The Abdus Salam
International Centre
for Theoretical Physics

2469-1

Workshop and Conference on Geometrical Aspects of Quantum States in Condensed Matter

1 - 5 July 2013

Momentum space topology in Standard Model and in condensed matter

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Momentum space topology in Standard Model & condensed matter

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RUSSIAN ACADEMY OF SCIENCES
L.D Landau
INSTITUTE FOR THEORETICAL PHYSICS

Geometrical Aspects of Quantum States in Condensed Matter, 1-5 July 2013

1. Gapless & gapped topological media

2. Fermi surface as topological object

3. Fermi points (Weyl, Majorana & Dirac points) & nodal lines

* superfluid 3He-A, topological semimetals , cuprate superconductors , graphene
  vacuum of Standard Model of particle physics in massless phase
* gauge fields, gravity, chiral anomaly as emergent phenomena; quantum vacuum as 4D graphene
* exotic fermions: quadratic, cubic & quartic dispersion; dispersionless fermions; Horava gravity

4. Flat bands & Fermi arcs in topological matter

* surface flat bands: 3He-A, semimetals , cuprate superconductors , graphene , graphite
* towards room-temperature superconductivity
* 1D flat band in the vortex core
* Fermi-arc on the surface of topological matter with Weyl points

5. Fully gapped topological media

* superfluid 3He-B, topological insulators , chiral superconductors, quantum spin Hall insulators,
  vacuum of Standard Model of particle physics in present massive phase, vacua of lattice QCD
* Majorana edge states & zero modes on vortices (planar phase , topological insulator & 3He-B)

3+1 sources of effective Quantum Field Theories in many-body system & quantum vacuum

I think it is safe to say that no one understands **Quantum Mechanics**

Richard Feynman

Thermodynamics is the only physical theory of universal content

Albert Einstein

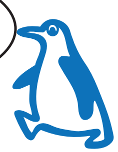
Symmetry: conservation laws, translational invariance, spontaneously broken symmetry, Grand Unification, ...

Topology: winding number
one can't comb hair on a ball smooth, anti-Grand-Unification

Lev Landau

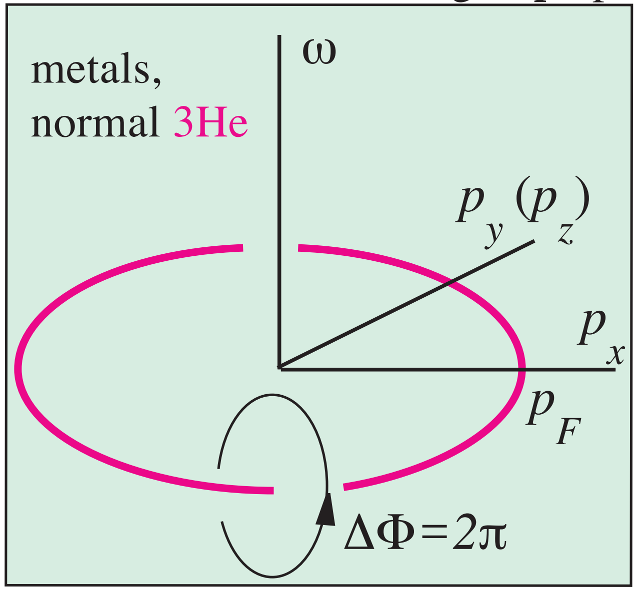
effective theories of quantum liquids: two-fluid hydrodynamics of superfluid ^{4}He & Fermi liquid theory of liquid ^{3}He

missing ingredient in Landau theories

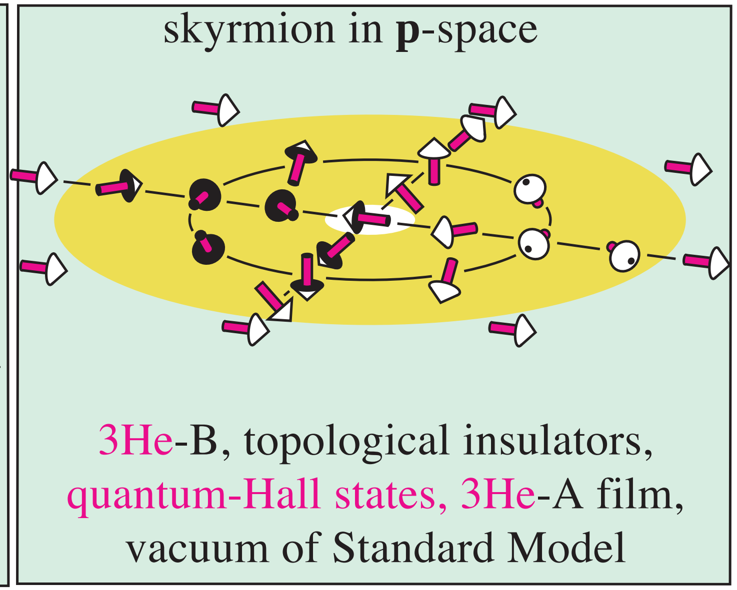


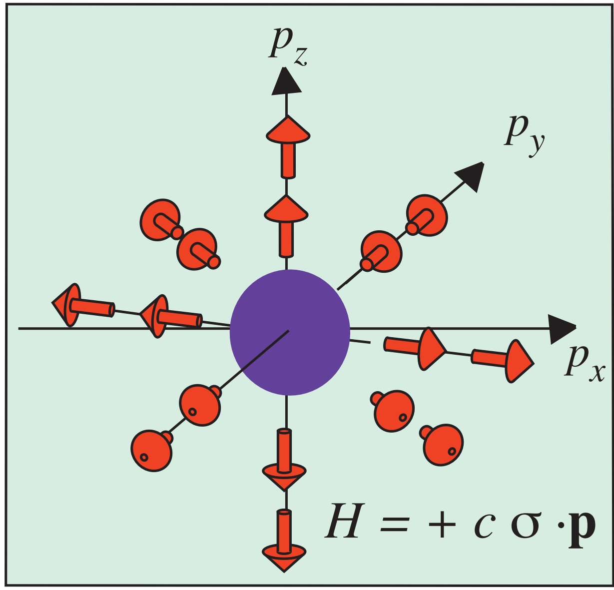
Topological classes as defects in momentum space

Fermi surface: vortex ring in **p**-space



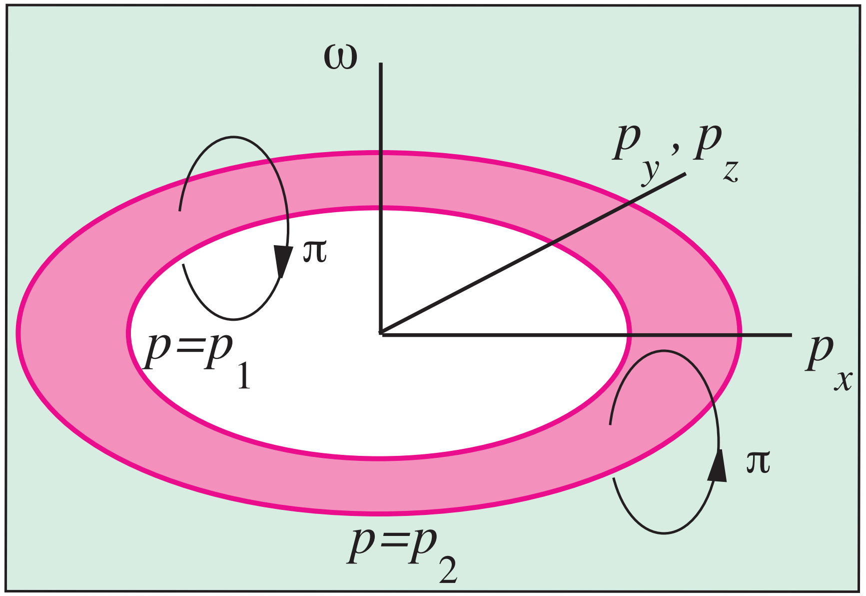
fully gapped topological matter:



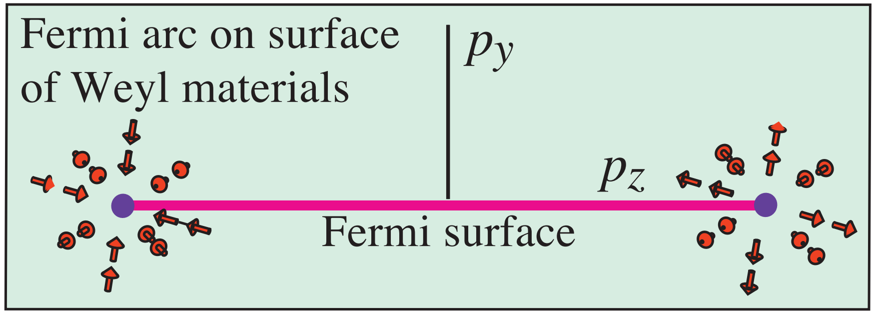


Weyl point - hedgehog in **p**-space

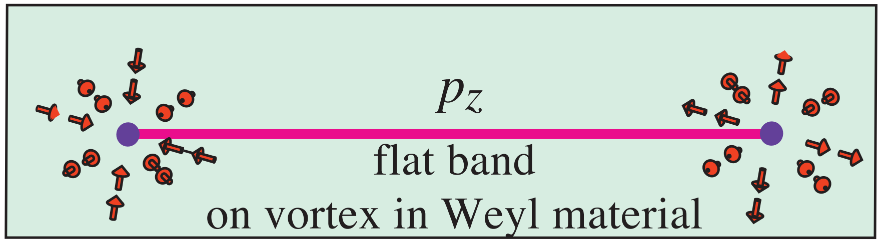
3He-A, vacuum of SM, topological semimetals (Abrikosov)



Khodel-Shaginyan flat band: π-vortex in **p**-space



Dirac strings in **p**-space terminating on monopole



bulk - edge correspondence:

topology in bulk protects
gapless fermions on surface of fully gapped systems;
higher order nodes in nodal topological materials

2D Quantum Hall insulator & 3He-A film	gapless chiral edge states (GV 1992)
3D topological insulator	gapless Dirac fermions (Volkov-Pankratov 1985)
superfluid 3He-B	gapless Majorana fermions (Salomaa-GV 1988)
3He-A, Weyl semimetal with point nodes	Fermi arc (nodal line) on surface (Tutsumi et al 2011)
graphene with point nodes	dispersionless 1D flat band (Ryu-Hatsugai 2002)
semimetal with Fermi lines	2D flat band on the surface (Heikkila-GV 2010)

bulk - defect correspondence:

topology in bulk protects gapless fermions inside topological defect

relativistic string	fermion zero modes in core (Jackiw-Rossi 1981)
3He-A with point nodes	1D flat band in the core (Kopnin-Salomaa 1991)
2D p+ip superconductor	Majorana fermions in the core (GV 1999)

two major universality classes of gapless fermionic vacua

Landau theory of Fermi liquid	Standard Model + gravity
vacuum with Fermi surface: metals, normal ^{3}He	**vacuum with Fermi (Weyl) point: ^{3}He-A, planar phase, Weyl semimetal, vacuum of SM**

gravity emerges from
Fermi (Weyl) point
analog of
Fermi surface

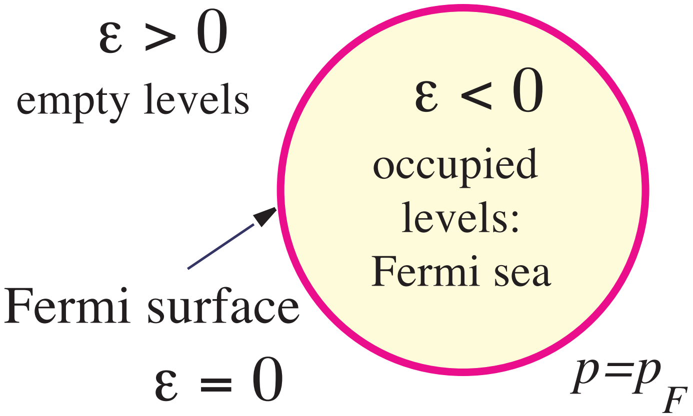
$$\rightarrow \boxed{g^{\mu\nu}(p_\mu - eA_\mu - e\tau \cdot \mathbf{W}_\mu)(p_\nu - eA_\nu - e\tau \cdot \mathbf{W}_\nu) = 0}$$

Theory of topological matter:
Nielsen, So, Ishikawa, Matsuyama, Haldane, Yakovenko, Horava, Kitaev, Ludwig, Schnyder, Ryu, Furusaki, S-C Zhang, Kane, Liang Fu, ...

Fermi surface as topological object

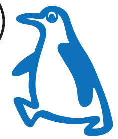
Energy spectrum of
non-interacting gas of fermionic atoms

$$\varepsilon(p) = \frac{p^2}{2m} - \mu = \frac{p^2}{2m} - \frac{p_F^2}{2m}$$



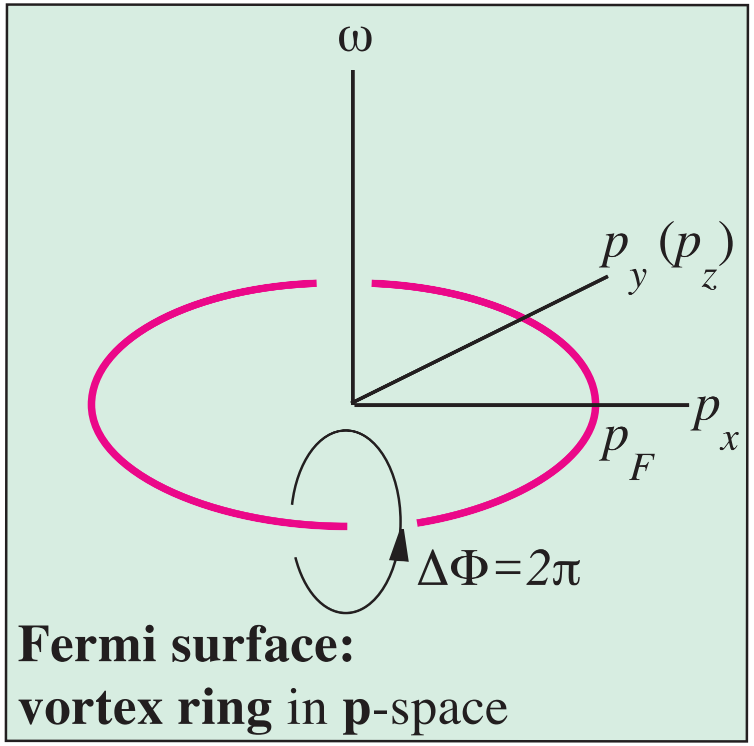
is Fermi surface a domain wall
in momentum space?

no!
it is a **vortex ring**



Green's function

$$G^{-1} = i\omega - \varepsilon(p)$$



phase of Green's function

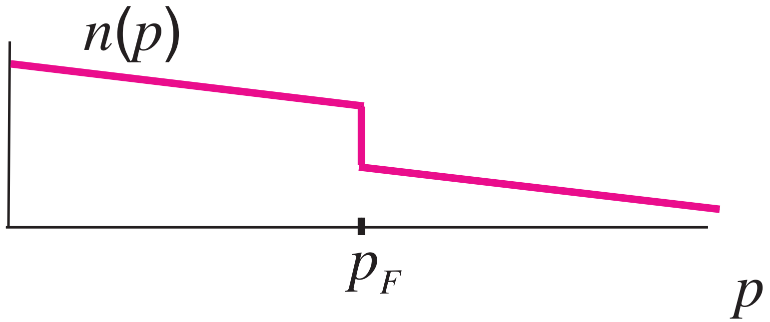
$$G(\omega,\mathbf{p}) = |G| e^{i\Phi}$$

has winding number $N = 1$

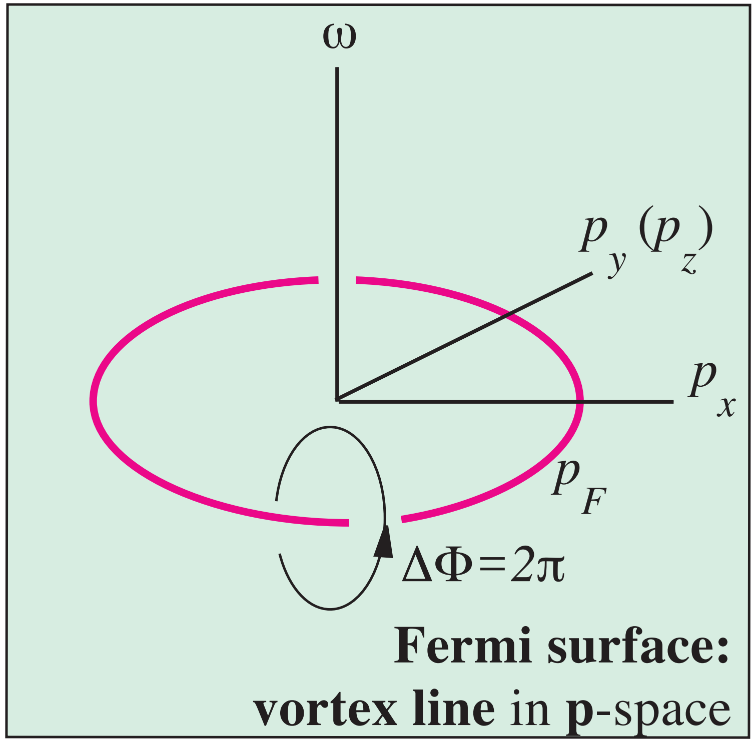
Migdal jump, non-Fermi liquids & p-space topology

