)

A given T-mesh $\mathcal{M}$ is $\mathrm{DC}_{p,q}$ if for any two different T-splines there is a direction (either horizontal or vertical) such that the two local index vectors in that direction are overlapping and do not coincide

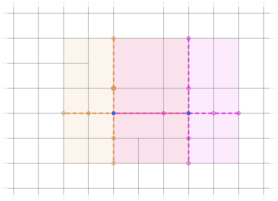
Dual Compatible T-splines

Definition of Dual Compatible (DC)

A given T-mesh $\mathcal{M}$ is $DC_{p,q}$ if for any two different T-splines there is a direction (either horizontal or vertical) such that the two local index vectors in that direction are overlapping and do not coincide

Equivalent definition in 2D

A given T-mesh $\mathcal{M}$ is $DC_{p,q}$ if any pair of T-splines has overlapping index vectors, either horizontally or vertically.



Dual Compatible T-splines

Definition of Dual Compatible (DC)

A given T-mesh $\mathcal{M}$ is $\mathrm{DC}_{p,q}$ if for any two different T-splines there is a direction (either horizontal or vertical) such that the two local index vectors in that direction are overlapping and do not coincide

Given any anchor A we associate a B-spline $B^A_{p,q} = B_p[\boldsymbol{\xi}^A]\, B_q[\boldsymbol{\eta}^A]$ and

the "dual" function: $\lambda^A_{p,q} = \lambda_p[\boldsymbol{\xi}^A]\, \lambda_q[\boldsymbol{\eta}^A].$

This is a dual basis for tensor product B-splines:

$$\int_{\mathbb{R}^2} B^A_{p,q} \lambda^A_{p,q} = \int_{\mathbb{R}} B_p[\boldsymbol{\xi}^A] \lambda_p[\boldsymbol{\xi}^A] \int_{\mathbb{R}} B_q[\boldsymbol{\eta}^A] \lambda_q[\boldsymbol{\eta}^A] = 1,$$

$$A \neq B \Rightarrow \int_{\mathbb{R}^2} B^A_{p,q} \lambda^B_{p,q} = \int_{\mathbb{R}} B_p[\boldsymbol{\xi}^A] \lambda_p[\boldsymbol{\xi}^B] \int_{\mathbb{R}} B_q[\boldsymbol{\eta}^A] \lambda_q[\boldsymbol{\eta}^B] = 0,$$

[deBoor, 1976] , [Lyche, JAT, 1978] ,[L. L. Schumaker BOOK, 2007] .

Dual Compatible T-splines

Definition of Dual Compatible (DC)

A given T-mesh $\mathcal{M}$ is $\text{DC}_{p,q}$ if for any two different T-splines there is a direction (either horizontal or vertical) such that the two local index vectors in that direction are overlapping and do not coincide

Given any anchor A we associate a B-spline $B^A_{p,q} = B_p[\xi^A]\, B_q[\eta^A]$ and

the "dual" function: $\lambda^A_{p,q} = \lambda_p[\xi^A]\, \lambda_q[\eta^A]$.

If the T-mesh is $\text{DC}_{p,q}$, we still have

$$\int_{\mathbb{R}^2} B^A_{p,q} \lambda^A_{p,q} = 1$$

$$A \neq B \Rightarrow \int_{\mathbb{R}^2} B^A_{p,q} \lambda^B_{p,q} = \int_{\mathbb{R}} B_p[\xi^A] \lambda_p[\xi^B] \int_{\mathbb{R}} B_q[\eta^A] \lambda_q[\eta^B] = 0,$$

Then $\{\lambda^A_{p,q}\}_A$ is still a dual basis !!.

Properties of DC T-splines

Properties of DC T-splines

Linear independence

The T-splines generated by $\mathcal{M} \in \mathrm{DC}_{p,q}$ are a basis for $\mathrm{T}_{p,q}(\mathcal{M})$.

Partition of unity

The basis above is a partition of unity.

Approximation

The operator $\Pi : L^2 \to \mathrm{T}_{p,q}(\mathcal{M})$ defined by

$$f \mapsto \Pi(f) = \sum_{\text{anchors } A} \left(\int_{\mathbb{R}^2} \lambda^A_{p,q} f \right) B^A_{p,q}$$

satisfies $||f - \Pi(f)||_{L^2} \leq Ch^{\min(p,q)+1} |f|_{H^{\min(p,q)+1}}$.

Properties of DC T-splines

Linear independence

The T-splines generated by $\mathcal{M} \in \mathrm{DC}_{p,q}$ are a basis for $\mathrm{T}_{p,q}(\mathcal{M})$.

Dim:

$$\sum_{\text{anchors } A} C^A B^A_{p,q}(\cdot) = 0$$

$$\Downarrow$$

$$\int_{\mathbb{R}^2} \lambda^B_{p,q} \left(\sum_{\text{anchors } A} C^A B^A_{p,q} \right) = 0$$

$$\Downarrow$$

$$C^B = 0$$

Partition of unity

The basis above is a partition of unity.

Properties of DC T-splines

Linear independence

The T-splines generated by $\mathcal{M} \in \mathrm{DC}_{p,q}$ are a basis for $\mathrm{T}_{p,q}(\mathcal{M})$.

Partition of unity

The basis above is a partition of unity.

Approximation

The operator $\Pi : L^2 \rightarrow \mathrm{T}_{p,q}(\mathcal{M})$ defined by

$$f \mapsto \Pi(f) = \sum_{\text{anchors } A} \left(\int_{\mathbb{R}^2} \lambda_{p,q}^A f \right) B_{p,q}^A$$

satisfies $||f - \Pi(f)||_{L^2} \leq C h^{\min(p,q)+1} |f|_{H^{\min(p,q)+1}}$.

Properties of DC T-splines

Linear independence

The T-splines generated by $\mathcal{M} \in \mathrm{DC}_{p,q}$ are a basis for $\mathrm{T}_{p,q}(\mathcal{M})$.

Partition of unity

The basis above is a partition of unity.

