The Abdus Salam International Centre for Theoretical Physics

2469-3

Workshop and Conference on Geometrical Aspects of Quantum States in Condensed Matter

1 - 5 July 2013

Conformal Invariance in 2D Hydrodynamics

Eldad Bettelheim
The Hebrew University of Jerusalem

Strada Costiera, 11 - 34151 - Trieste - Italy • Tel. +39 0402240111 • Fax. +39 040224163 • sci_info@ictp.it • www.ictp.it
ICTP is governed by UNESCO, IAEA, and Italy, and it is a UNESCO Category 1 Institute

Conformal Invariance in 2D Hydrodynamics

E. Bettelheim

March 12th, 2013

Acknowledgements:
O. Agam
G. Falkovich
P. B. Wiegmann
A. Zamolodchikov

Talk Outline

Motivation: Classical Stochastic Hydrodynamics

Calogero Phase space liquid

Stochastic Quantization in Hydrodynamic (Coulomb)

Integrable Structure

Conclusion & Outlook

Reference: EB, arXiv:1306.3782

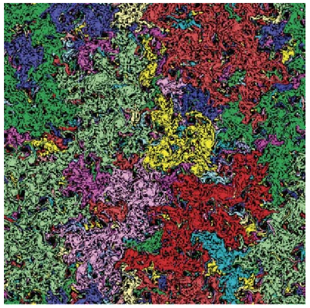
Motivation: Classical Stochastic Hydrodynamics

Classical Stochastic Hydrodynamics

1. Adding noise to classically stochastic hydrodynamics in 2D is a rich problem, which includes **Turbulence**
2. What kind of symmetries can we expect in this low-dimensional non-equilibrium problem?
3. A picture emerges from numerical simulations which points to CFT-type conformal invariance.

Reference to Numerical results

- Bernard, Boffetta, Celani, Falkovich (Nature Physics (2006) 2, 124128), Showed that zero voriticty isolines conform with $c = 0$ Percolation hulls.



- Polyakov had earlier suggested a connection to CFT (with another regime in mind):
  Try to match solutions of $\langle \vec{v}(x_1)\vec{v}(x_2)\ldots\vec{v}(x_n)\rangle$ with CFT correlators

Strategy

1. Look at the quantum problem instead
2. Search for hidden symmetries of quantum hydrodynamics.
3. Compare to classical stochastics.

Phase Space Stochastic Quantization

- We shall make use of Stochastic quantization (SQ) in phase space:

$$\frac{dp}{d\tau} = \imath\dot{q} - \frac{\delta H}{\delta p} + d\xi_p, \quad \frac{dq}{d\tau} = -\imath\dot{p} - \frac{\delta H}{\delta q} + d\xi_q,$$

- Take $\tau \to \infty$, whereupon $\frac{dp}{d\tau} \stackrel{d}{=} \frac{dq}{d\tau} \stackrel{d}{=} 0$
- Classically, take $\frac{d}{d\tau} \to 0$, $t \to \imath t$:

$$0 = \dot{q} - \frac{\delta H}{\delta p} + d\xi_p, \quad 0 = -\dot{p} - \frac{\delta H}{\delta q} + d\xi_q,$$

- The relation is formal.

Calogero Phase space liquid

Calogero-Sutherland Model

The Calogero-Sutherland model is an N-particle system on a line with inverse square law interactions:

$$H = \sum_i \left(p_i^2 + x_i^2\right) + \sum_{i \neq j} \frac{\theta(\theta - \hbar)}{(x_i - x_j)^2}$$

Calogero as Hamiltonian Reduction

The interaction can be thought of as a purely phase volume effect. Indeed, the dynamics are equivalent to the motion of the eigenvalues of X:

$$\dot{X} = P, \quad \dot{P} = -X$$

$$[X, P] = 2\imath\theta \left[\mathbb{1} - (N+1)|N\rangle\langle N|\right]$$

The system is symmetric under $X \to U^\dagger X U$, $P \to U^\dagger P U$ with $U^\dagger = U^{-1}$, and $U|N\rangle = e^{i\phi}|N\rangle$.