* Singularity at Fermi surface is robust to perturbations:
winding number $N=1$ *cannot change continuously,*
interaction (perturbative) cannot destroy singularity

* Typical singularity: Migdal jump



* Other types of singularity with the same winding number:
Luttinger Fermi liquid,
marginal Fermi liquid, pseudo-gap ...



$$G(\omega,\mathbf{p})=|G|e^{\,i\Phi}$$

* Example: zeroes in $G(\omega,\mathbf{p})$ have the same $N=1$ as poles

$$G(\omega,\mathbf{p}) = \frac{Z(p,\omega)}{i\omega - \varepsilon(p)} \qquad Z(p,\omega) = (\omega^2 + \varepsilon^2(p))^\gamma$$

zeroes in $G(\omega,\mathbf{p})$
for $\gamma > 1/2$

* Important for interacting systems, where quasiparticles are ill defined

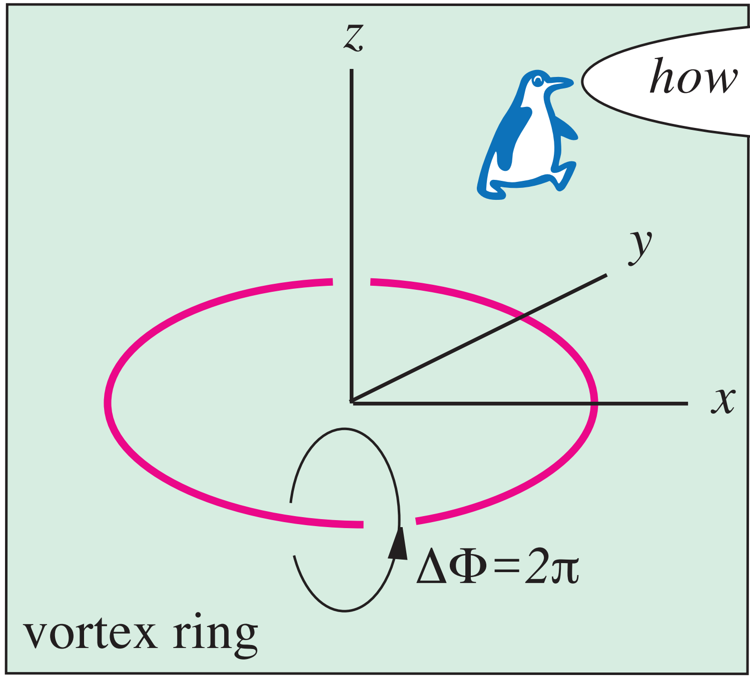
* Fermi surface exists in superfluids/superconductors, examples: 3He-A in flow &
Gubankova-Schmitt-Wilczek, PRB74 (2006) 064505,
but Luttinger theorem is not applied

quantized vortex in **r**-space $\equiv$ **Fermi surface** in **p**-space

homotopy group π_1

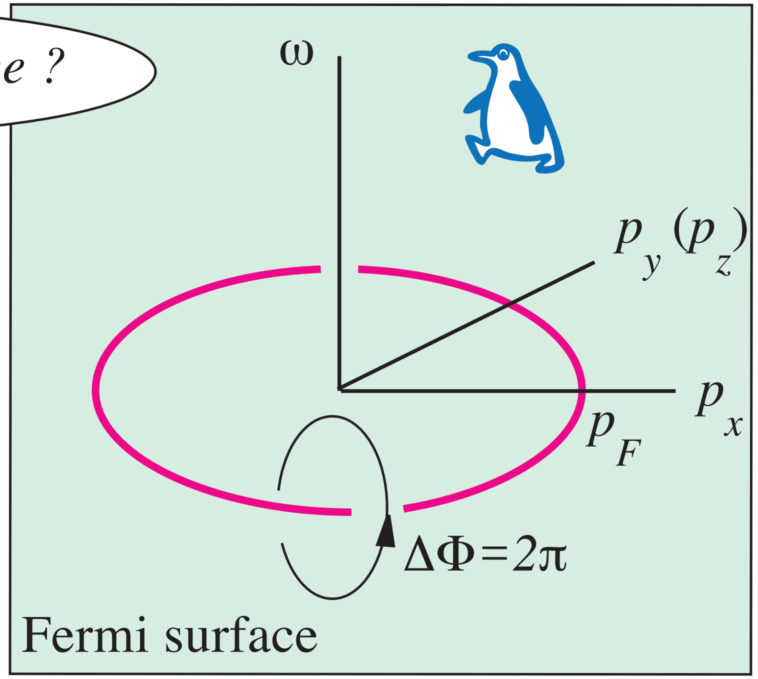
how is it in **p**-*space ?*

Topology in **r**-space



winding
number
$N_1 = 1$

Topology in **p**-space



$\Psi(\mathbf{r})=|\Psi|\, e^{i\Phi}$

scalar order parameter
of superfluid & superconductor

$G(\omega,\mathbf{p})=|G| e^{i\Phi}$

Green's function (propagator)

classes of mapping $S^1 \rightarrow U(1)$

manifold of
broken symmetry vacuum states

classes of mapping $S^1 \rightarrow GL(\mathrm{n,C})$

space of
non-degenerate complex matrices

non-topological flat bands due to interaction

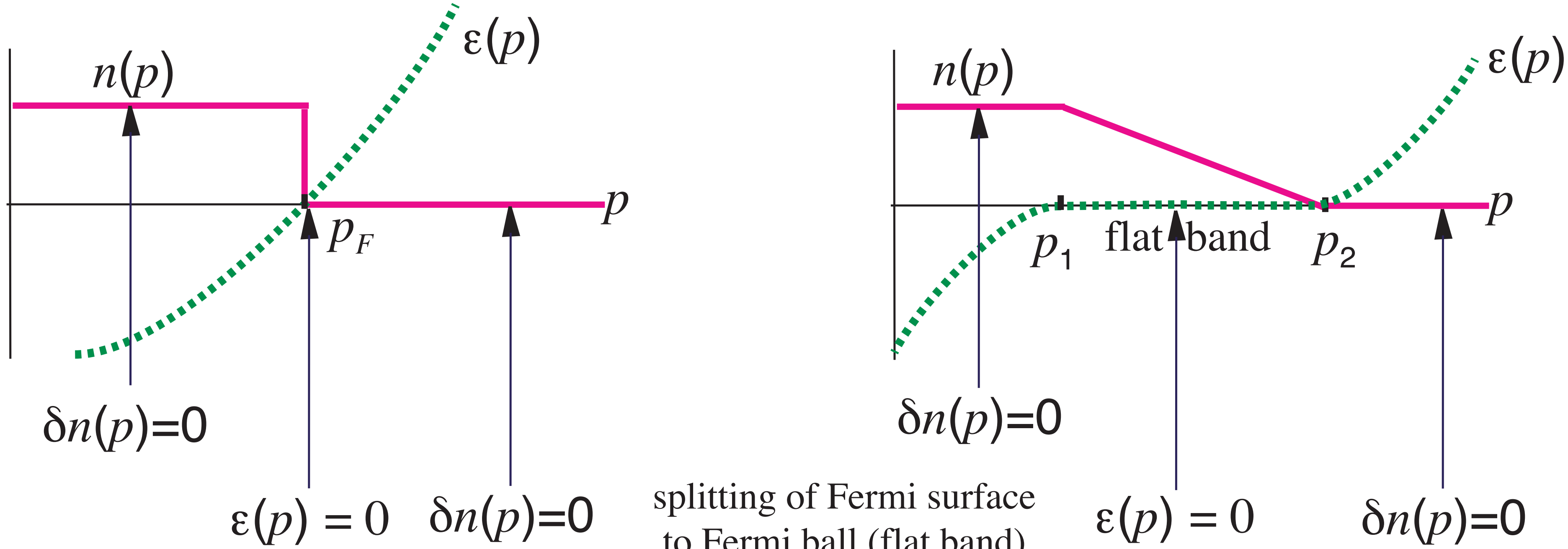
Khodel-Shaginyan fermion condensate

JETP Lett. **51**, 553 (1990)
GV, JETP Lett. **53**, 222 (1991)
Nozieres, J. Phys. (Fr.) **2**, 443 (1992)

$$E\{n(p)\}$$

$$\delta E\{n(p)\} = \int \varepsilon(p)\delta n(p) d^{\mathrm{d}}p = 0$$

solutions: $\varepsilon(p) = 0$ or $\delta n(p)=0$

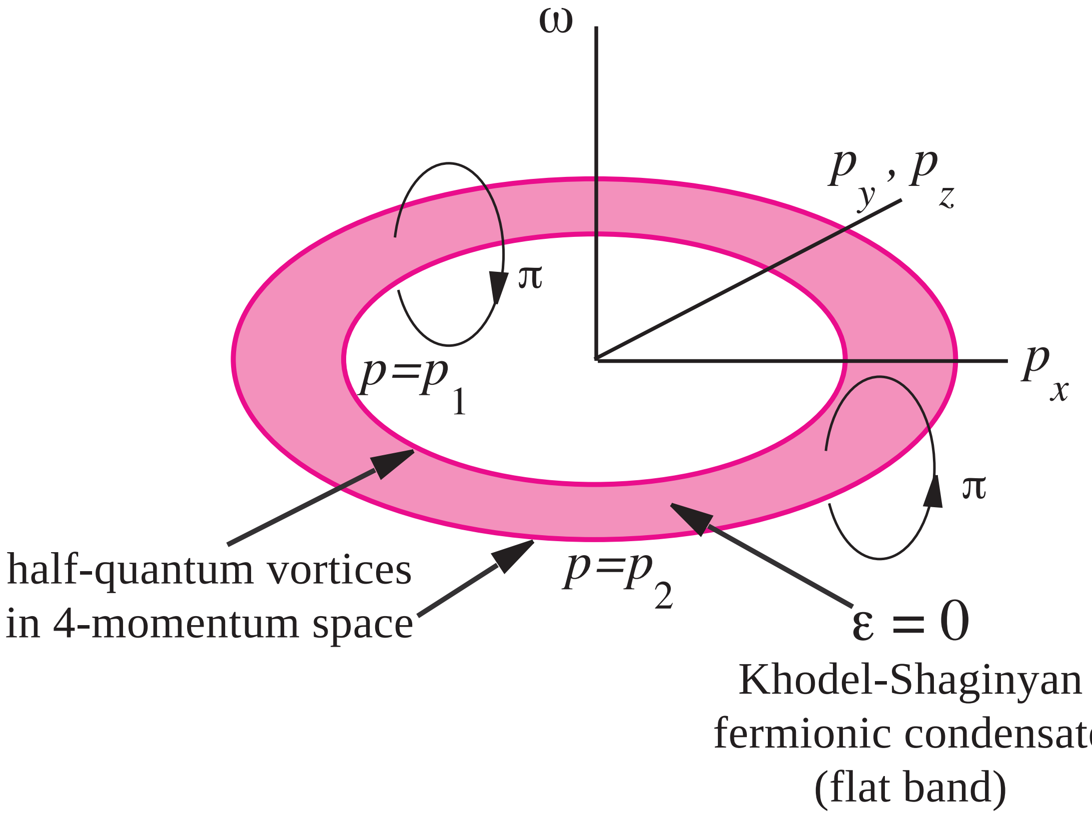


splitting of Fermi surface
to Fermi ball (flat band)

S.-S. Lee
Non-Fermi liquid from a charged black hole: A critical Fermi ball
PRD 79, 086006 (2009)

anti-de Sitter/conformal field theory correspondence

Flat band as momentum-space dark soliton terminated by half-quantum vortices

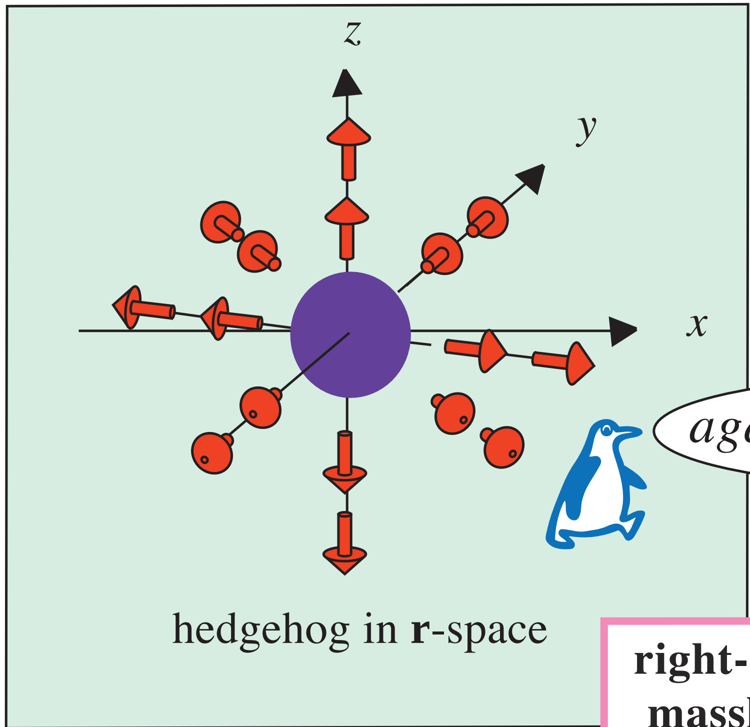


phase of Green's function changes by π across the "dark soliton"

3. Classes of Fermi points & nodal lines:

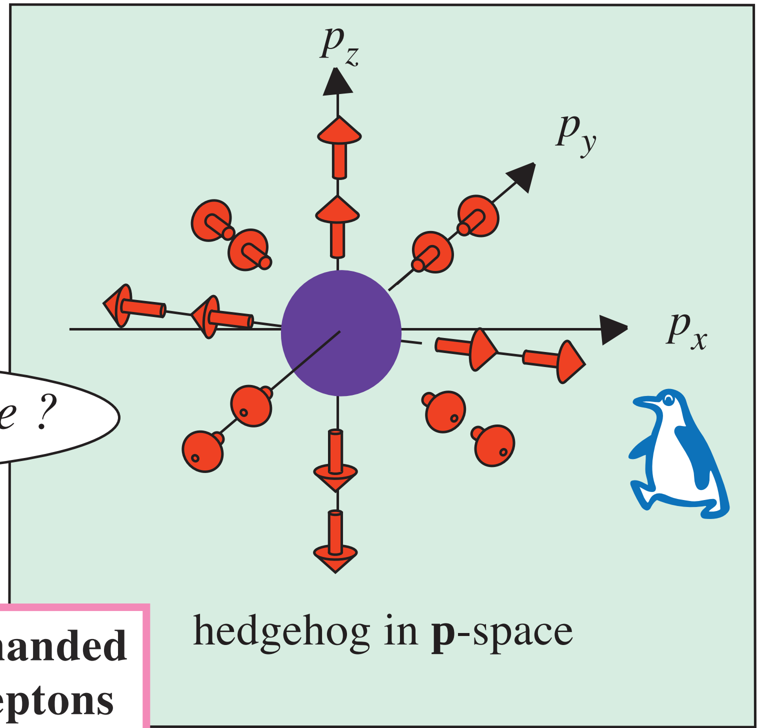
superfluid ^{3}He-A, Standard Model, semimetals, graphene, cuprate SC, ...
surface of ^{3}He-B & topological insulators

magnetic hedgehog



$$\sigma(\mathbf{r}) = \hat{\mathbf{r}}$$

Weyl point
or *Berry phase Dirac monopole*



$$\sigma(\mathbf{p}) = \hat{\mathbf{p}}$$

again no difference ?

right-handed and left-handed massless quarks and leptons are elementary particles in Standard Model