Dim:

$$\sum_{\text{anchors } A} C^A B^A_{p,q}(\cdot) = 1$$

$$\Downarrow$$

$$\int_{\mathbb{R}^2} \lambda^B_{p,q} \left(\sum_{\text{anchors } A} C^A B^A_{p,q} \right) = \int_{\mathbb{R}^2} \lambda^B_{p,q}$$

$$\Downarrow$$

$$C^B = 1$$

Properties of DC T-splines

Linear independence

The T-splines generated by $\mathcal{M} \in \mathrm{DC}_{p,q}$ are a basis for $\mathrm{T}_{p,q}(\mathcal{M})$.

Partition of unity

The basis above is a partition of unity.

Approximation

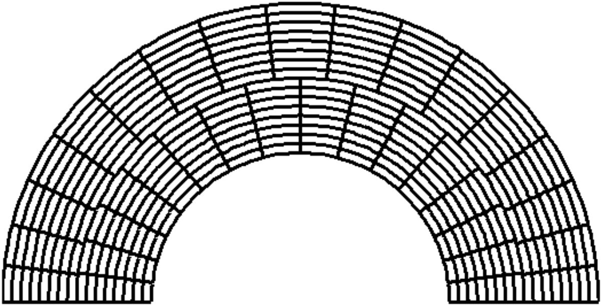
The operator $\Pi : L^2 \to \mathrm{T}_{p,q}(\mathcal{M})$ defined by

$$f \mapsto \Pi(f) = \sum_{\text{anchors } A} \left(\int_{\mathbb{R}^2} \lambda_{p,q}^A f \right) B_{p,q}^A$$

satisfies $||f - \Pi(f)||_{L^2} \leq C h^{\min(p,q)+1} |f|_{H^{\min(p,q)+1}}$.

The origin of the concept of DC T-splines

The idea originates from the analysis of “IsoGeometric analysis using T-splines on two-patch geometries” [Beirão da Veiga, Buffa, Cho, GS, CMAME 2011]



simplest possible T-spline space and geometry.

AS is equivalent to DC

AS is equivalent to DC

Theorem [Beirão da Veiga,Buffa, Cho, GS, CMAME 2012] - [Beirão da Veiga,Buffa, GS, Vázquez, M^3AS,2013]

$AS_{p,q} = DC_{p,q}$.

To summarize, for a given T-mesh $\mathcal{M}$:

$$\mathcal{M} \in AS_{p,q} \Leftrightarrow \mathcal{M} \in DC_{p,q} \Rightarrow \begin{cases} \text{T-splines are a basis} \\ \text{T-splines are a p.o.u.} \\ T_{p,q}(\mathcal{M}) \text{ has optimal approx.} \end{cases}$$

AS is equivalent to DC

Theorem [Beirão da Veiga,Buffa, Cho, GS, CMAME 2012] - [Beirão da Veiga,Buffa, GS, Vázquez, M^3AS,2013]

$\mathrm{AS}_{p,q} = \mathrm{DC}_{p,q}$.

To summarize, for a given T-mesh $\mathcal{M}$:

$$\mathcal{M} \in \mathrm{AS}_{p,q} \Leftrightarrow \mathcal{M} \in \mathrm{DC}_{p,q} \Rightarrow \begin{cases} \text{T-splines are a basis} \\ \text{T-splines are a p.o.u.} \\ T_{p,q}(\mathcal{M}) \text{ has optimal approx.} \end{cases}$$

Proving theorem for arbitrary-degree T-splines

Induction argument:

$$\text{if } \wp_{0,0} \quad \text{and} \quad \begin{array}{lcl} \forall p' \,|\, 0 \leq p' < p, \ \wp_{p',q} & \Rightarrow & \wp_{p,q}, \\ \forall q' \,|\, 0 \leq q' < q, \ \wp_{p,q'} & \Rightarrow & \wp_{p,q}. \end{array} \quad \text{then} \quad \forall p, q, \wp_{p,q}.$$

For example $\wp_{p,q} =$ " Let $\mathcal{M} \in AS_{p,q}$ and T a T-splines defined on $\mathcal{M}$. Then there are no vertices of $\mathcal{M}$ in the interior of the $(p+1) \times (q+1)$ elements of T."

Proving theorem for arbitrary-degree T-splines

Induction argument:

$$\text{if } \wp_{0,0} \quad \text{and} \quad \begin{array}{ccc} \forall p' \,|\, 0 \leq p' < p, \ \wp_{p',q} & \Rightarrow & \wp_{p,q}, \\ \forall q' \,|\, 0 \leq q' < q, \ \wp_{p,q'} & \Rightarrow & \wp_{p,q}. \end{array} \quad \text{then} \quad \forall p, q, \ \wp_{p,q}.$$

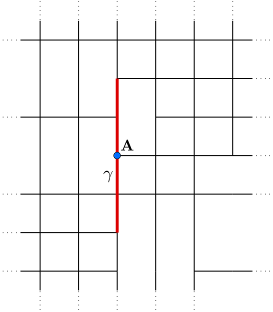
For example $\wp_{p,q} =$ " Let $\mathcal{M} \in AS_{p,q}$ and T a T-splines defined on $\mathcal{M}$. Then there are no vertices of $\mathcal{M}$ in the interior of the $(p+1) \times (q+1)$ elements of T."

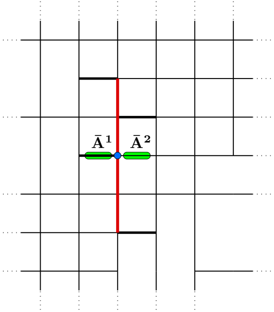
Proving theorem for arbitrary-degree T-splines

Induction argument:

$$\text{if } \wp_{0,0} \quad \text{and} \quad \begin{array}{lcl} \forall p' \,|\, 0 \le p' < p,\ \wp_{p',q} & \Rightarrow & \wp_{p,q}, \\ \forall q' \,|\, 0 \le q' < q,\ \wp_{p,q'} & \Rightarrow & \wp_{p,q}. \end{array} \quad \text{then} \quad \forall p, q,\ \wp_{p,q}.$$

For example $\wp_{p,q} =$ " Let $\mathcal{M} \in AS_{p,q}$ and T a T-splines defined on $\mathcal{M}$. Then there are no vertices of $\mathcal{M}$ in the interior of the $(p+1) \times (q+1)$ elements of T."