Phase Space Picture of Calogero

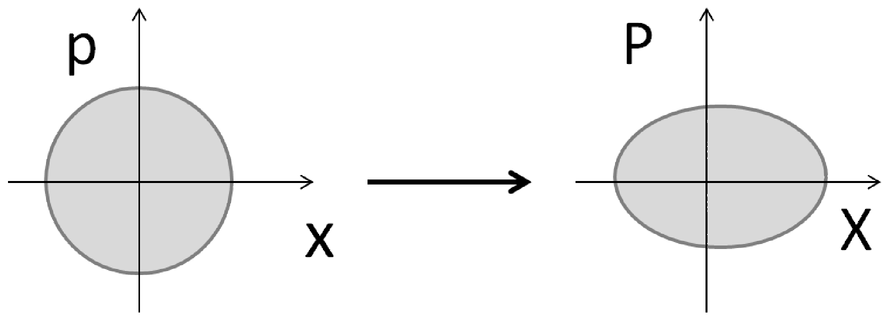
X and P are coordinates in phase space. Since they are Matrices, we consider their Wigner transform, and the Wigner distribution

$$\hat{X} = \sum_{i,j} X_{i,j} |i\rangle\langle j|, \quad \langle x|i\rangle = H_i(x) e^{-\frac{x^2}{\theta}}$$

$$X(x,p) = \int \langle x - y/2|\hat{X}|x + y/2\rangle e^{i\frac{yp}{\theta}}$$

Phase Space Picture of Calogero

Susskind gave the following picture:



$$\rho(X,P) = \frac{\mathbb{1}_N(x,p)}{\{X,P\}} = \frac{\mathbb{1}_N(x,p)}{\mathcal{WT}([X,P])} = f(x,p)$$

The density is a time independent function of Lagrangian coordinates, i.e. **incompressibility**.

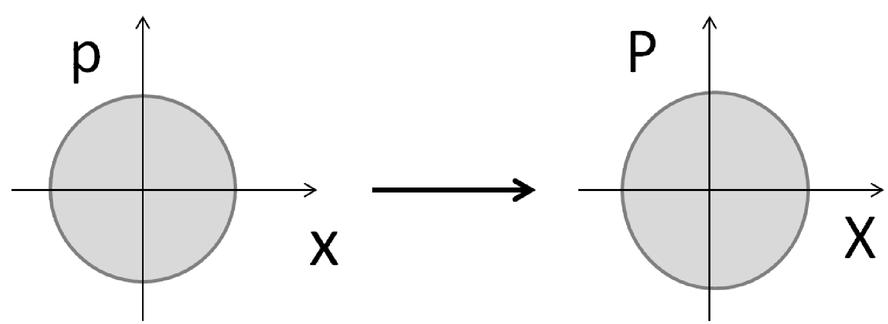
Ground State

The ground state is given by:

$$\hat{X} = \hat{x}\cos(t) + \hat{p}\sin(t), \quad \hat{P} = \hat{p}\cos(t) - \hat{x}\sin(t)$$

where:

$$\hat{x} + \imath\hat{p} = \sum_{i=0}^{N} \sqrt{i}|i-1\rangle\langle i| = \mathbb{1}_{N\times N} a \mathbb{1}_{N\times N}$$



Stochastic Quantization in Hydrodynamic (Coulomb) Gauge

Hamiltonian Reductions

We want to quantize

$$\dot{\hat{Z}} = \imath \hat{Z} + \imath [\lambda, \hat{Z}]$$

where $\hat{Z} = \hat{X} + \imath \hat{P}$. With constraint:

$$[\hat{Z}, \hat{Z}^\dagger] = 2\theta \left(\mathbb{1} - (N+1)|N\rangle\langle N|\right)$$

And a gauge choice determining λ

This is a **Hamiltonian Reduction**.

Stochastic Quantization of Hamiltonian Reductions

Quantize (Parisi) by introducing **Auxiliary time,** τ:

$$\frac{d\hat{Z}}{d\tau} = -\frac{dS}{d\hat{Z}} + d\hat{\Xi}$$
$$= \dot{\hat{Z}} - \imath\hat{Z} + d\hat{\Xi}$$

(Ohba) Introduce Lagrange multipliers, $\hat{\nu}$ associated with the constraint, $\hat{\phi}$, and Lagrange multiplier, $\hat{\lambda}$, associated with the gauge condition, $\hat{\chi}$.