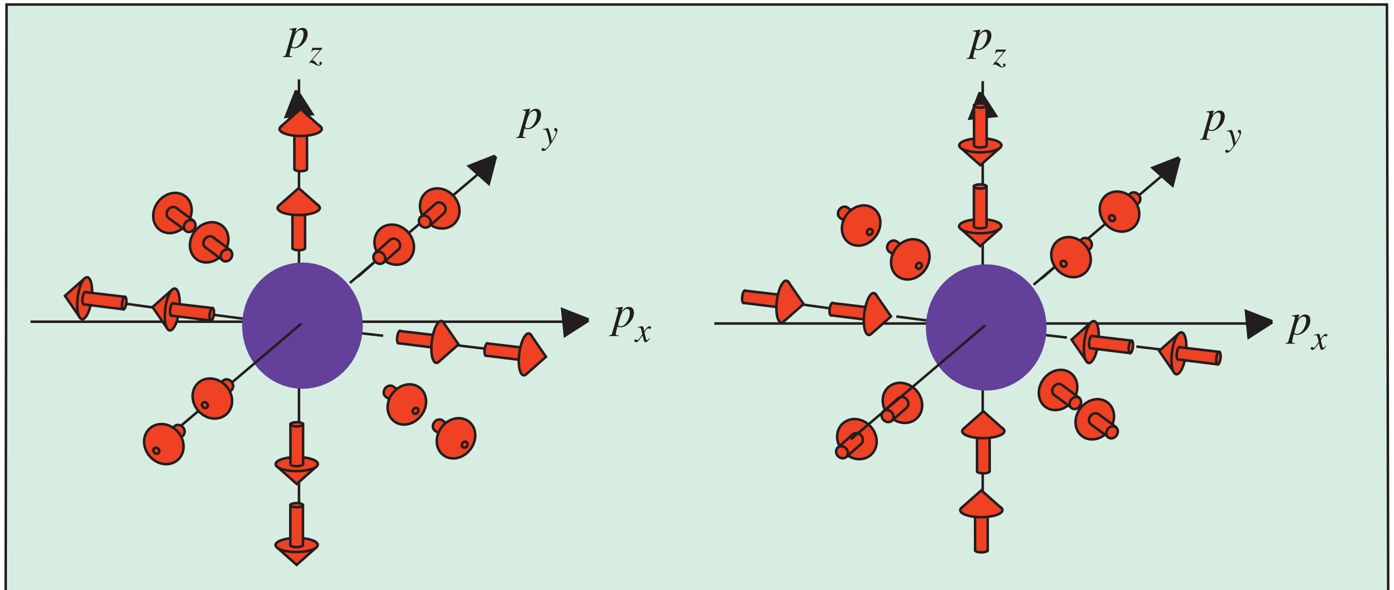
Landau CP symmetry is emergent

close to Fermi point

$$H = + c\,\sigma \cdot \mathbf{p}$$

right-handed Weyl electron =
hedgehog in **p**-space with spines = spins

Topological invariant for right and left elementary particles



hedgehog with spines (spins) outward ($N_3 = +1$)

hedgehog with spines (spins) inward ($N_3 = -1$)

right

$$H = +c\,\sigma \cdot \mathbf{p}$$
$$\mathbf{g}(\mathbf{p}) = +\,c\mathbf{p}$$

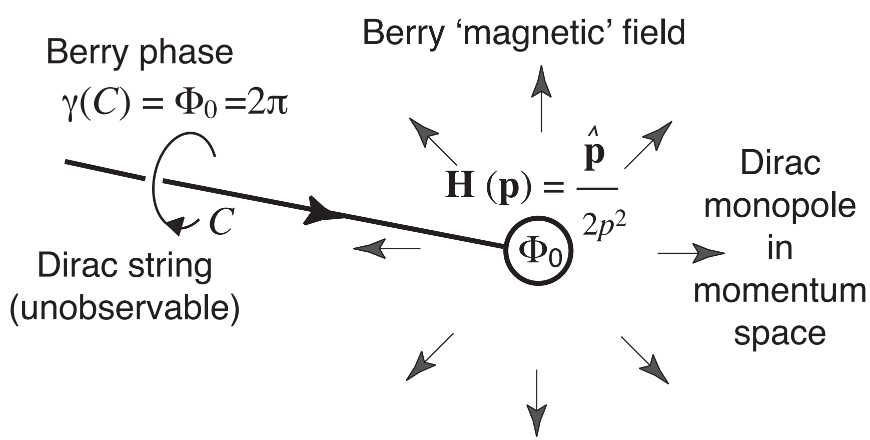
$$\boxed{H = \sigma \cdot \mathbf{g}(\mathbf{p})}$$

left

$$H = -\,c\,\sigma \cdot \mathbf{p}$$
$$\mathbf{g}(\mathbf{p}) = -\,c\mathbf{p}$$

$$N_3 = \frac{1}{8\pi}\, e_{ijk} \int dS^i\, \hat{\mathbf{g}} \cdot (\partial^j \hat{\mathbf{g}} \times \partial^k \hat{\mathbf{g}})$$

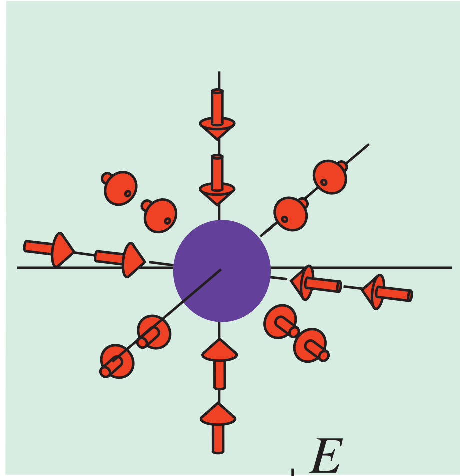
over 2D surface around Fermi point



Weyl fermions in 3+1 gapless topological cond-mat

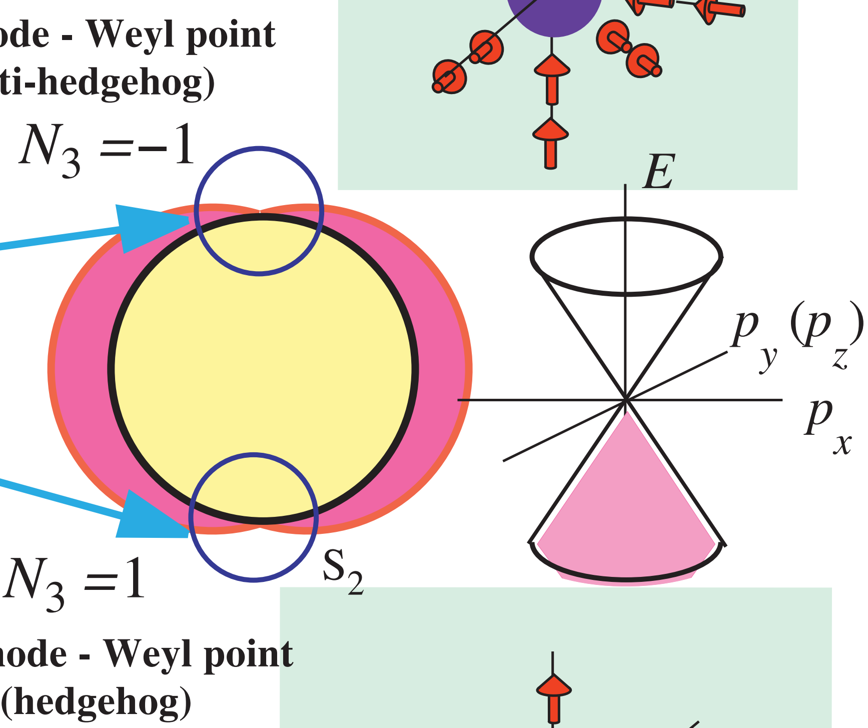
topologically protected Weyl points in:

topological semi-metal or Weyl metals (Abrikosov-Beneslavskii 1971), ^{3}He-A (1982), triplet Fermi gases, $CoSb_3$ (arXiv:1204.5905)

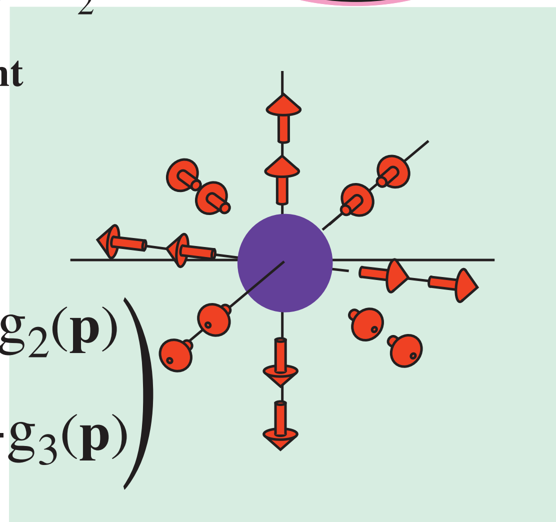


$$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \; \hat{\mathbf{g}} \cdot (\nabla_{p_i} \hat{\mathbf{g}} \times \nabla_{p_j} \hat{\mathbf{g}})$$

over 2D surface S in 3D p-space



$$p^2 = p_x^{\,2} + p_y^{\,2} + p_z^{\,2}$$



$$\mathrm{H} = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix}$$

emergence of relativistic chiral Weyl fermions near Fermi points

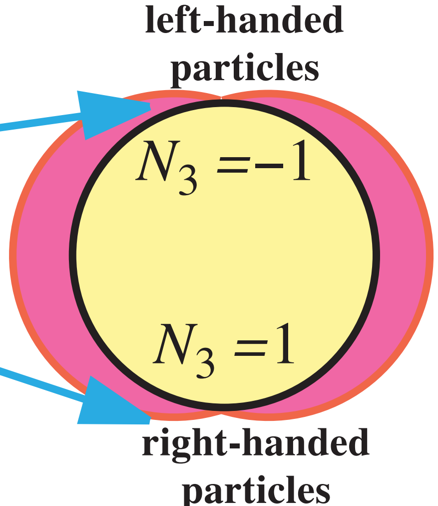
original non-relativistic Hamiltonian

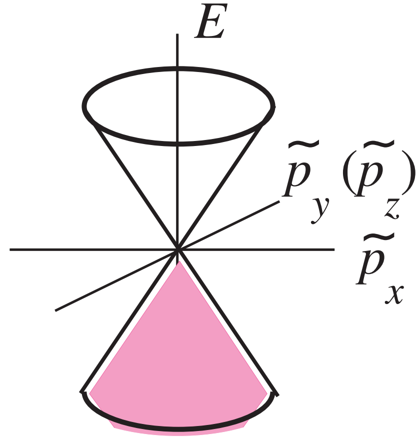
$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

close to nodes, i.e. in low-energy corner
relativistic chiral fermions emerge

$$H = N_3\, c\, \boldsymbol{\tau} \cdot (\mathbf{p} - \mathbf{p}_0)$$

$$E = -c\tilde{p}$$





top. invariant determines chirality in low-energy corner

chirality is emergent ??

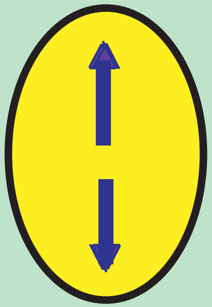
relativistic invariance as well

what else is emergent ?

bosonic collective modes in two generic fermionic vacua

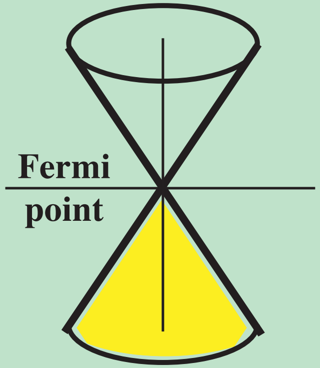
Landau theory of Fermi liquid

collective Bose modes
of fermionic vacuum:
propagating
oscillation of **shape**
of Fermi surface



Landau, ZhETF **32**, 59 (1957)

Standard Model + gravity

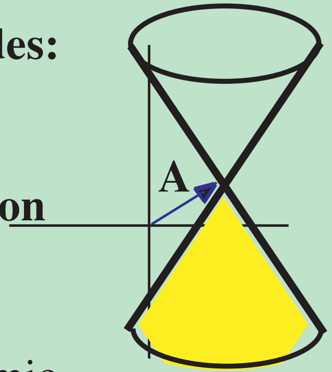


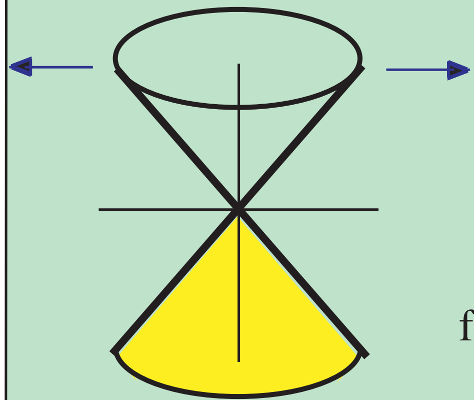
collective Bose modes:

propagating
oscillation of **position**
of Fermi point

$$\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A}$$

form effective dynamic
electromagnetic field

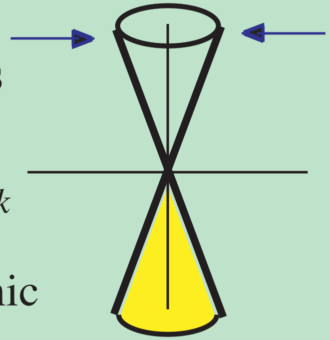




propagating
oscillation of **slopes**

$$E^2 = c^2 p^2 \rightarrow g^{ik} p_i p_k$$

form effective dynamic
gravity field



two generic quantum field theories of interacting bosonic & fermionic fields

relativistic quantum fields & gravity emerging near Weyl point

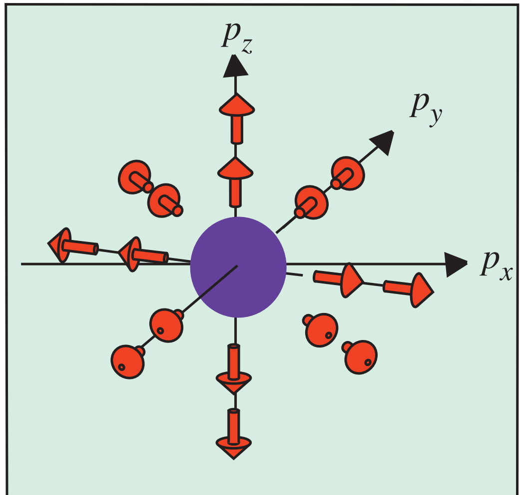
Atiyah-Bott-Shapiro construction:
linear expansion of Hamiltonian near the nodes in terms of Dirac Γ-matrices

primary object:
tetrad
$e_a^{\ \mu}$

$$\boxed{E = v_F\,(p - p_F)}$$ emergent relativity $$\boxed{H = e_a^{\ k}\,\Gamma^a \cdot (p_k - p_k^0)}$$

linear expansion near Fermi surface — linear expansion near Weyl point

secondary object:
metric
$g^{\mu\nu} = \eta^{ab}\, e_a^{\ \mu}\, e_b^{\ \nu}$



hedgehog in **p**-space

$$\boxed{g^{\mu\nu}(p_\mu - eA_\mu - e\tau \cdot \mathbf{W}_\mu)(p_\nu - eA_\nu - e\tau \cdot \mathbf{W}_\nu) = 0}$$

effective metric:
emergent gravity

effective
electromagnetic
field

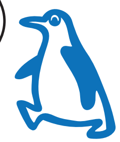
effective
$SU(2)$ gauge
field

effective
electric charge
$e = +1$ or -1

effective
isotopic spin

all ingredients of Standard Model :
chiral fermions & gauge fields
emerge in low-energy corner
together with spin, Dirac Γ–matrices,
gravity & physical laws:
Lorentz & gauge invariance,
equivalence principle, etc

gravity & gauge fields
are collective modes
of vacua with Weyl point

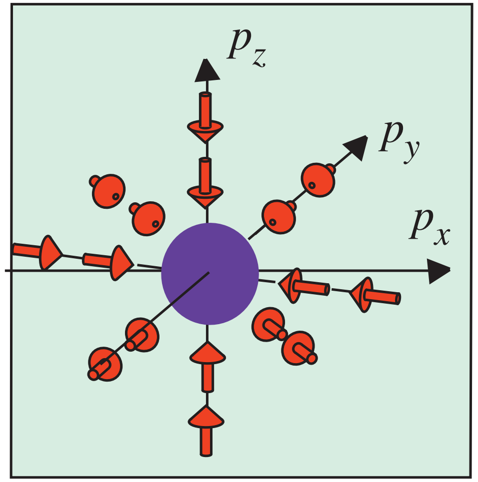


Einstein-Cartan-Sciama-Kibble theory
with tetrads, spin connection & torsion

Chiral Weyl fermions in Standard Model

Family #1 of quarks and leptons

left particles



hedgehog with spines (spins) inward ($N_3 = -1$)

quarks

+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6

$SU(3)_C$

+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3

$SU(2)_L$

leptons

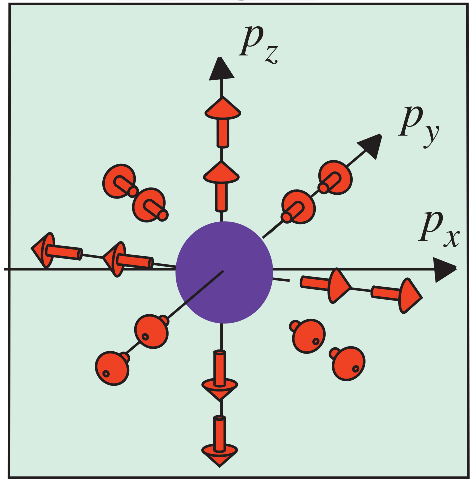
0 ν_L -1/2	-1 e_L -1/2

0 ν_R 0	-1 e_R -1

$$H = -\, c\, \sigma \cdot \mathbf{p} \qquad N_3 = -1$$

$$H = +\, c\, \sigma \cdot \mathbf{p} \qquad N_3 = +1$$

right particles



hedgehog with spines (spins) outward ($N_3 = +1$)

$$N_3 = \frac{1}{24\pi^2}\, e_{\mu\nu\lambda\gamma}\, \mathbf{tr} \int dS^\gamma\, \mathbf{G}^\mu\, \mathbf{G}^{-1}\, \mathbf{G}^\nu\, \mathbf{G}^{-1} \mathbf{G}^\lambda\, \mathbf{G}^{-1}$$

over 3D surface S in 4D momentum space

general topological invariant
in terms of Green's function
for interacting systems

Standard Model topological invariant

Topological invariant protected by symmetry

$$N_K = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \, \mathbf{tr} \int_{\text{over } S^3} dV \, \mathbf{K} \, \mathbf{G} \nabla^\mu \mathbf{G}^{-1} \, \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G} \nabla^\lambda \mathbf{G}^{-1}$$

$\mathbf{G}$ is Green's function, K is symmetry operator

$$\mathbf{GK} = +/- \, \mathbf{KG}$$

for Standard Model vacuum K is Z_2 center group

$$K = \exp 2\pi i \tau_3$$

τ_3 weak isotopic spin of SU(2)