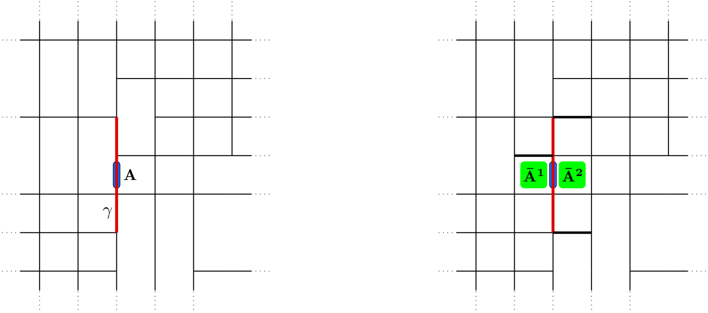


Proving theorem for arbitrary-degree T-splines

Induction argument:

$$\text{if } \wp_{0,0} \quad \text{and} \quad \begin{array}{l} \forall p' \,|\, 0 \leq p' < p,\ \wp_{p',q} \quad \Rightarrow \quad \wp_{p,q}, \\ \forall q' \,|\, 0 \leq q' < q,\ \wp_{p,q'} \quad \Rightarrow \quad \wp_{p,q}. \end{array} \quad \text{then} \quad \forall p, q,\ \wp_{p,q}.$$

For example $\wp_{p,q} =$ " Let $\mathcal{M} \in AS_{p,q}$ and T a T-splines defined on $\mathcal{M}$. Then there are no vertices of $\mathcal{M}$ in the interior of the $(p+1) \times (q+1)$ elements of T."

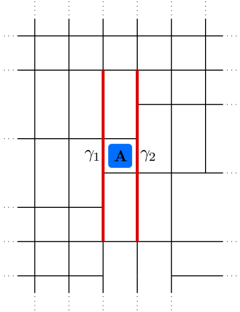


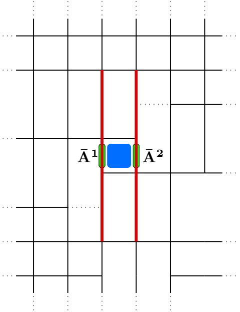
Proving theorem for arbitrary-degree T-splines

Induction argument:

if $\wp_{0,0}$ and $\begin{array}{lcl} \forall p' \,|\, 0 \le p' < p,\ \wp_{p',q} & \Rightarrow & \wp_{p,q}, \\ \forall q' \,|\, 0 \le q' < q,\ \wp_{p,q'} & \Rightarrow & \wp_{p,q}. \end{array}$ then $\forall p, q,\ \wp_{p,q}$.

For example $\wp_{p,q} =$ " Let $\mathcal{M} \in AS_{p,q}$ and T a T-splines defined on $\mathcal{M}$. Then there are no vertices of $\mathcal{M}$ in the interior of the $(p+1) \times (q+1)$ elements of T."



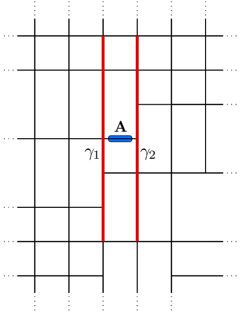


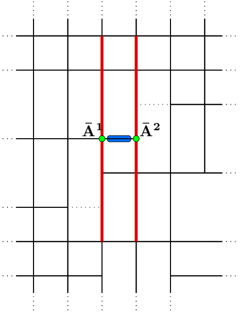
Proving theorem for arbitrary-degree T-splines

Induction argument:

$$\text{if } \wp_{0,0} \quad \text{and} \quad \begin{array}{lcl} \forall p' \,|\, 0 \leq p' < p,\ \wp_{p',q} & \Rightarrow & \wp_{p,q}, \\ \forall q' \,|\, 0 \leq q' < q,\ \wp_{p,q'} & \Rightarrow & \wp_{p,q}. \end{array} \quad \text{then} \quad \forall p,q,\ \wp_{p,q}.$$

For example $\wp_{p,q} =$ " Let $\mathcal{M} \in AS_{p,q}$ and T a T-splines defined on $\mathcal{M}$. Then there are no vertices of $\mathcal{M}$ in the interior of the $(p+1) \times (q+1)$ elements of T."





$\mathrm{DC}_{p,q}$ is easy to generalize

Abstract definition of $\mathrm{DC}_{p,q}$

A set of B-splines fulfills $\mathrm{DC}_{p,q}$ if given two different B-splines in the set there is a direction such that the two local index vectors in that direction are overlapping and do not coincide

The definition above can be extended to:

- trivariate splines
- LR-splines [Dokken, Lyche, Pettersen, CAGD, 2013]

It allows to use the classical dual basis and get:

- partition of unity (with proper scaling of the basis)
- linear independence
- Greville abscissae = interior knots average
- approximation properties

provided the polynomials belong to the space.

Abstract definition of $DC_{p,q}$

A set of B-splines fulfills $DC_{p,q}$ if given two different B-splines in the set there is a direction such that the two local index vectors in that direction are overlapping and do not coincide

The definition above can be extended to:

- trivariate splines
- LR-splines [Dokken, Lyche, Pettersen, CAGD, 2013]

It allows to use the classical dual basis and get:

- partition of unity (with proper scaling of the basis)
- linear independence
- Greville abscissae = interior knots average
- approximation properties

provided the polynomials belong to the space.

Abstract definition of $DC_{p,q}$

A set of B-splines fulfills $DC_{p,q}$ if given two different B-splines in the set there is a direction such that the two local index vectors in that direction are overlapping and do not coincide

The definition above can be extended to:

- trivariate splines
- LR-splines [Dokken, Lyche, Pettersen, CAGD, 2013]

It allows to use the classical dual basis and get:

- partition of unity (with proper scaling of the basis)
- linear independence
- Greville abscissae = interior knots average
- approximation properties

provided the polynomials belong to the space.

Differential forms based on AS T-splines

Discrete differential forms

Goal: T-splines discretizations of the sequence:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \mathbf{H}(\mathbf{curl};\Omega) \xrightarrow{\textbf{curl}} \mathbf{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)\,.$$

It is a generalization of the finite element sequence:

$$\text{Nodal FE} \xrightarrow{\textbf{grad}} \text{Edge FE} \xrightarrow{\textbf{curl}} \text{Face FE} \xrightarrow{\mathrm{div}} \text{Discont. FE}\,.$$

Applications: electromagnetics, Darcy's flow, Navier-Stokes...