$$\frac{d\hat{Z}}{d\tau} = -\frac{dS}{d\hat{Z}} + \hat{\lambda}\frac{d\hat{\phi}}{d\hat{Z}^\dagger} + \hat{\nu}\frac{d\hat{\chi}}{d\hat{Z}^\dagger} + d\hat{\Xi} \qquad (1)$$

Gauge Choice

1. The usual gauge choice is to let $\hat{X}$ be diagonal, $[\hat{x}, \hat{X}] = 0$.
   Where $\hat{x}, \hat{p}$ are ground state matrices.
2. This choice leads to Calogero, $H = \sum_i \left(p_i^2 + x_i^2\right) + \sum_{i \neq j} \frac{\theta(\theta - \hbar)}{(x_i - x_j)^2}$, as a Hamiltonian reduction .
3. This anisotropic choice is inconvenient. Choose instead the **Coulomb gauge**:

$$[\hat{x}, \hat{X}] + [\hat{p}, \hat{P}] = 0$$

Hydrodynamics in Coulomb gauge

1. The equations are:
   $\frac{dZ}{d\tau} = \dot{Z} - \imath Z + \imath[\lambda, Z] + [a, \nu] + d\Xi$
2. Considering small perturbations:
   $\hat{Z} = \hat{a}_N + V(\hat{a}_N^\dagger) + \ldots$
3. The equation of motion turns out to be:
   $\dot{V} = \imath V + d\xi + \ldots$
4. In vector form, $V \rightarrow \delta\vec{R}$ and
   $\vec{v} = \hat{z} \times \delta\vec{R}, \quad \delta\rho = \vec{\nabla} \cdot \delta\vec{R}, \quad \vec{\nabla} \times \delta\vec{R} = 0,$
   which leads to
   $$\omega = \frac{\theta}{m}\delta\rho, \quad \vec{\nabla} \cdot \vec{v} = 0,$$
   which is equivalent to hydrodynamics.

Vortex Dynamics

1. Taking curl of Euler $\dot{v} + v \cdot \nabla v + \nabla P = 0$:

$$\dot{\omega} + v \cdot \nabla \omega = 0, \quad \dot{\rho} + v \cdot \nabla \rho = 0,$$

   leads to a solution $\delta\rho \propto \omega$.
2. In our case it is only true integrated over any region whose boundary has small deformation:

$$\delta N_D = \int_D \rho(\vec{r}) d^2\vec{r} = \frac{m}{\theta} \oint_{\partial D} \vec{v} \cdot d\vec{r} = \frac{m}{\theta} \Gamma_D.$$

3. In particular quantized vortices - $n_D = \frac{m}{\hbar}\Gamma_D = \frac{\hbar}{\theta}\delta N_D$

Integrable Structure

Mode Expansion

1. An infinite number of conserved quantites exist and can be written through the mode expansion:
$$V = \sum_n \alpha_n a^{\dagger n},$$
where $Z = a + V + \dots$
2. The expressions are complicated, but one may expand in $\frac{\hbar}{L}$. Assuming vortices one obtains $V \sim \frac{\hbar}{L}$. One obtains a set of conserved quantities whose leading orders commute.

Conserved quantities

1. $H_1 = \sum_n \alpha_n^\dagger \alpha_n + \ldots$
2. $H_2 =$

$$= \sum \sqrt{\theta}\,(\alpha_n^\dagger \alpha_m^\dagger \alpha_{n+m} + hc) + (\theta - \hbar) n \alpha_n^\dagger \alpha_n + \ldots$$

3. The simultaneous diagonalization of H_1 and H_2 is a problem already solved in collective field theory of Calogero and is related to Jack polynomials, CFT with $c = 1 - 12\left(\frac{\sqrt{\theta}}{\sqrt{\hbar}} - \frac{\sqrt{\hbar}}{\sqrt{\theta}}\right)^2$

Conclusion & Outlook

Conclusion

1. Calogero in a different gauge is related to hydrodynamics
2. The integrability of the Calogero expresses itself in the problem of diagonalizing the Hamiltonian written in terms of modes of deformation
3. The diagonalization problem is given through Jacks related to the representation theory of

$$c = 1 - 12\left(\sqrt{\frac{\theta}{\hbar}} - \sqrt{\frac{\hbar}{\theta}}\right)^2$$

Outlook

1. Beyond the the separated vortices limit: Hall viscosity?
2. Relation to the fractional quantum Hall effect
3. Relation to Classical Stochastic Dynamics