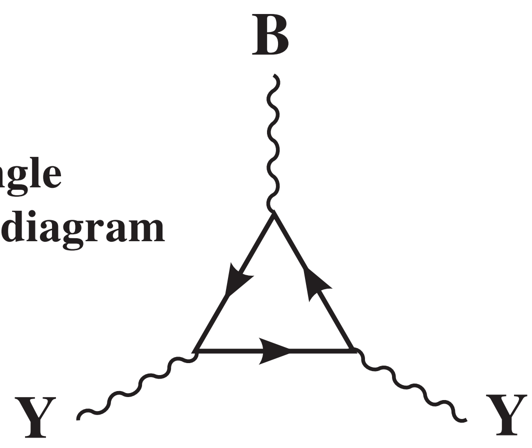
$$N_K = 16 \, n_g$$

16 massless Weyl particles in one generation are protected by combined symmetry and topology

Weyl point (Dirac monopole) as origin of chiral anomaly

example:
baryon production from vacuum by hypermagnetic field

triangle
Feynman diagram



chiral anomaly equation
(Adler, Bell, Jackiw)

$$\dot{B} = \frac{1}{4\pi^2} N_B\, \mathbf{B}_Y \cdot \mathbf{E}_Y$$

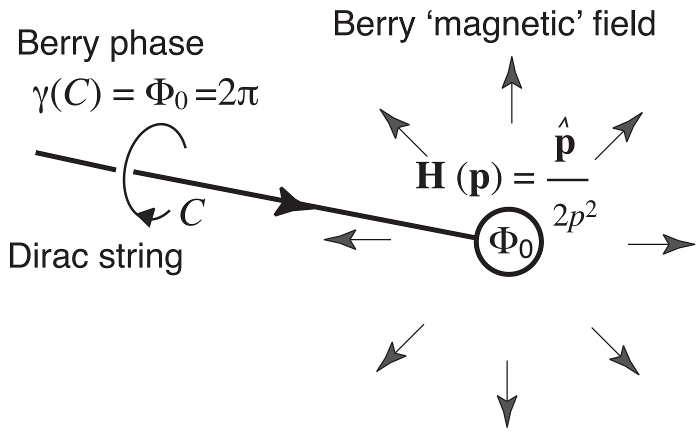
$$N_B = \frac{1}{24\pi^2} e_{\mu\nu\lambda}\, \mathbf{tr} \int_{\text{over } S^3} dV\, \mathbf{B}\mathbf{Y}^2\, \mathbf{G} \nabla^\mu \mathbf{G}^{-1}\, \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G} \nabla^\lambda \mathbf{G}^{-1}$$

matrix of baryonic charge **matrix of hypecharge**

$$\dot{B} = \frac{1}{4\pi^2}\, \mathbf{B}_Y \cdot \mathbf{E}_Y \sum_a B_a C_a Y_a^2$$

B_a -- **baryonic charge**
Y_a -- **hypercharge**
C_a = +1 **for right**
-1 **for left**

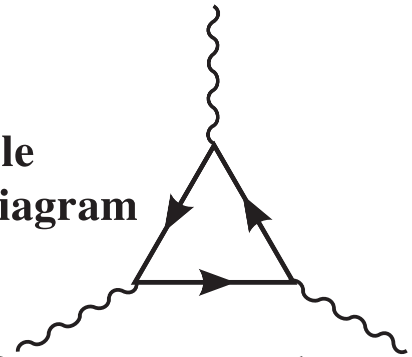
Dirac monopole in p-space



earlier indications of axial anomaly in 3He-A from Weyl point:
anomaly in orbital dynamics & paradox of angular & linear momenta
Leggett-Takagi, Ann. Phys. **110**, 353 (1978); GV & Mineev, JETP **56**, 579 (1982)

Symmetry protected invariants and chiral anomaly

prefactors of ***r***-space topological terms
are ***p***-space topological invariant



chiral anomaly from 3+1 Weyl points

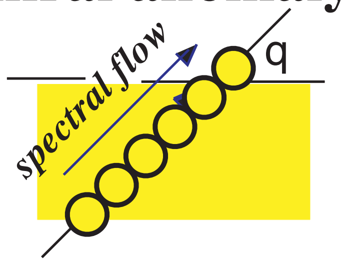
$$K_{abc} = \frac{1}{24\pi^2}\, e_{\mu\nu\lambda}\, \mathbf{tr}\int_{\text{over } S^3} dV\, \mathbf{K}_a\mathbf{K}_b\mathbf{K}_c\, \mathbf{G}\, \nabla^{\mu}\, \mathbf{G}^{-1}\, \mathbf{G}\nabla^{\nu}\, \mathbf{G}^{-1}\mathbf{G}\, \nabla^{\lambda}\, \mathbf{G}^{-1}$$

prefactor of Chern-Simons term in 2+1 gapped topological matter

$$K_{ab} = \frac{1}{24\pi^2}\, e_{\mu\nu\lambda}\, \mathbf{tr}\int_{\text{over 2+1 p-space}} dV\, \mathbf{K}_a\mathbf{K}_b\, \mathbf{G}\, \nabla^{\mu}\, \mathbf{G}^{-1}\, \mathbf{G}\nabla^{\nu}\, \mathbf{G}^{-1}\mathbf{G}\, \nabla^{\lambda}\, \mathbf{G}^{-1}$$

experimental verification of chiral anomaly equation

measurement of *Kopnin force*



momentum from vacuum of fermion zero modes

spectral flow produces

baryons from vacuum

$$\dot{\mathbf{P}} = \sum \mathbf{P}_a \dot{n}_a$$

$\mathbf{P}_a$ -- **momentum of Weyl point (fermionic charge)**

e_a -- **effective electric charge**

$$\dot{\mathbf{P}} = (1/4\pi^2)\, \mathbf{B}\cdot\mathbf{E} \sum_a \mathbf{P}_a C_a e_a^2$$

applied to ^{3}He-A

C_a = +1 **for right**

-1 **for left**

chiral anomaly equation (Adler, Bell, Jackiw)

$$\dot{B} = \sum B_a \dot{n}_a$$

B_a -- **baryonic charge**

Y_a -- **hypercharge**

$$\dot{B} = (1/4\pi^2)\, \mathbf{B}_Y\cdot\mathbf{E}_Y \sum_a B_a C_a Y_a^2$$

applied to Standard Model

C_a = +1 **for right**

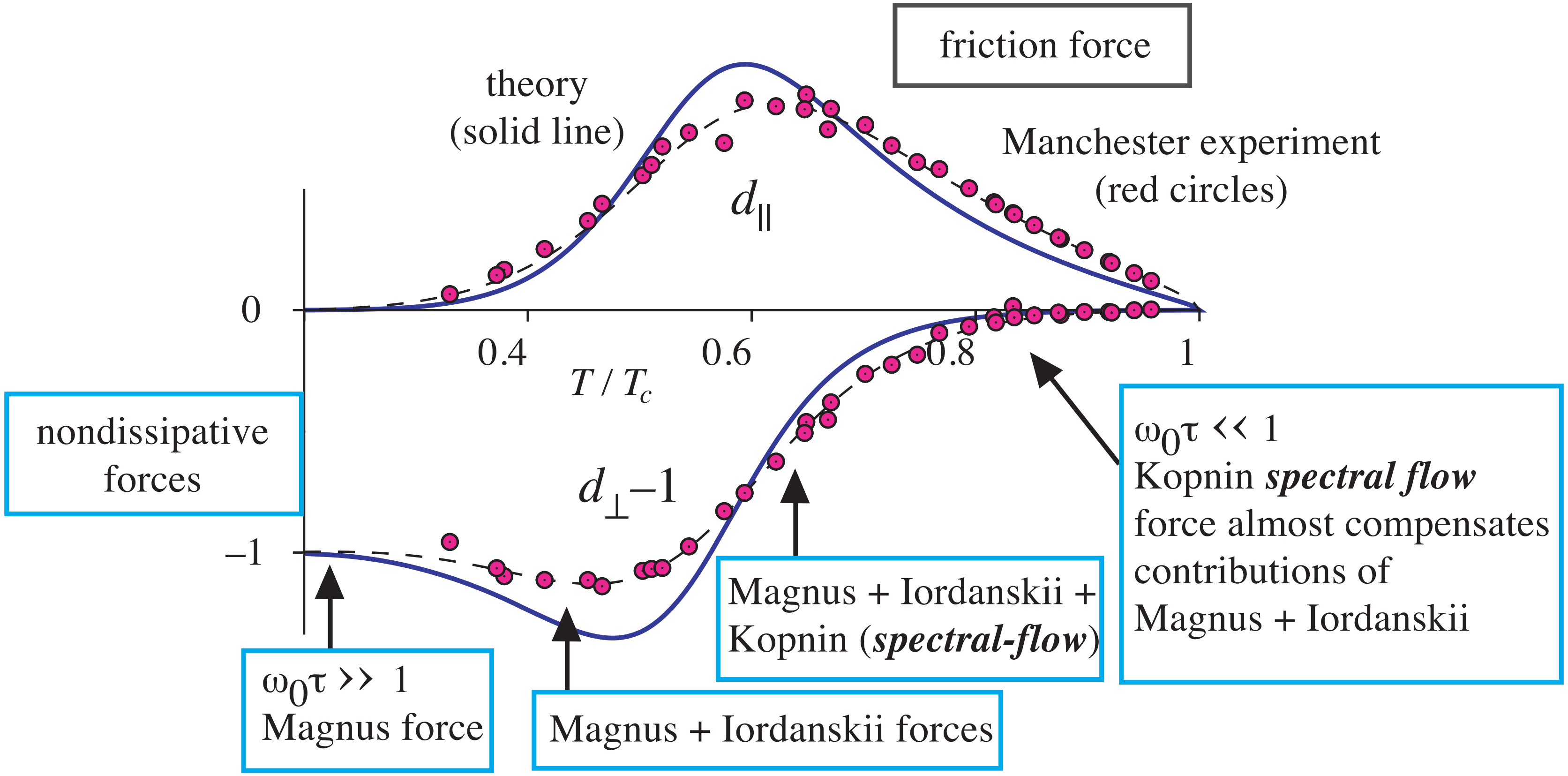
-1 **for left**

quasiparticles move from vacuum to the positive energy world, where they are scattered by quasiparticles in bulk and transfer momentum from vortex to normal component

this is the source of Kopnin spectral flow force

Bevan, Manninen, Cook, Hook, Hall, Vachaspati & GV
Momentum creation by vortices in superfluid 3He as a model of primordial baryogenesis, Nature **386**, 689 (1997)

Experimental chiral anomaly: spectral flow force on a vortex



superfluid Reynolds number

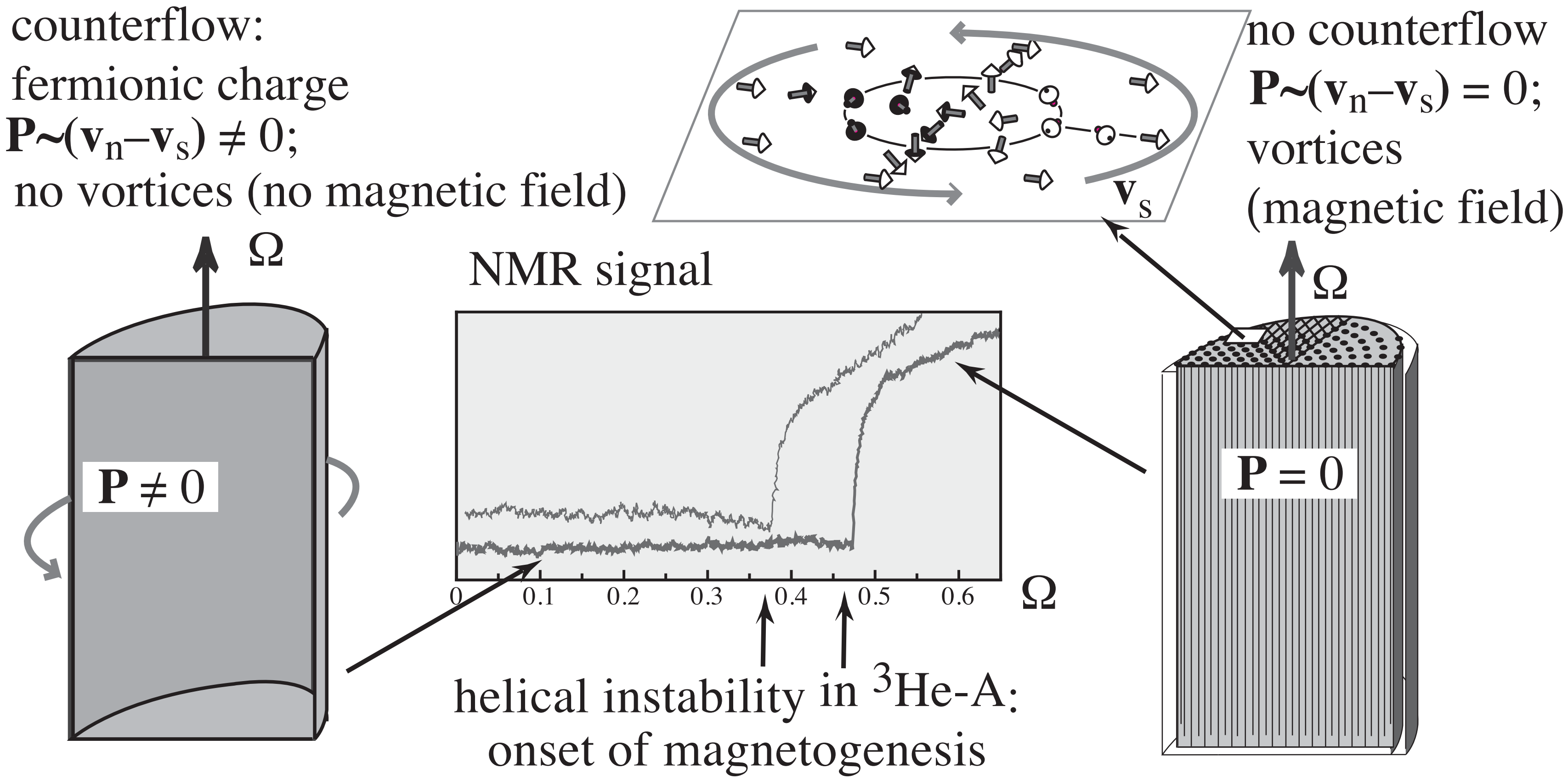
$$\boxed{\mathrm{Re}_\alpha(T) = \frac{1 - d_\perp}{d_\|} \approx \omega_0\tau}$$

$\mathrm{Re}_\alpha(T_c) = 0$ $\mathrm{Re}_\alpha(T \sim 0.6\, T_c) \sim 1$ $\mathrm{Re}_\alpha(0) = \infty$

laminar vortex flow turbulent vortex flow

Chiral magnetic effect (CME) leads to observed helical instability

$$\mathrm{L} = \mathbf{A}\cdot\mathbf{B}\,(\mu_L - \mu_R)\,/\,8\pi^2 \qquad \mathbf{J} = \mathbf{B}\,(\mu_L - \mu_R)\,/\,8\pi^2$$

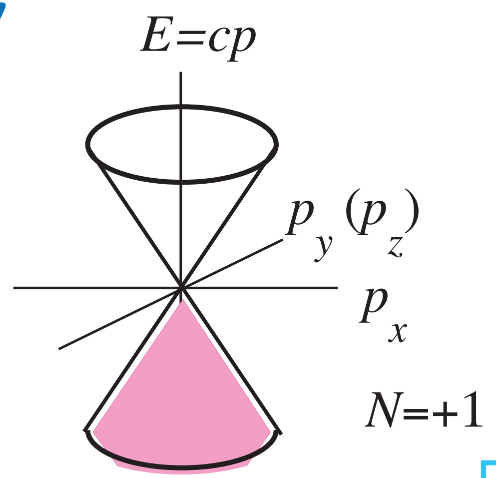


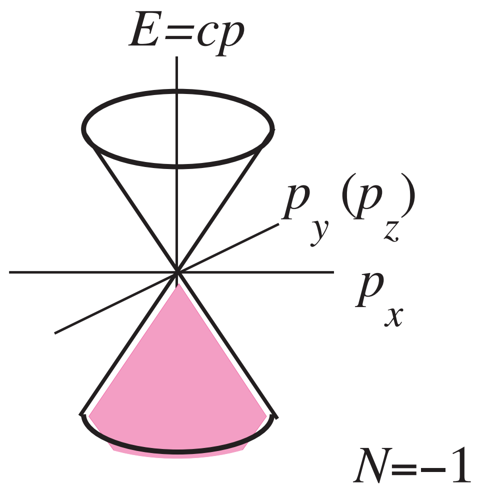
Helsinki 1996

From massless Weyl particles to massive Dirac particles

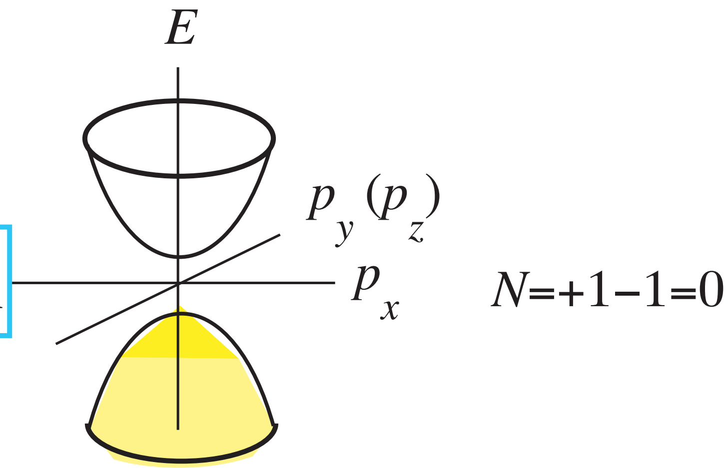
where are massive Dirac particles?