[Buffa, GS, Vazquez, 2010-2012] , [Buffa, De Falco, GS, 2011] , [Evans, Hughes 2012-2013] ...

Today I consider only the 2D sequence:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \boldsymbol{H}(\mathrm{curl}\,,\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega)\,.$$

Discrete differential forms

Goal: T-splines discretizations of the sequence:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \mathbf{H}(\mathbf{curl};\Omega) \xrightarrow{\textbf{curl}} \mathbf{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)\,.$$

It is a generalization of the finite element sequence:

$$\text{Nodal FE} \xrightarrow{\textbf{grad}} \text{Edge FE} \xrightarrow{\textbf{curl}} \text{Face FE} \xrightarrow{\mathrm{div}} \text{Discont. FE}\,.$$

Applications: electromagnetics, Darcy's flow, Navier-Stokes...

[Buffa, GS, Vazquez, 2010-2012] , [Buffa, De Falco, GS, 2011] , [Evans, Hughes 2012-2013] ...

Today I consider only the 2D sequence:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \boldsymbol{H}(\mathrm{curl}\,,\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega)\,.$$

Discrete differential forms

Goal: T-splines discretizations of the sequence:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \mathbf{H}(\mathbf{curl};\Omega) \xrightarrow{\textbf{curl}} \mathbf{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)\,.$$

It is a generalization of the finite element sequence:

$$\text{Nodal FE} \xrightarrow{\textbf{grad}} \text{Edge FE} \xrightarrow{\textbf{curl}} \text{Face FE} \xrightarrow{\mathrm{div}} \text{Discont. FE}\,.$$

Applications: electromagnetics, Darcy's flow, Navier-Stokes...

[Buffa, GS, Vazquez, 2010-2012] , [Buffa, De Falco, GS, 2011] , [Evans, Hughes 2012-2013] ...

Today I consider only the 2D sequence:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \boldsymbol{H}(\mathrm{curl}\,,\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega)\,.$$

Univariate B-splines: derivatives

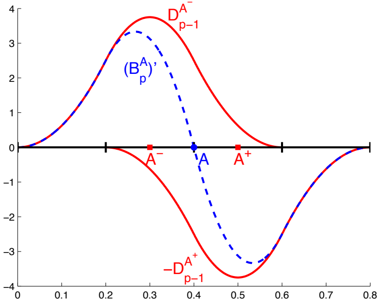
$S_p(\Xi)$: space of B-splines of degree p on the **open knot vector** Ξ.

$$\Xi = \{\xi_1 = \cdots = \xi_{p+1} < \xi_{p+2} \leq \cdots \leq \xi_n < \xi_{n+1} = \cdots = \xi_{n+p+1}\}$$

B-splines **derivatives** are splines as well, in the space $S_{p-1}(\Xi')$:

$$\Xi' = \{\xi_2 = \cdots = \xi_{p+1} < \xi_{p+2} \leq \cdots \leq \xi_n < \xi_{n+1} = \cdots = \xi_{n+p}\}$$

The derivative is $(B_p^A)' = D_{p-1}^{A^-} - D_{p-1}^{A^+}$, with $D_{p-1}^{A^\pm} = \frac{p}{|supp(B_{p-1}^{A^\pm})|} B_{p-1}^{A^\pm}$.



Tensor-product spline differential forms [Buffa, GS, Vazquez, 2010-2012]

Recall the 2D diagram:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \mathbf{H}(\mathrm{curl}\,;\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega)\,.$$

In the **parametric domain** $\hat{\Omega}$ =square, we have the exact sequence:

$$S_{p,p}/\mathbb{R} \xrightarrow{\textbf{grad}} (S_{p-1,p}, S_{p,p-1}) \xrightarrow{\mathrm{curl}} S_{p-1,p-1}$$

$$\hat{X}^0 = S_{p,p} \equiv S_p(\Xi_1) \otimes S_p(\Xi_2)$$

Tensor-product spline differential forms [Buffa, GS, Vazquez, 2010-2012]

Recall the 2D diagram:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \mathbf{H}(\mathrm{curl}\,;\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega)\,.$$

In the **parametric domain** $\hat{\Omega}$ =square, we have the exact sequence:

$$S_{p,p}/\mathbb{R} \xrightarrow{\textbf{grad}} (S_{p-1,p}, S_{p,p-1}) \xrightarrow{\mathrm{curl}} S_{p-1,p-1}$$

$$\hat{X}^0 = S_{p,p} \equiv S_p(\Xi_1) \otimes S_p(\Xi_2)$$

$$\hat{X}^1 = (S_{p-1,p}, S_{p,p-1}) \equiv (S_{p-1}(\Xi_1') \otimes S_p(\Xi_2)) \times (S_p(\Xi_1) \otimes S_{p-1}(\Xi_2'))$$

Tensor-product spline differential forms [Buffa, GS, Vazquez, 2010-2012]

Recall the 2D diagram:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \mathbf{H}(\mathrm{curl}\,;\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega)\,.$$

In the **parametric domain** $\hat{\Omega}$ =square, we have the exact sequence:

$$S_{p,p}/\mathbb{R} \xrightarrow{\textbf{grad}} (S_{p-1,p}, S_{p,p-1}) \xrightarrow{\mathrm{curl}} S_{p-1,p-1}$$

$$\hat{X}^0 = S_{p,p} \equiv S_p(\Xi_1) \otimes S_p(\Xi_2)$$

$$\hat{X}^1 = (S_{p-1,p}, S_{p,p-1}) \equiv (S_{p-1}(\Xi_1') \otimes S_p(\Xi_2)) \times (S_p(\Xi_1) \otimes S_{p-1}(\Xi_2'))$$

$$\hat{X}^2 = S_{p-1,p-1} \equiv S_{p-1}(\Xi_1') \otimes S_{p-1}(\Xi_2')$$

The next step is to map them to the physical domain.