Dirac particle - composite object: mixture of left and right Weyl particles





$T_{ew} \sim 1$ TeV$\sim 10^{16}$K



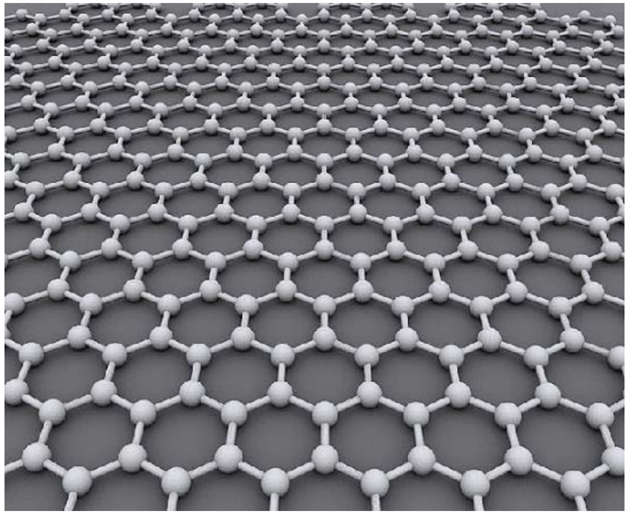
$$E^2 = c^2p^2 + m^2$$

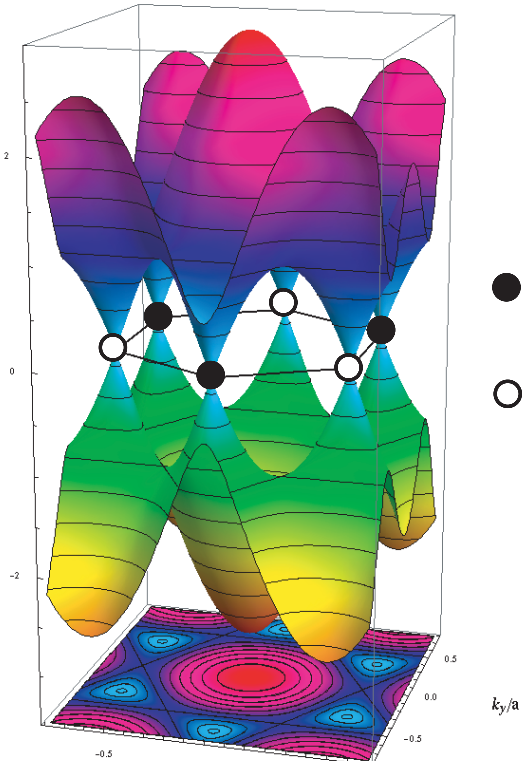
is Dirac vacuum topologically trivial ?

fully gapped vacua can be also topologically nontrivial (3He-B, topological insulators, ...)

p-space analogs of graphene

emergence of 2+1 gapless relativistic fermions in 2D graphene





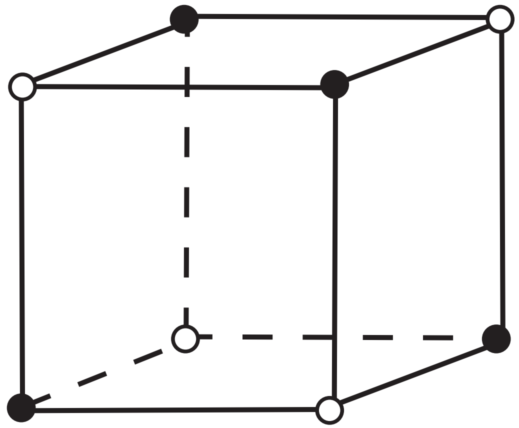
● Weyl/Dirac point $N = +1$

○ Weyl/Dirac point $N = -1$

p-space analog of 4D graphene in lattice QCD

Kaplan 1993, 2011, Creutz JHEP 04 (2008) 017

quantum vacuum as crystal



p-space analog of 3D graphene: superconductor α - phase

emergence of 2+1 relativistic fermions due to topology of graphene nodes

$$N = \frac{1}{4\pi i}\, \mathbf{tr}\, [\mathbf{K} \oint dl\, \mathbf{H}^{-1} \nabla_l \mathbf{H}]$$

K - symmetry operator,
commuting or anti-commuting with **H**

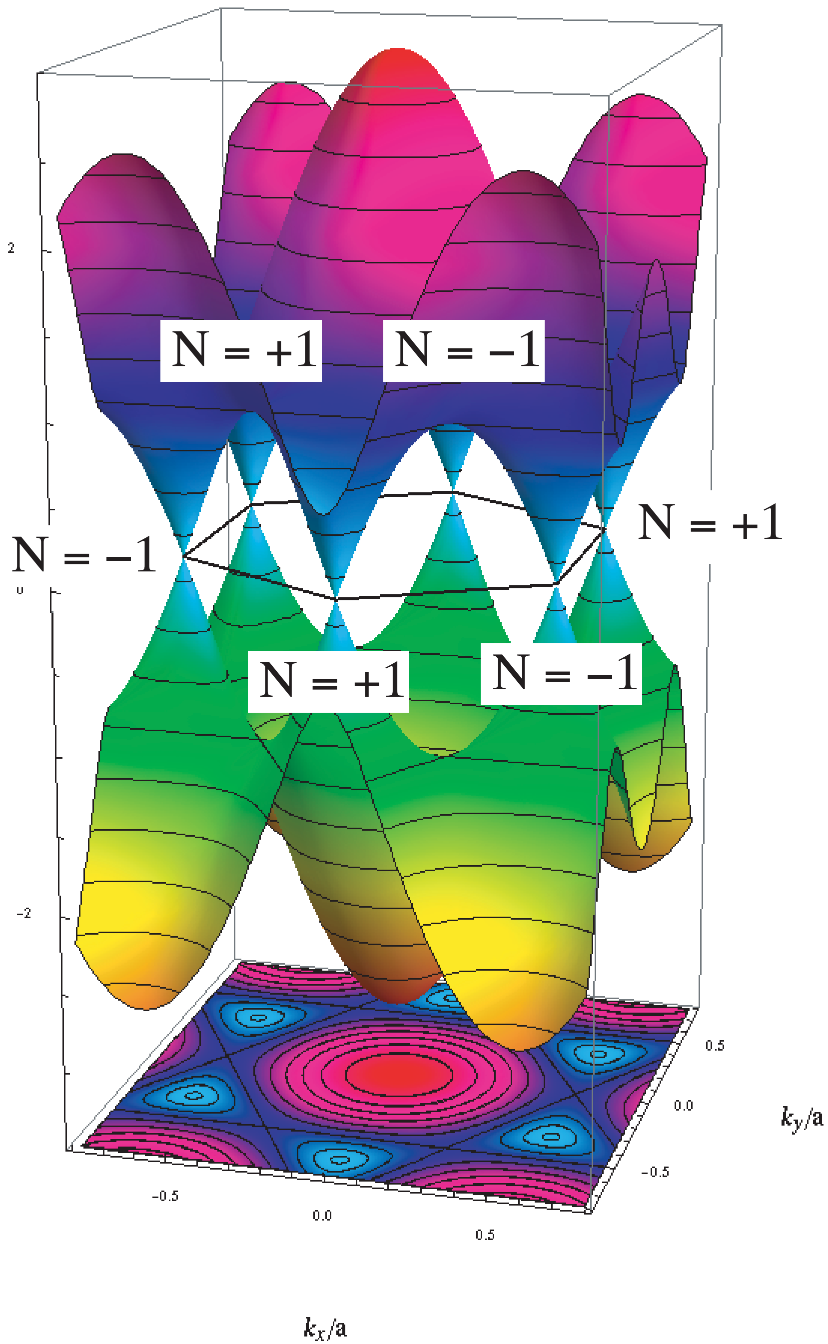
close to nodes:

$$\mathbf{H}_{\mathrm{N}\,=\,+1} = \tau_x p_x + \tau_y p_y$$

$$\mathbf{H}_{\mathrm{N}\,=\,-1} = \tau_x p_x - \tau_y p_y$$

$$\mathbf{K} = \tau_z$$

E/γ_0



for real interacting systems
the Hamiltonian $\mathbf{H}(\mathbf{p})$ is substituted
by inverse Green's function at zero frequency
$\mathbf{G}^{-1}(\omega{=}0,\mathbf{p})$

SU(2) gauge fields emerging near Weyl & Dirac points

Atiyah-Bott-Shapiro construction:
linear expansion of Hamiltonian near the nodes in terms of Dirac Γ-matrices

$$\boxed{H = e_a{}^k \, \Gamma^a \cdot (p_k - eA_k - e\tau \cdot \mathbf{W}_k)}$$

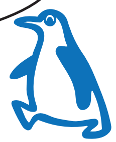
effective tetrad: emergent gravity

effective electromagnetic field

effective isotopic spin

effective $SU(2)$ gauge field

$SU(2)$ gauge field is collective mode of vacuum with Weyl point



isotopic spin comes from spin or from band indices

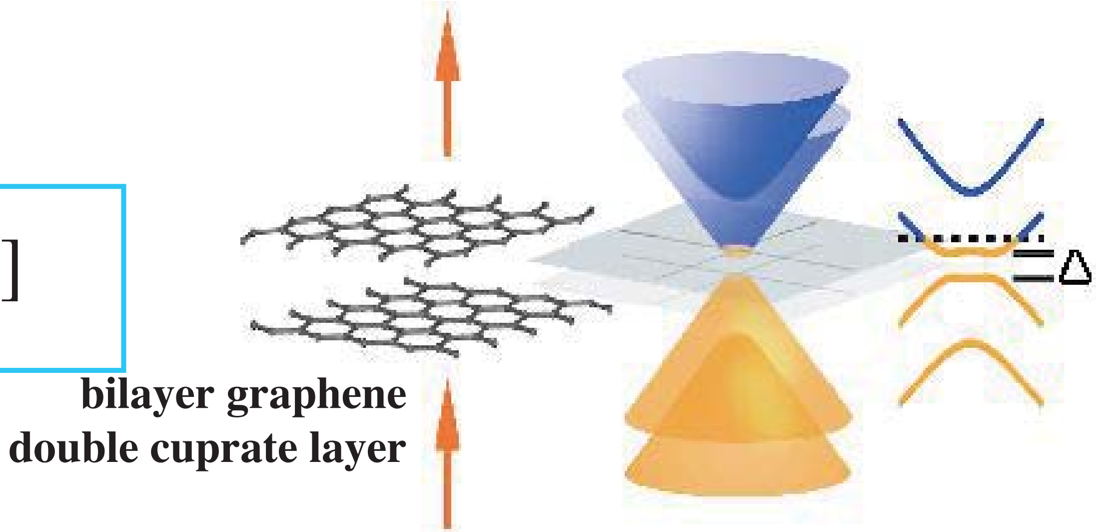
SU(2) field near Dirac points in graphene

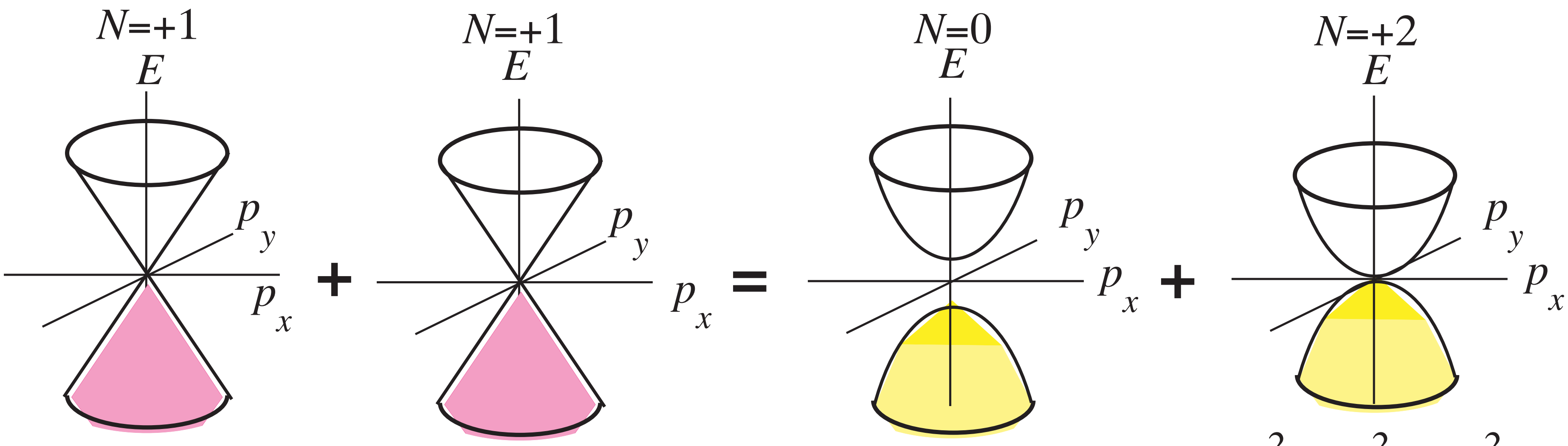
$$\mathbf{H}_{\mathrm{N}\,=\,+1} = \tau_x(p_x - A_x - \sigma \cdot \mathbf{W}_x) + \tau_y(p_y - A_y - \sigma \cdot \mathbf{W}_y)$$

Summation of topological charges in action

exotic fermions:
massless fermions with quardatic,
cubic & quartic dispersion
semi-Dirac fermions

$$\boxed{N = \frac{1}{4\pi i}\,\mathbf{tr}\,[\mathbf{K}\oint dl\,\mathbf{H}^{-1}\nabla_l\,\mathbf{H}]}$$





$E=cp$ $E=cp$ $E^2 = 2c^2p^2 + 4m^2$ $E^2 = (p^2/\,2m)^2$

relativistic massless fermions

Dirac fermion

nonrelativistic Dirac fermion
with quadratic dispersion

re-entrant violation of Lorentz invariance & neutrino physics Klinkhamer & GV arXiv:1109.6624

nonlinear Dirac fermions

N=1: Dirac fermions with linear dispersion

$$H = \begin{pmatrix} 0 & p_x + ip_y \\ p_x - ip_y & 0 \end{pmatrix} = \sigma_x p_x + \sigma_y p_y$$

N=2: Dirac fermions with quadratic dispersion

$$H = \begin{pmatrix} 0 & (p_x + ip_y)^2 \\ (p_x - ip_y)^2 & 0 \end{pmatrix}$$

N=3: Dirac fermions with cubic dispersion

$$H = \begin{pmatrix} 0 & (p_x + ip_y)^3 \\ (p_x - ip_y)^3 & 0 \end{pmatrix}$$

Dirac fermions with nonlinear dispersion

$$H = \begin{pmatrix} 0 & (p_x + ip_y)^N \\ (p_x - ip_y)^N & 0 \end{pmatrix}$$

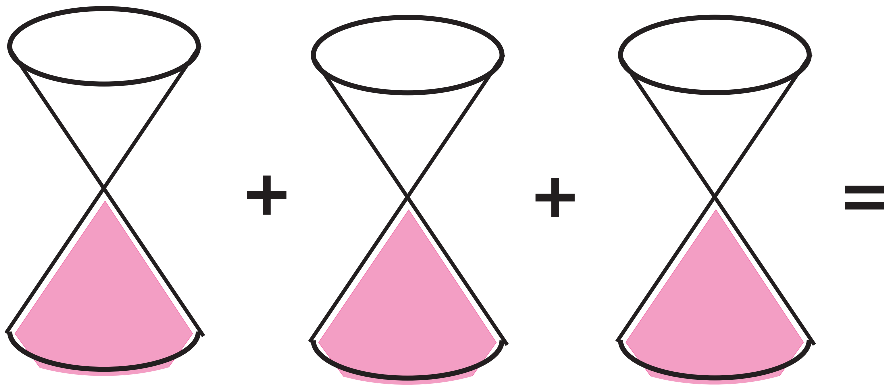
multiple Fermi point

T. Heikkila & GV arXiv:1010.0393

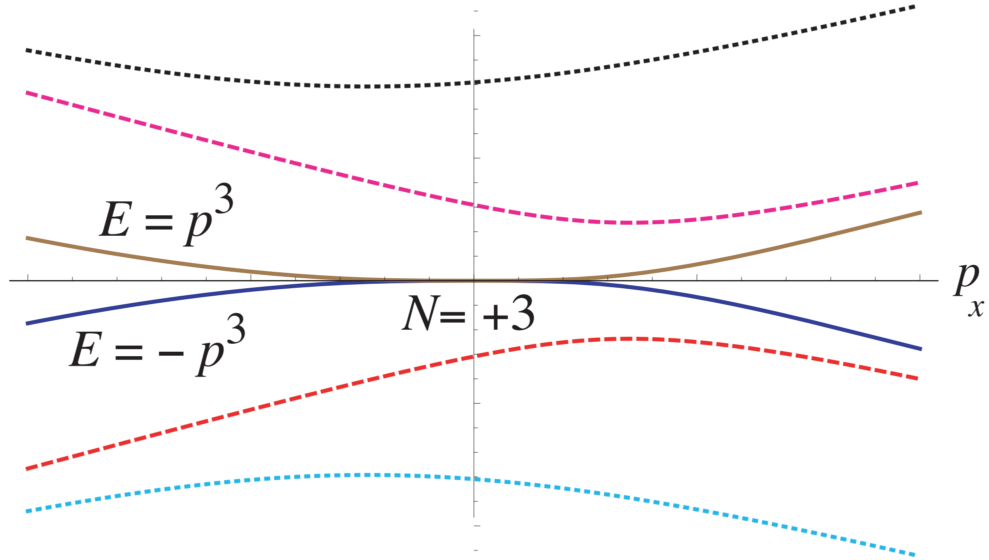
cubic spectrum in trilayer graphene

$$N = 1 + 1 + 1 = 3$$

$N=+1$ $N=+1$ $N=+1$



$E=cp$ $E=cp$ $E=cp$



multilayered graphene

$$N = 1 + 1 + 1 + \ldots$$

spectrum in the outer layers

$$E = p^N$$

$$E = -p^N$$

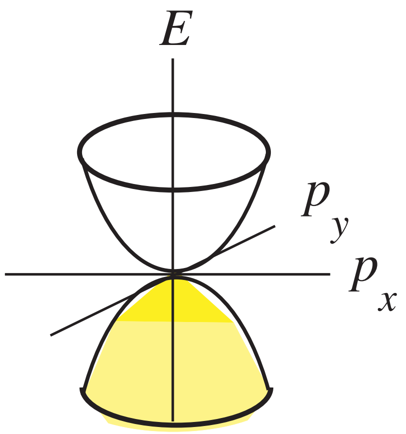
what kind of induced gravity emerges near degenerate Fermi point?