Tensor-product spline differential forms [Buffa, GS, Vazquez, 2010-2012]

Recall the 2D diagram:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \mathbf{H}(\mathrm{curl}\,;\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega)\,.$$

In the **parametric domain** $\hat{\Omega}$ =square, we have the exact sequence:

$$S_{p,p}/\mathbb{R} \xrightarrow{\textbf{grad}} (S_{p-1,p}, S_{p,p-1}) \xrightarrow{\mathrm{curl}} S_{p-1,p-1}$$

Use structure preserving push-forward.

Tensor-product spline differential forms [Buffa, GS, Vazquez, 2010-2012]

Recall the 2D diagram:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \mathbf{H}(\mathrm{curl}\,;\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega)\,.$$

In the **parametric domain** $\hat{\Omega}$ =square, we have the exact sequence:

$$S_{p,p}/\mathbb{R} \xrightarrow{\textbf{grad}} (S_{p-1,p}, S_{p,p-1}) \xrightarrow{\mathrm{curl}} S_{p-1,p-1}$$

Use structure preserving push-forward.

$$X^0/\mathbb{R} \xrightarrow{\textbf{grad}} X^1 \xrightarrow{\mathrm{curl}} X^2$$

Tensor-product spline differential forms [Buffa, GS, Vazquez, 2010-2012]

Recall the 2D diagram:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \mathbf{H}(\mathrm{curl}\,;\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega)\,.$$

In the **parametric domain** $\hat{\Omega}$ =square, we have the exact sequence:

$$S_{p,p}/\mathbb{R} \xrightarrow{\textbf{grad}} (S_{p-1,p}, S_{p,p-1}) \xrightarrow{\mathrm{curl}} S_{p-1,p-1}$$

Use structure preserving push-forward.

$$X^0/\mathbb{R} \xrightarrow{\textbf{grad}} X^1 \xrightarrow{\mathrm{curl}} X^2$$

$$X^0 = \{\phi : \phi \circ \mathbf{F} \in \hat{X}^0\}, \qquad \text{standard mapping,}$$

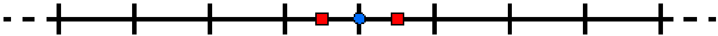
$$X^1 = \{\mathbf{u} : (D\mathbf{F})^\top(\mathbf{u} \circ \mathbf{F}) \in \hat{X}^1\}, \qquad \text{curl-conserving mapping,}$$

$$X^2 = \{\varphi : \det(D\mathbf{F})(\varphi \circ \mathbf{F}) \in \hat{X}^2\}, \qquad \text{integral preserving mapping.}$$

F is the geometry parametrization of Ω.

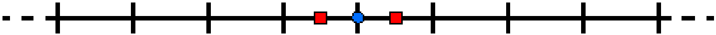
T-splines differential forms: odd degree

Recall the univariate derivative: $(B_p^A)' = D_{p-1}^{A^-} - D_{p-1}^{A^+}$:

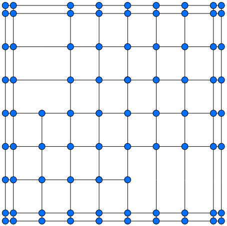


T-splines differential forms: odd degree

Recall the univariate derivative: $(B_p^A)' = D_{p-1}^{A^-} - D_{p-1}^{A^+}$:



Start with an AS T-mesh $\mathcal{M}^0$, and the space $\hat{Y}^0 = T_{p,p}(\mathcal{M}^0)$:

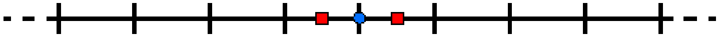


Odd degree: anchors are associated with the **vertices**.

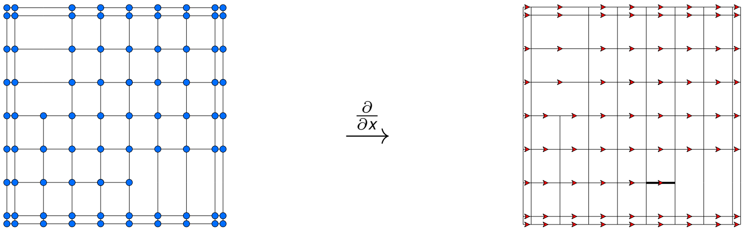
The space $\hat{Y}^1$ must contain the gradient of these functions.

T-splines differential forms: odd degree

Recall the univariate derivative: $(B_p^A)' = D_{p-1}^{A^-} - D_{p-1}^{A^+}$:



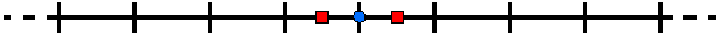
Start with an analysis suitable T-mesh $\mathcal{M}^0$, and the space $\hat{Y}^0 = T_{p,p}(\mathcal{M}^0)$.



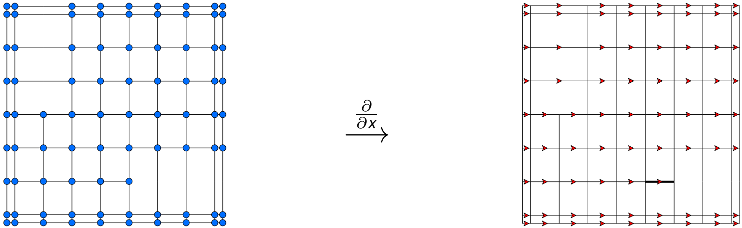
Degree (even,odd): anchors associated with the **horizontal edges.**

T-splines differential forms: odd degree

Recall the univariate derivative: $(B_p^A)' = D_{p-1}^{A^-} - D_{p-1}^{A^+}$:



Start with an analysis suitable T-mesh $\mathcal{M}^0$, and the space $\hat{Y}^0 = T_{p,p}(\mathcal{M}^0)$.



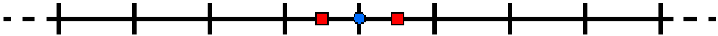
Degree (even,odd): anchors associated with the **horizontal edges.**

New mesh, $\mathcal{M}^1_{\partial x}$: added one extension of all **horizontal T-junctions.**

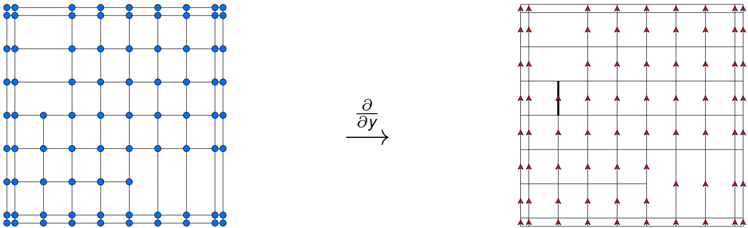
The first component of $\hat{Y}^1$ is the T-spline space: $T_{p-1,p}(\mathcal{M}^1_{\partial x})$.

T-splines differential forms: odd degree

Recall the univariate derivative: $(B_p^A)' = D_{p-1}^{A^-} - D_{p-1}^{A^+}$:



Start with an analysis suitable T-mesh $\mathcal{M}^0$, and the space $\hat{Y}^0 = T_{p,p}(\mathcal{M}^0)$.



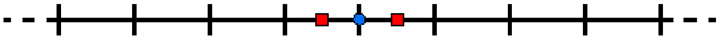
Degree (odd,even): anchors associated with the **vertical edges.**

New mesh, $\mathcal{M}^1_{\partial y}$: added one extension of all **vertical T-junctions.**

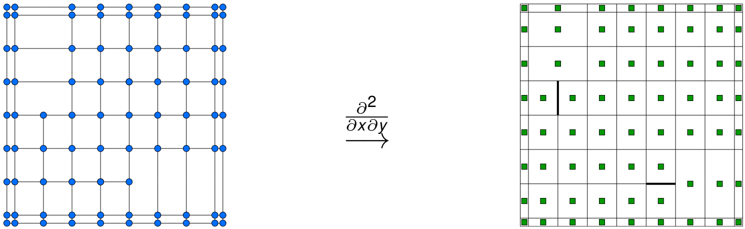
The second component of $\hat{Y}^1$ is the T-spline space: $T_{p,p-1}(\mathcal{M}^1_{\partial y})$.

T-splines differential forms: odd degree

Recall the univariate derivative: $(B_p^A)' = D_{p-1}^{A^-} - D_{p-1}^{A^+}$:



Start with an analysis suitable T-mesh $\mathcal{M}^0$, and the space $\hat{Y}^0 = T_{p,p}(\mathcal{M}^0)$.



Degree (even,even): anchors associated with the **elements**.

New mesh, $\mathcal{M}^2$: added one extension of all **T-junctions**.

The third space is $\hat{Y}^2 = T_{p-1,p-1}(\mathcal{M}^2)$.

T-splines differential forms: odd degree (II)

We have constructed the meshes and spaces for the sequence

$$T_{p,p}(\mathcal{M}^0)/\mathbb{R} \xrightarrow{\textbf{grad}} (T_{p-1,p}(\mathcal{M}^1_{\partial x}), T_{p,p-1}(\mathcal{M}^1_{\partial y})) \xrightarrow{\text{curl}} T_{p-1,p-1}(\mathcal{M}^2)$$

If $\mathcal{M}^0$ is AS, then $\mathcal{M}^1_{\partial x}$, $\mathcal{M}^1_{\partial y}$ and $\mathcal{M}^2$ are AS.

Theorem ([Buffa, GS, Vazquez, submitted to JCP])

If $\mathcal{M}^0$ is analysis suitable, then the sequence is exact.

- based on polynomial characterization of T-spline spaces,
- we are currently constructing the stable commuting interpolators,
- the Bézier mesh is the same in the four spaces,
- works for any p, odd and even,
- mapping to the **physical domain** done as for B-splines/NURBS.

T-splines differential forms: odd degree (II)

We have constructed the meshes and spaces for the sequence

$$T_{p,p}(\mathcal{M}^0)/\mathbb{R} \xrightarrow{\textbf{grad}} (T_{p-1,p}(\mathcal{M}^1_{\partial x}), T_{p,p-1}(\mathcal{M}^1_{\partial y})) \xrightarrow{\text{curl}} T_{p-1,p-1}(\mathcal{M}^2)$$

If $\mathcal{M}^0$ is AS, then $\mathcal{M}^1_{\partial x}$, $\mathcal{M}^1_{\partial y}$ and $\mathcal{M}^2$ are AS.

Theorem ([Buffa, GS, Vázquez, submitted to JCP])

If $\mathcal{M}^0$ is analysis suitable, then the sequence is exact.

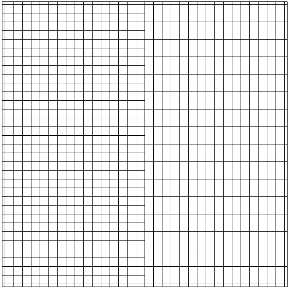
- based on polynomial characterization of T-spline spaces,
- we are currently constructing the stable commuting interpolators,
- the Bézier mesh is the same in the four spaces,
- works for any p, odd and even,
- mapping to the **physical domain** done as for B-splines/NURBS.

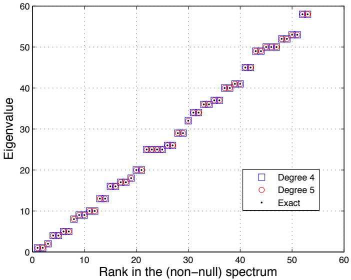
Numerical tests in 2D: the square domain

We test $\hat{Y}^1$ on the eigenvalue problem

$$\mathbf{curl}\,\mathrm{curl}\,\mathbf{u} = \omega^2 \mathbf{u}, \qquad \omega \neq 0,\ \mathbf{u} \neq \mathbf{0}.$$

Unit square domain , with T-mesh as below:





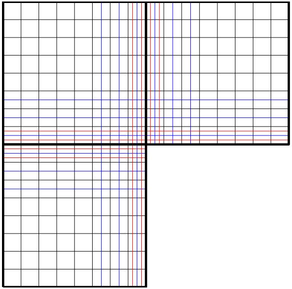
- The computed spectrum is **spurious free** for any degree.
- Good approximation of eigenvalues with the right multiplicity.