route to topological flat band on the surface of 3D material

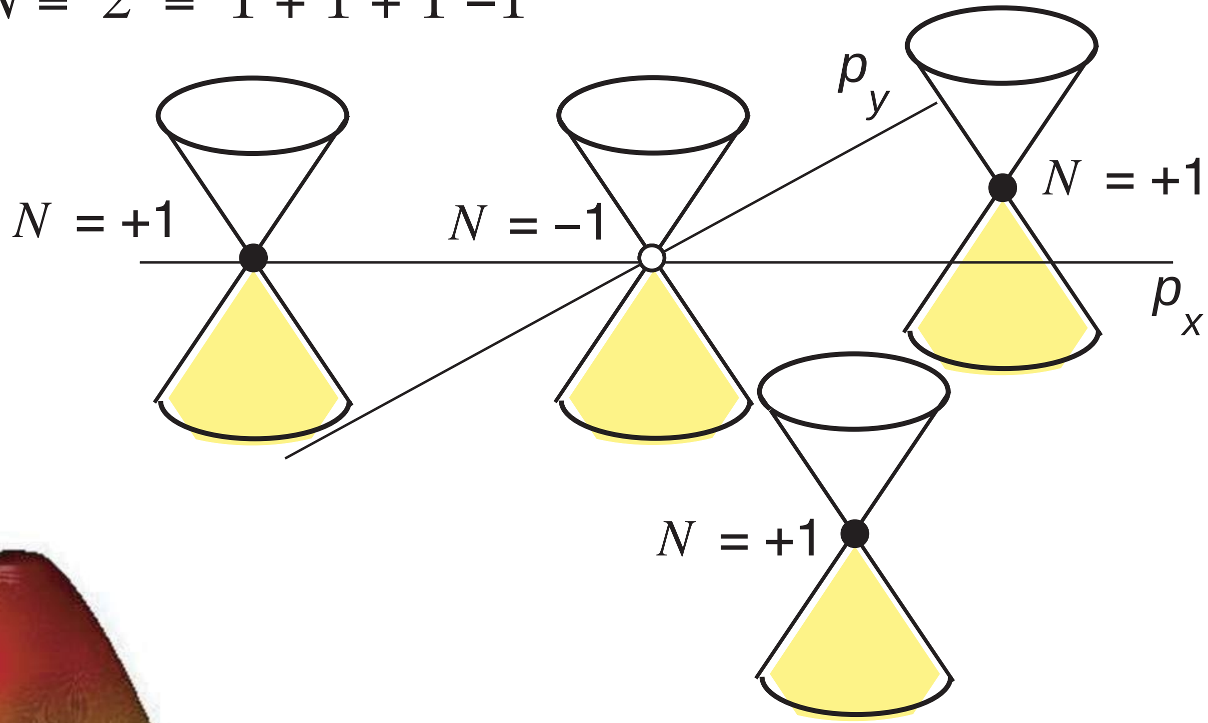
Splitting of Dirac and Weyl points

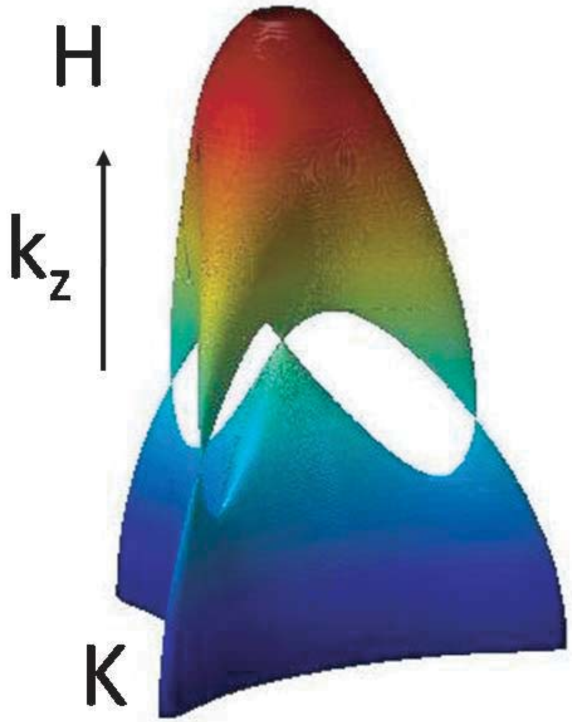
Splitting of quadratic point or trigonal warping

$$N = 2 = 1 + 1 + 1 - 1$$



$N = +2$





Horava anisotropic scaling gravity

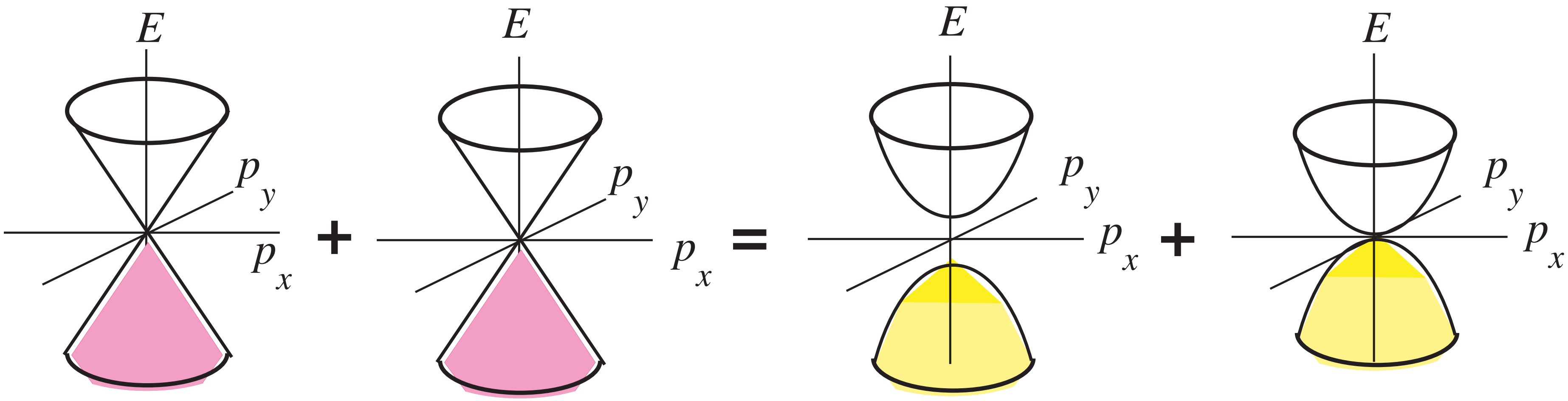
anisotropic z=3 scaling: $x = b\,x'\,,\ \ t = b^3 t'$

$$S_{grav} = \int \underset{b^3}{d^3x}\ \underset{b^3}{dt}\ \underset{b^{-6}}{R^3}$$

Horava anisotropic z=2 scaling in bilayered graphene

$$\boxed{N = \frac{1}{4\pi i}\,\mathbf{tr}\left[\mathbf{K}\oint dl\,\mathbf{H}^{-1}\nabla_l\,\mathbf{H}\right]}$$

N=+1 | N=+1 | N=0 | N=+2



$E=cp$ | $E=cp$ | $E^2 = 2c^2p^2 + 4m^2$ | $E^2 = (p^2/\,2m)^2$

2+1 massless Dirac fermions | massive fermions | massless Dirac fermions with quadratic dispersion

Fermions in 2+1 bylayer graphene

single layer

$$H = \begin{pmatrix} 0 & p_x + ip_y \\ p_x - ip_y & 0 \end{pmatrix} = \sigma_x p_x + \sigma_y p_y = \begin{array}{c} \text{zweibein} \\ \begin{pmatrix} 0 & (\mathbf{e}_1 + i\,\mathbf{e}_2)\cdot(\mathbf{p} - \mathbf{A}) \\ (\mathbf{e}_1 - i\,\mathbf{e}_2)\cdot(\mathbf{p} - \mathbf{A}) & 0 \end{pmatrix} \end{array}$$

double layer

$$H = \begin{pmatrix} 0 & (p_x + ip_y)^2 \\ (p_x - ip_y)^2 & 0 \end{pmatrix} = \begin{array}{c} \text{zweibein} \\ \begin{pmatrix} 0 & [(\mathbf{e}_1 + i\,\mathbf{e}_2)\cdot(\mathbf{p} - \mathbf{A})]^2 \\ [(\mathbf{e}_1 - i\,\mathbf{e}_2)\cdot(\mathbf{p} - \mathbf{A})]^2 & 0 \end{pmatrix} \end{array}$$

anisotropic scaling: $x = b\,x'$, $t = b^2\ t'$

Horava-Lifshitz gravity:
Horava, Quantum gravity at a Lifshitz point
PRD **79**, 084008 (2009)

Heisenberg-Euler action in massive QED

non-linear corrections to Maxwell equations due to vacuum polarization

$$S_{\mathrm{HE}} = \int d^3x\, dt\, [(B^2 - E^2)^2 /M^4 + (\mathbf{B}\cdot\mathbf{E})^2 /M^4]$$

M is rest energy of electrons $\quad B,E << M^2$

What is the Heisenberg-Euler action for relativistic massless QED emerging in condensed matter ?

What is the Heisenberg-Euler action for massless QED with exotic Dirac fermions with quadratic and cubic spectrum ?

relation to Horava quantum gravity with anisotropic scaling

isotropic QED emerging in 3He-A and single layer graphene

isotropic scaling: $x = b\,x'$, $t = b\,t'$, $B = b^{-2}\,B'$, $E = b^{-2}\,E'$, $S = S'$

3+1 isotropic QED & emerging in Weyl superfluids & semimetals

$$S_{\mathrm{QED}} = \int \underset{b^3}{d^3x}\; \underset{b}{dt}\; \underset{b^{-4}}{(B^2 - E^2)}\; \underset{b^{-4}}{\ln}\; 1/(B^2 - E^2)$$

imaginary action (Schwinger pair production) at $B^2 < E^2$

2+1 isotropic QED emerging in single layer graphene

$$S_{\mathrm{QED}} = \int \underset{b^2}{d^2x}\; \underset{b}{dt}\; \underset{b^{-3}}{(B^2 - E^2)^{3/4}}$$

imaginary action (Schwinger pair production) at $B^2 < E^2$

2+1 anisotropic QED emerging in bylayer graphene

$$H = \begin{pmatrix} 0 & (p_x + ip_y)^2/2m \\ (p_x - ip_y)^2/2m & 0 \end{pmatrix} = \begin{pmatrix} 0 & [(\mathbf{e}_1 + i\,\mathbf{e}_2)(\mathbf{p} - \mathbf{A})]^2/2m \\ [(\mathbf{e}_1 - i\,\mathbf{e}_2)(\mathbf{p} - \mathbf{A})]^2/2m & 0 \end{pmatrix}$$

Heisenberg-Euler action

anisotropic scaling: $x = b\,x'$, $t = b^2\,t'$, $B = b^{-2}\,B'$, $E = b^{-3}\,E'$, $S = S'$

$$S = 1/m \int \underset{b^2}{d^2x}\ \underset{b^2}{dt}\ \underset{b^{-4}}{B^2}\ g(\mu)$$

$g(\mu)$ – dimensionless fuction of dimensionless parameter $\mu = m^2 \underset{b^{-6}}{E^2} / \underset{b^{6}}{B^3}$

magnetic field asymptote

$$S_B = 1/m \int d^2x\ dt\ B^2\ ln\ 1/B^2$$

electric field asymptote

$$S_E = 1/m \int \underset{b^2}{d^2x}\ \underset{b^2}{dt}\ (-\,m^2 \underset{b^{-4}}{E^2})^{2/3}$$

Schwinger pair production $\sim E^{4/3}$

pair production mainly occurs at $\mu > 1$ **i.e at** $E^2 > B^3/m^2$

Schwinger pair production

$$\mathrm{Im}\, S = \pi^{-2} B^2 \mu \int_0^1 dx \exp(-\, \mu\, f(x)\,)$$

$$\mu = m^2 E^2 / B^3 \qquad\qquad f(x) = x - (1+x)/2 \ln(1+x) - (1-x)/2 \ln(1-x)$$

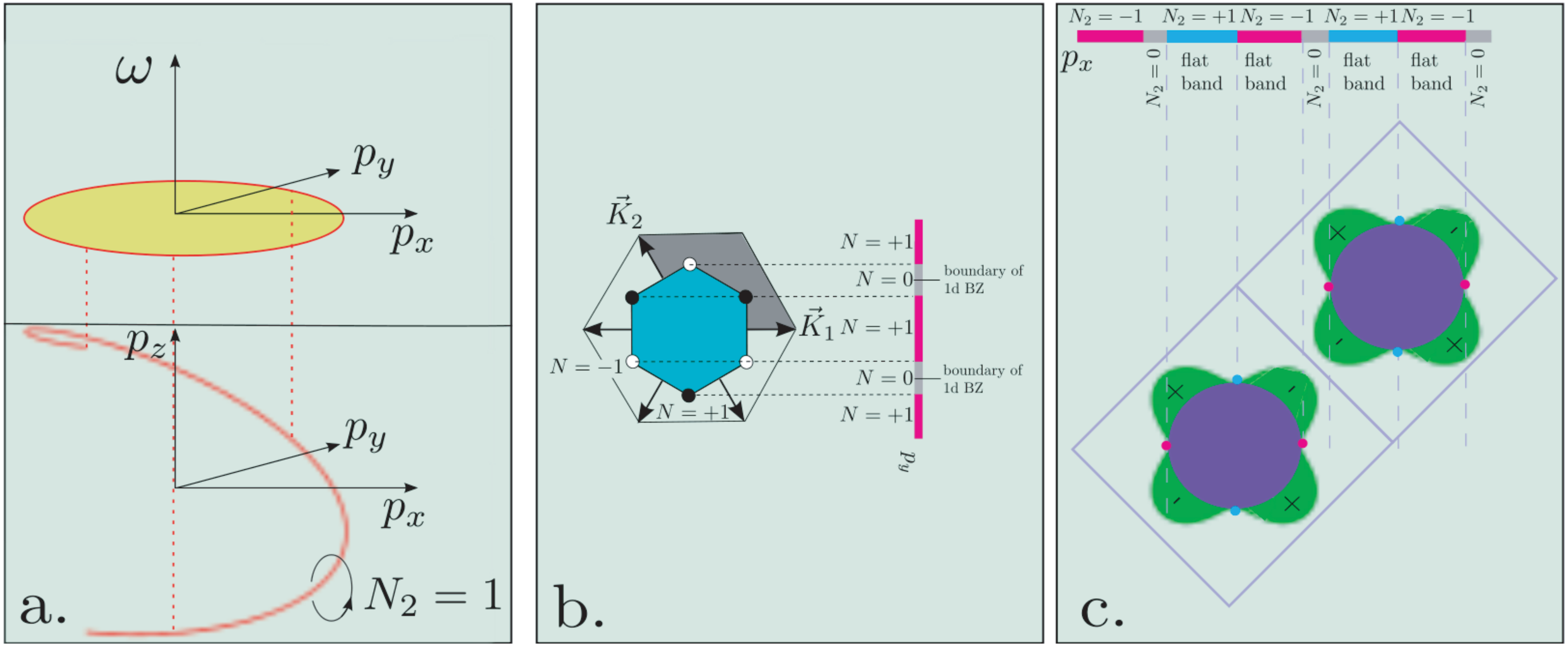
pair production mainly occurs at $\mu > 1$ **i.e at** $E^2 > B^3/m^2$

at $\mu >> 1 \quad f(x) = x^3/6$

Schwinger pair production $\quad \mathrm{Im}\, S \sim E^{4/3} m^{1/3}$

M.I. Katsnelson & G.E. Volovik,
Quantum electrodynamics with anisotropic scaling:
Heisenberg-Euler action and Schwinger pair production in the bilayer graphene,
Pis'ma ZhETF 95, 457 (2012); arXiv:1203.1578

4. Flat bands & Fermi arcs in topological matter



nodal spiral in multilayered graphene
generates flat band with zero energy
in the top and bottom layers

Hekilla, Kopnin, GV

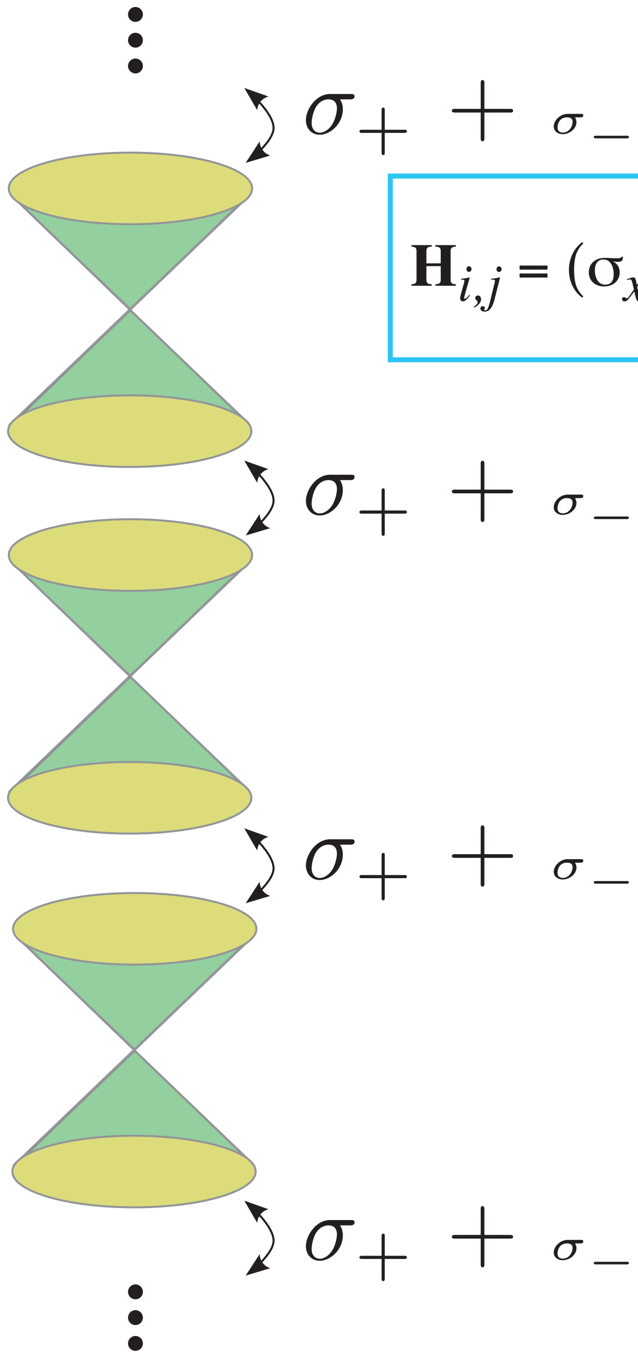
nodes in graphene
generate flat band on zigzag edge

nodal lines
in cuprate superconductors
generate flat band on side surface

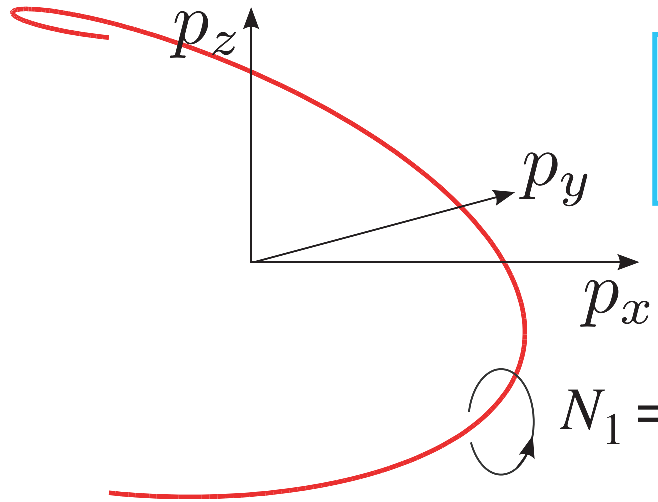
Shinsei Ryu

approximate flat band on side surface
of graphite

formation of nodal spiral in bulk (together with flat band on the surface) by stacking of graphene layers



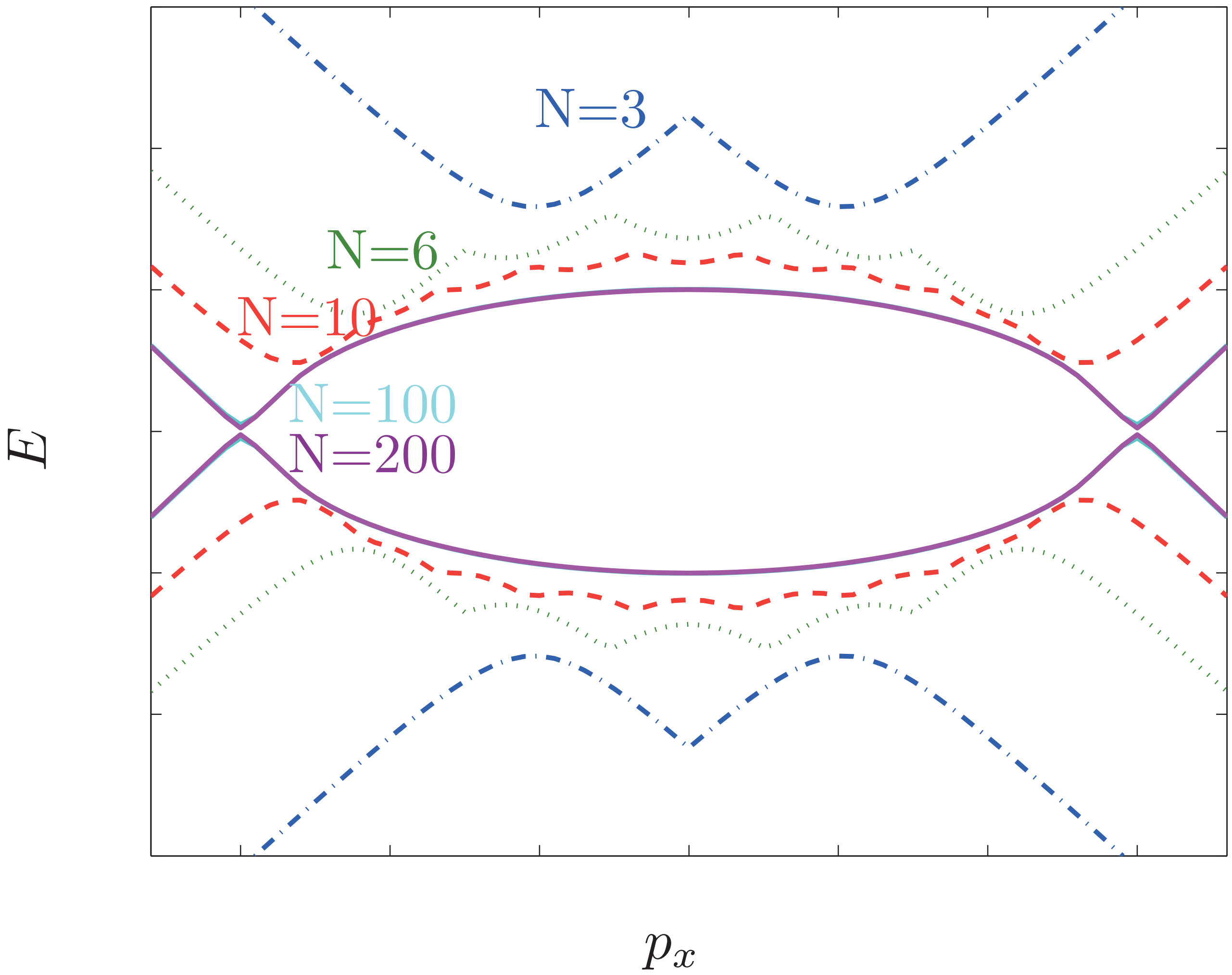
$$\mathbf{H}_{i,j} = (\sigma_x p_x + \sigma_y p_y)\delta_{i,j} + (\sigma_+ t_+ + \sigma_- t_-)\delta_{i,j+1}$$



$$t_+ > t_-$$

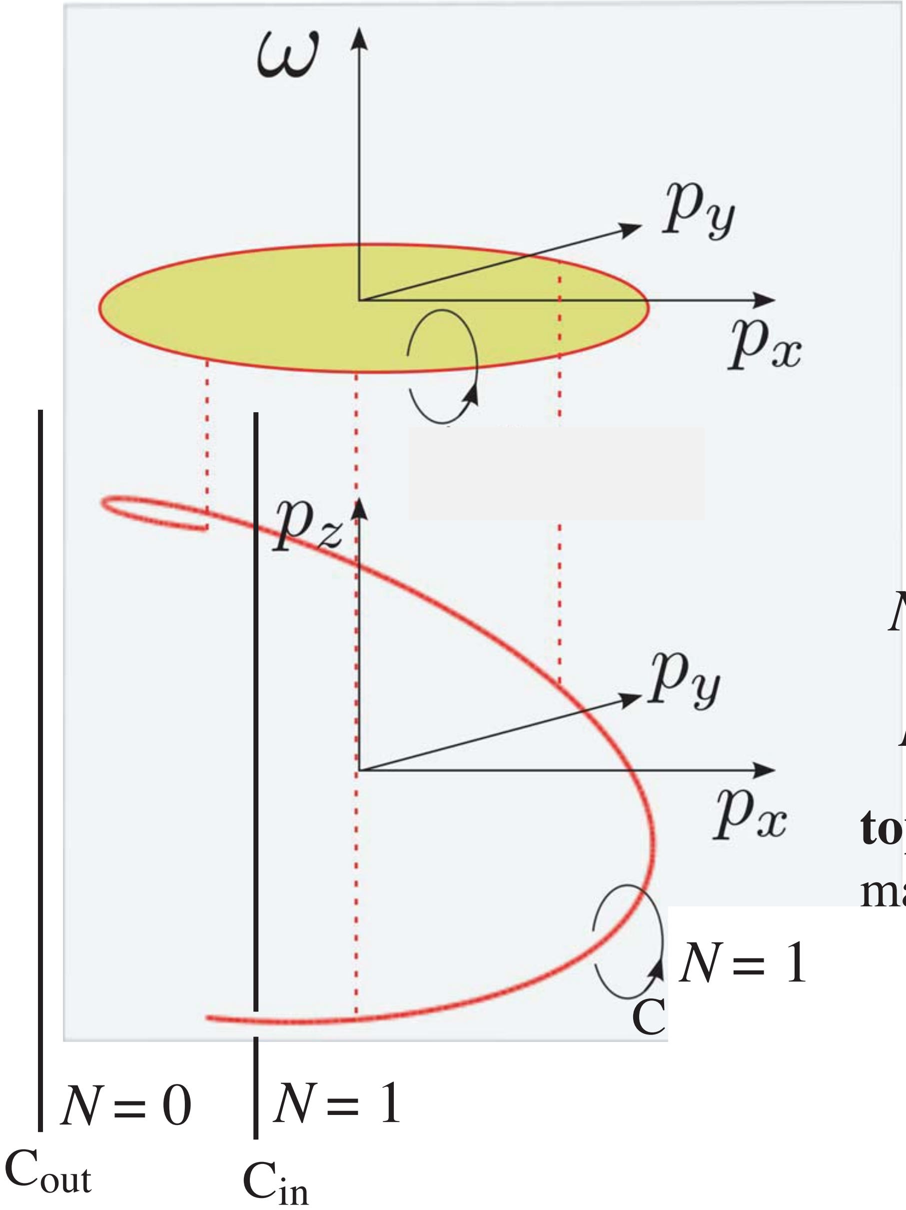
$$N_1 = \frac{1}{4\pi i}\,\mathbf{tr}\left[\mathbf{K}\oint dl\,\mathbf{H}^{-1}\nabla_l\,\mathbf{H}\right]$$

Emergence of nodal line from gapped branches by stacking graphene layers



example of topological bulk-surface correspondense:
Nodal spiral generates flat band on the surface

projection of spiral on the surface determines boundary of flat band



$$\boxed{N = \frac{1}{4\pi i}\,\mathbf{tr}\,[\mathbf{K}\oint_C dl\,\mathbf{H}^{-1}\nabla_l\,\mathbf{H}]}$$

at each (p_x, p_y) except the boundary of circle one has 1D fully gapped state (insulator)

$N_{out}(p_x, p_y) = 0$ trivial 1D insulator

$N_{in}(p_x, p_y) = 1$ topological 1D insulator

topological insulator has 1D gapless **edge state**
manifold (p_x, p_y) of edge states forms **flat band**

Nodal spiral generates flat band on the surface

projection of nodal spiral on the surface determines boundary of flat band

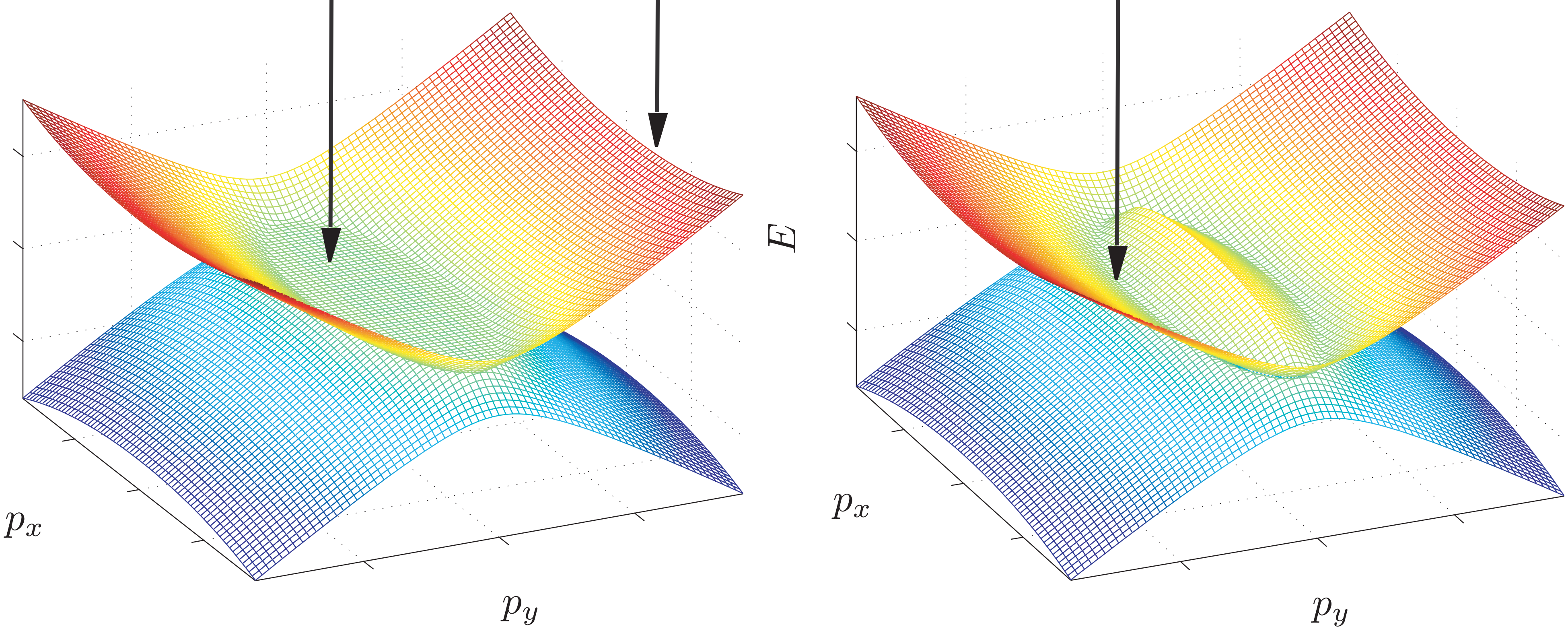
lowest energy states:

bulk states

surface states form the flat band

energy spectrum in bulk
(projection to p_x , p_y plane)

nodal line



Modified nodal spiral in rhombohedral graphite: spiral of Fermi surfaces (McClure 1969)