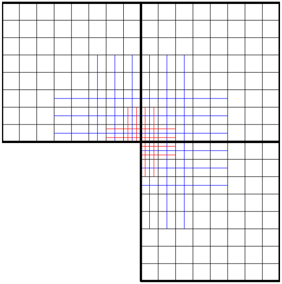
2D numerical tests: the L-shaped domain

We discretize with the space Y^1 the eigenvalue problem

$$\mathbf{curl}\,\mathrm{curl}\,\mathbf{u} = \omega^2 \mathbf{u}, \qquad \omega \neq 0,\ \mathbf{u} \neq \mathbf{0}.$$

L-shaped domain, in a mesh refined towards the corner.



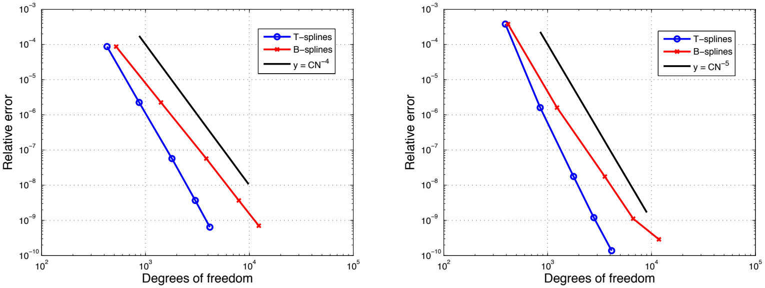


2D numerical tests: the L-shaped domain

We discretize with the space Y^1 the eigenvalue problem

$$\mathbf{curl}\,\mathrm{curl}\,\mathbf{u} = \omega^2 \mathbf{u}, \qquad \omega \neq 0,\ \mathbf{u} \neq \mathbf{0}.$$

L-shaped domain, in a mesh refined towards the corner.



- Convergence results for degrees 4 and 5.
- Better approximation properties than B-splines.

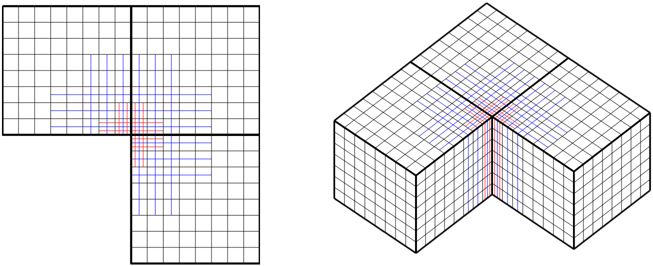
Three dimensional domains

We extend to 3D, assuming that we don't need local refinement in z.

$$\hat{Y}^0/\mathbb{R} \xrightarrow{\textbf{grad}} \hat{Y}^1 \xrightarrow{\textbf{curl}} \hat{Y}^2 \xrightarrow{\text{div}} \hat{Y}^3$$

The idea is to use 2D T-splines in the first two directions.

In the third direction we use univariate B-splines.



Three dimensional domains

We extend to 3D, assuming that we don't need local refinement in z.

$$\hat{Y}^0/\mathbb{R} \xrightarrow{\textbf{grad}} \hat{Y}^1 \xrightarrow{\textbf{curl}} \hat{Y}^2 \xrightarrow{\text{div}} \hat{Y}^3$$

The idea is to use 2D T-splines in the first two directions.

In the third direction we use univariate B-splines.

These are the spaces that form the sequence:

$\hat{Y}^0 = T_{p,p} \otimes S_p(\Xi)$,

$\hat{Y}^1 = (T_{p-1,p} \otimes S_p(\Xi), T_{p,p-1} \otimes S_p(\Xi), T_{p,p} \otimes S_{p-1}(\Xi'))$,

$\hat{Y}^2 = (T_{p,p-1} \otimes S_{p-1}(\Xi'), T_{p-1,p} \otimes S_{p-1}(\Xi'), T_{p-1,p-1} \otimes S_p(\Xi))$,

$\hat{Y}^3 = T_{p-1,p-1} \otimes S_{p-1}(\Xi')$.

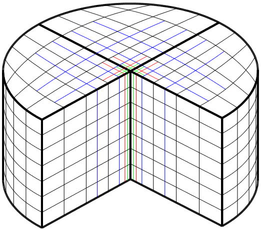
3D numerical results: three quarters of a cylinder

We discretize with the space Y^1 the model problem:

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} + \mathbf{u} = \mathbf{f},$$

with exact solution $\mathbf{u} = \mathbf{grad}(r^{2/3} \sin 2\theta/3 \sin(\pi z))$.

Refinement towards the reentrant edge, to catch the singularity.



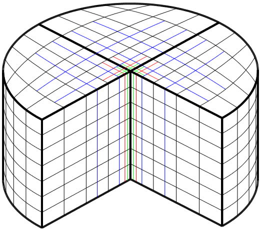
3D numerical results: three quarters of a cylinder

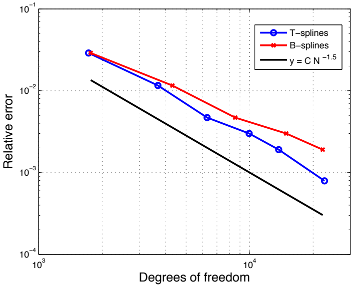
We discretize with the space Y^1 the model problem:

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} + \mathbf{u} = \mathbf{f},$$

with exact solution $\mathbf{u} = \mathbf{grad}(r^{2/3} \sin 2\theta/3 \sin(\pi z))$.

Refinement towards the reentrant edge, to catch the singularity.





T-splines of degree 3: better convergence than B-splines in terms of dofs.

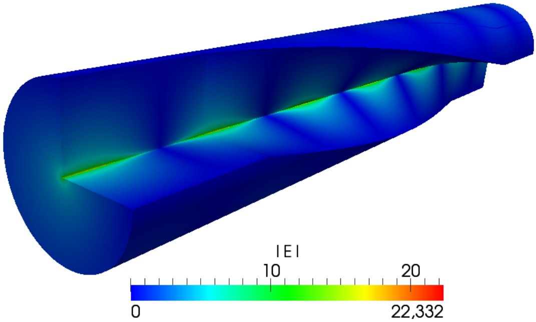
3D numerical results: twisted waveguide

Propagation of the first TE mode in a **twisted waveguide.**

Discretized with T-splines with $p = 3$, and 7936 dofs.

With 6 elements in the z-direction, C^2 or C^0 tang. continuity.

Accurate transmission and reflection output.[Buffa, GS, Vázquez, submitted to JCP]



[http://geopdes.sourceforge.net]

Collaborator on recent researches I have presented

- Lourenco Beirao da Veiga (University of Milan)
- Andrea Bressan (University of Pavia)
- Annalisa Buffa (IMATI-CNR)
- Massimiliano Martinelli (University of Pavia)
- Christoph Schwab (ETH)
- Rafael Vazquez (IMATI-CNR)

References

On IGA (Lecture 1-2):

- J. A. Cottrell, T. J. R. Hughes, Y. Bazilevs, Isogeometric Analysis: toward integration of CAD and FEA, John Wiley & Sons, 2009.
- L. Beirao da Veiga, D. Cho, G. Sangalli, Anisotropic NURBS approximation in Isogeometric Analysis, Comput. Methods Appl. Mech. Engrg. 209-212 (2012) 1–11.
- J. A. Evans, Y. Bazilevs, I. Babuska, T. J. R. Hughes, N-widths, sup-infs, and optimality ratios for the k-version of the isogeometric FEM, Comput. Methods Appl. Mech. Engrg. 198 (21-26) (2009) 1726–1741.
- M. Scott, M. Borden, C. Verhoosel, T. Sederberg, T. J. R. Hughes, Isogeometric finite element data structures based on Bezier extraction of T-splines, Internat. J. Numer. Methods Engrg. 88 (2) (2011) 126–156.
- C. de Falco, A. Reali, R. Vazquez, GeoPDEs: a research tool for IGA of PDEs, Adv. Eng. Softw. 42 (12) (2011) 1020–1034.
- Buffa, Sangalli, Schwab, Exponential Convergence of the hp version of Isogeometric analysis in 1D, Spectral and High Order Methods for Partial Differential Equations - From the ICOSAHOM '12 conference

References

On splines (Lecture 1-2):

- C. de Boor, A practical guide to splines, revised Edition, Vol. 27 of Applied Mathematical Sciences, Springer-Verlag, New York, 2001.

References

On spline-based discrete differential forms (Lecture 2-3):

- A. Buffa, J. Rivas, G. Sangalli, R. Vazquez, Isogeometric discrete differential forms in three dimensions, SIAM J. Numer. Anal. 49 (2) (2011) 818–844.
- A. Buffa, C. de Falco, G. Sangalli, Isogeometric Analysis: stable elements for the 2D Stokes equation, Internat. J. Numer. Methods Fluids 65 (11-12) (2011) 1407–1422.
- J. A. Evans, T. J. R. Hughes, Isogeometric divergence-conforming B-splines for the Darcy- Stokes-Brinkman equations., Math. Models Methods Appl. Sci. 23 (04) (2013) 671–741.
- J. A. Evans, T. J. R. Hughes, Isogeometric divergence-conforming B-splines for the Steady Navier-Stokes Equations, Math. Models Methods Appl. Sci. 23 (08) (2013) 1421–1478.
- J. A. Evans, T. J. R. Hughes, Isogeometric divergence-conforming B-splines for the Unsteady Navier-Stokes Equations, J. Comput. Phys. 241 (2013) 141–167.

References

On T-splines (Lecture 3)

- M. Scott, T-splines as a design-through-analysis technology, Ph.D. thesis, The University of Texas at Austin (2011).
- Y. Bazilevs, V. Calo, J. A. Cottrell, J. A. Evans, T. J. R. Hughes, S. Lipton, M. Scott, T. Sederberg, Isogeometric analysis using T-splines, Comput. Methods Appl. Mech. Engrg. 199 (5-8) (2010) 229–263.
- X. Li, J. Zheng, T. Sederberg, T. Hughes, M. Scott, On linear independence of T-spline blending functions, Comput. Aided Geom. Design 29 (1) (2012) 63–76.
- L. Beirao da Veiga, A. Buffa, G. Sangalli, R. Vazquez, Analysis-suitable T-splines of arbitrary degree: Definition, linear independence and approximation properties, Math. Models Methods Appl. Sci. (2013), DOI:10.1142/S0218202513500231.
- M.R. Dörfel, B. Jüttler, and B. Simeon. Adaptive isogeometric analysis by local h-refinement with T-splines. Comput. Methods Appl. Mech. Engrg., 199(5-8):264–275, 2010.

References

On other non-tensor-product spline spaces (Lecture 3)

- T. Dokken, T. Lyche, K. F. Pettersen, Polynomial splines over locally refined box-partitions, Comput. Aided Geom. Design 30 (3) (2013) 331–356.
- A.-V. Vuong, C. Giannelli, B. Jüttler, B. Simeon, A hierarchical approach to adaptive local refinement in isogeometric analysis, Comput. Methods Appl. Mech. Engrg. 200 (49-52) (2011) 3554–3567.
- H. Speleers, C. Manni, F. Pelosi, M. L. Sampoli, Isogeometric analysis with Powell-Sabin splines for advection-diffusion-reaction problems, Comput. Methods Appl. Mech. Engrg. 221/222 (2012) 132–148.

Conclusion

- we have defined arbitrary-degree AS T-splines,
- we have proved linear independence of AS T-splines, partition of unity property and fundamental approximation results,
- the theory is based on the characterization of AS T-splines as DC T-splines.
- we have defined T-splines based differential forms, based on arbitrary-degree AS T-meshes:
  - the Bézier mesh is the same for all the spaces in the diagram.
  - tested spurious free approximation of Maxwell eigenproblem.
  - better convergence than B-splines in terms of d.o.f.

Conclusion

- we have defined arbitrary-degree AS T-splines,
- we have proved linear independence of AS T-splines, partition of unity property and fundamental approximation results,
- the theory is based on the characterization of AS T-splines as DC T-splines.

- we have defined T-splines based differential forms, based on arbitrary-degree AS T-meshes:
  - the Bézier mesh is the same for all the spaces in the diagram.
  - tested spurious free approximation of Maxwell eigenproblem.
  - better convergence than B-splines in terms of d.o.f.

*** Thank you for your attention ***