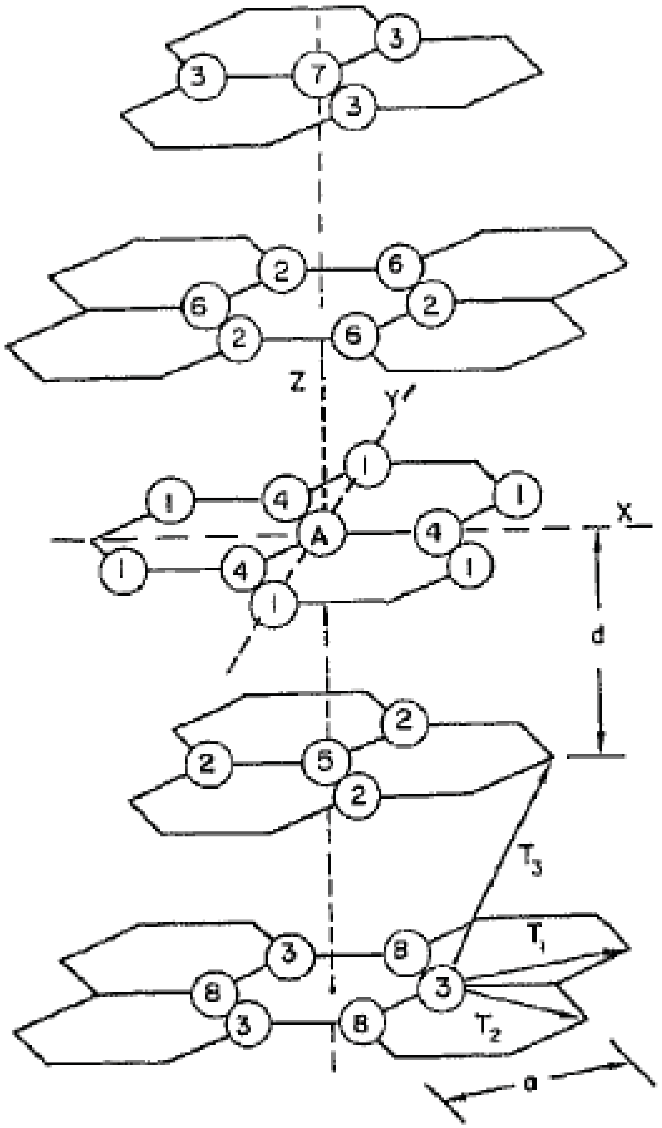
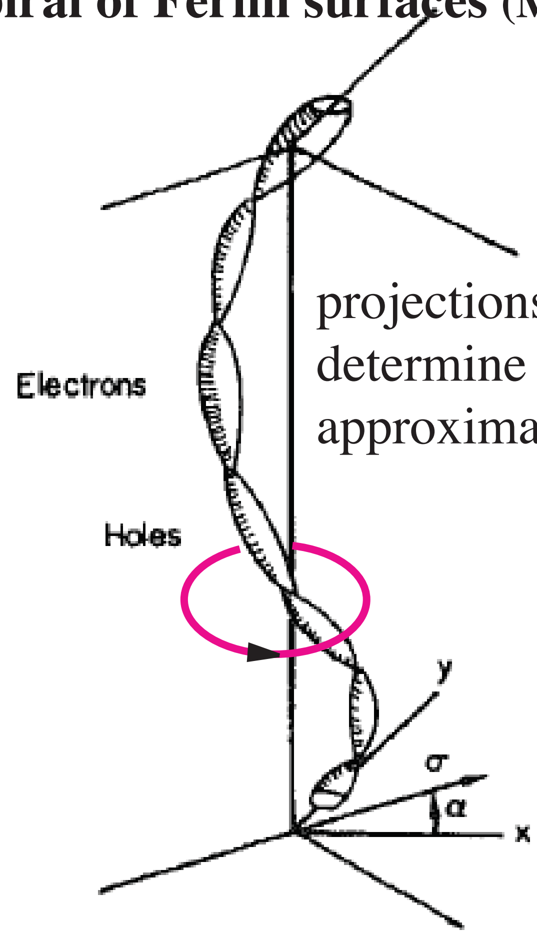


Fig. 1. The crystal lattice of rhombohedral graphite. The numbering of the groups of neighbors of the central *A* atom is explained in the text.

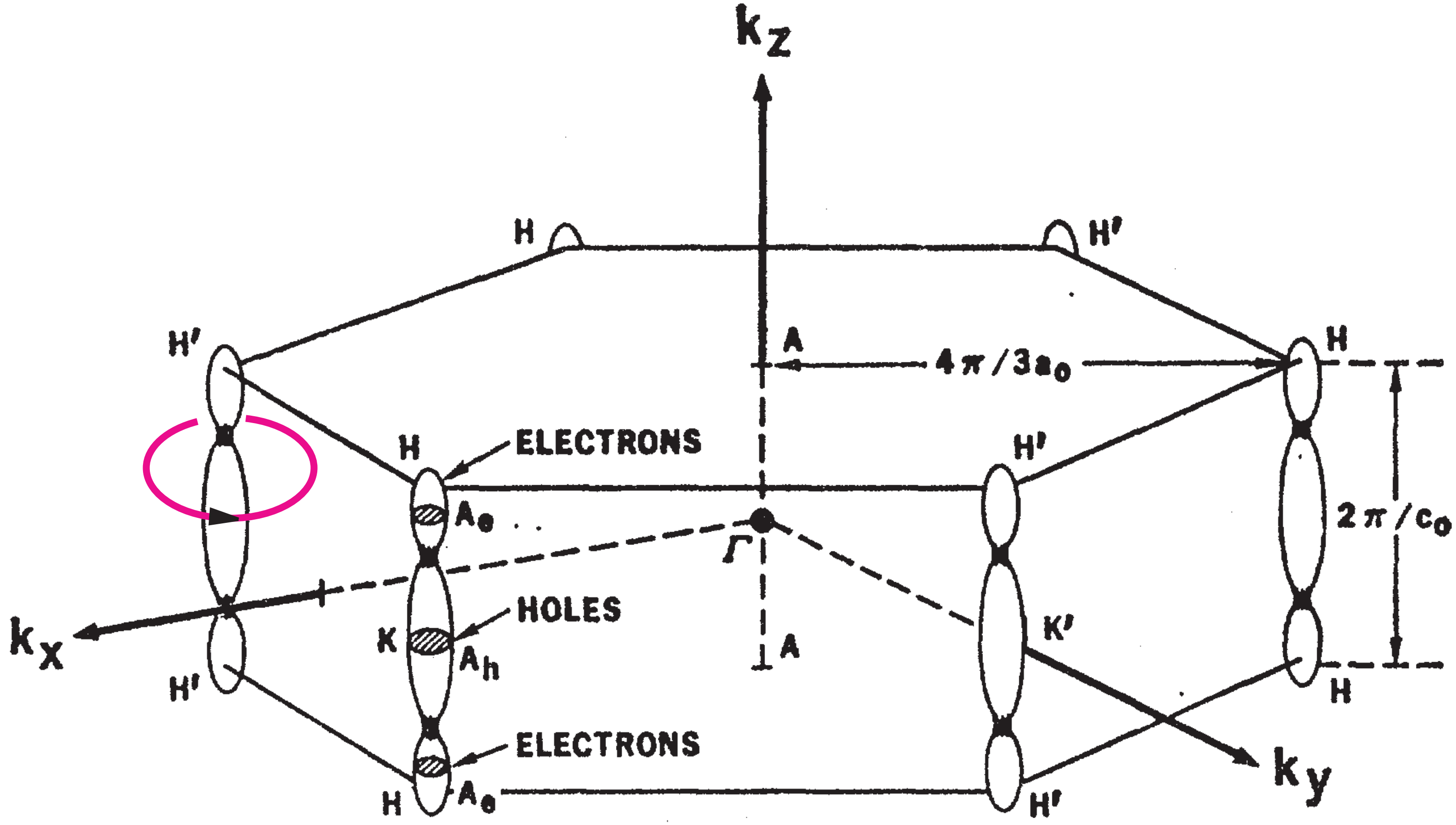


projections on surfaces determine approximate flat band

Fig. 2. The Fermi surface of rhombohedral graphite. The surface is centered on one of the six vertical zone edges. The widths of the surfaces have been magnified by more than an order of magnitude.

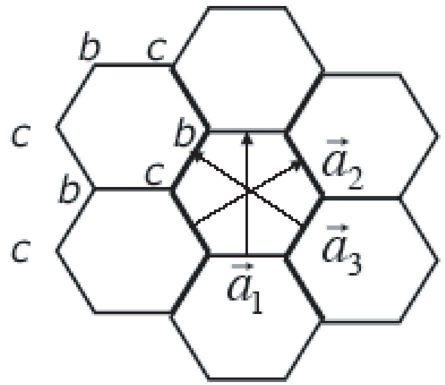
for conventional graphite: approximate flat band on the lateral surface

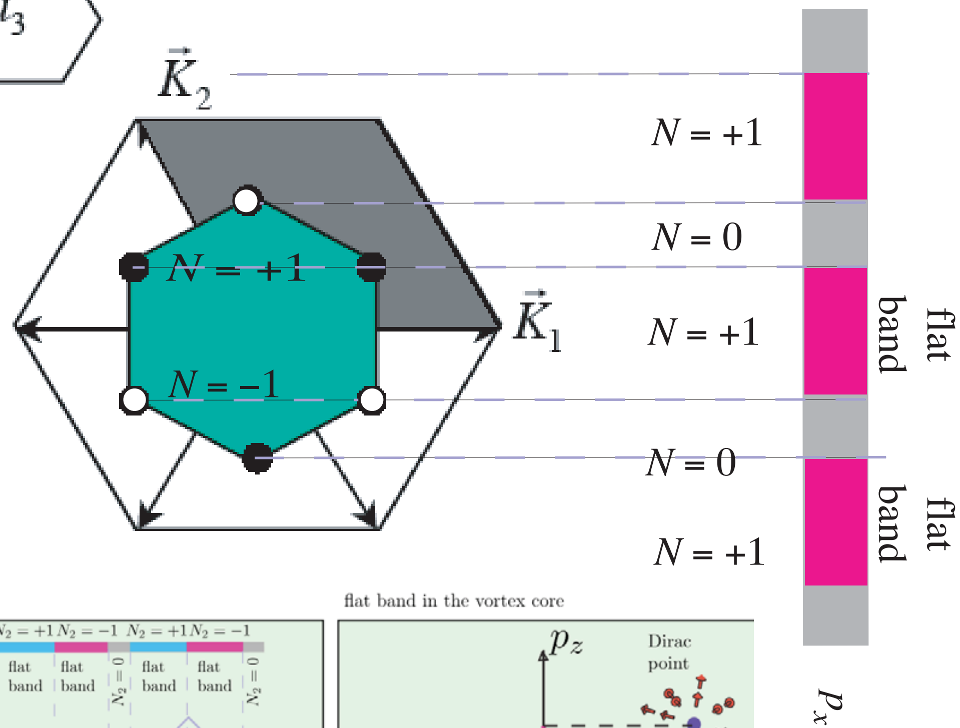
Nodal lines in graphite tranformed to chain of electron and hole FS



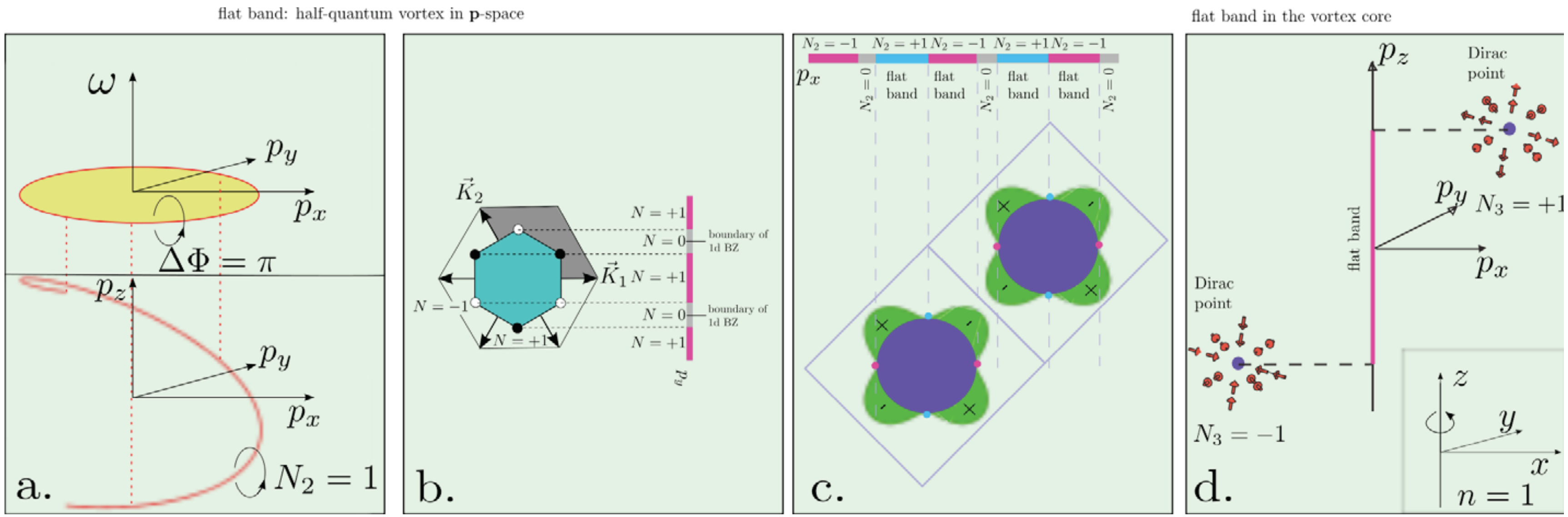
for conventional graphite:
approximate flat band
on the lateral surface

Flat band on the graphene edge

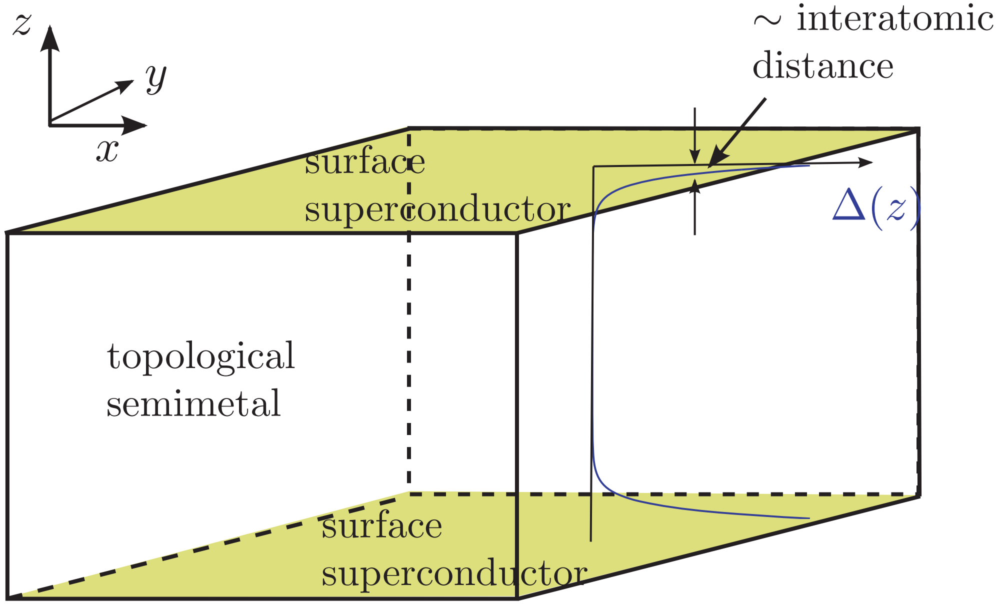


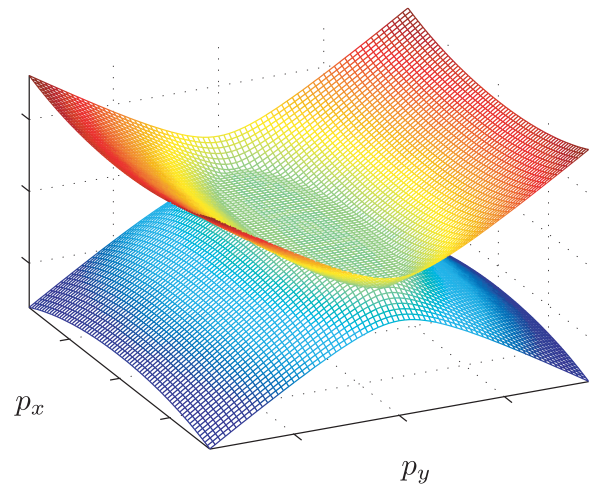


$$N = \frac{1}{4\pi i}\,\mathbf{tr}\left[\mathbf{K}\oint dl\,\mathbf{H}^{-1}\nabla_l\,\mathbf{H}\right]$$



Surface superconductivity in topological semimetals: route to room temperature superconductivity





Extremely high DoS of flat band gives high transition temperature:

normal superconductors: exponentially suppressed transition temperature

$$\boxed{T_c = T_F \exp\left(-1/g\nu\right)}$$

interaction ***DOS***

$$\boxed{1 = g \int \frac{d^2p}{2h^2} \frac{1}{E(p)}}$$

$$\boxed{\mathrm{DoS} = \nu(\varepsilon) \sim \varepsilon^{2/N-1}}$$

N is number of layers

flat band superconductivity: linear dependence of T_c on coupling g

$$\boxed{T_c \sim g S_{\mathrm{FB}}}$$

coupling ***flat band area***

$N = 4$: $\nu(\varepsilon) \sim \varepsilon^{-1/2}$ Kopaev (1970); Kopaev-Rusinov (1987)

evidence of room-temperature superconductivity?

2000: Kopelevich Y, Esquinazi P, Torres J H S and Moehlecke S
J. of Low Temp. Phys. **11**, 691–702

2002: Kempa H, Esquinazi P and Kopelevich Y
Phys. Rev. B **65** 241101

2007: Kopelevich Y and Esquinazi P
J. of Low Temp. Phys. **146** 629

2012: Scheike T, Böhlmann W, Esquinazi P, Barzola--Quiquia J, Ballestar A and Setzer A.
Advanced Materials **24** 5826

2013: Scheike T, Esquinazi P, Setzer A and Böhlmann W , arXiv:1301.4395

Ballestar A, Barzola-Quiquia J, Scheike T and Esquinazi P
New J. Phys. **15** 023024

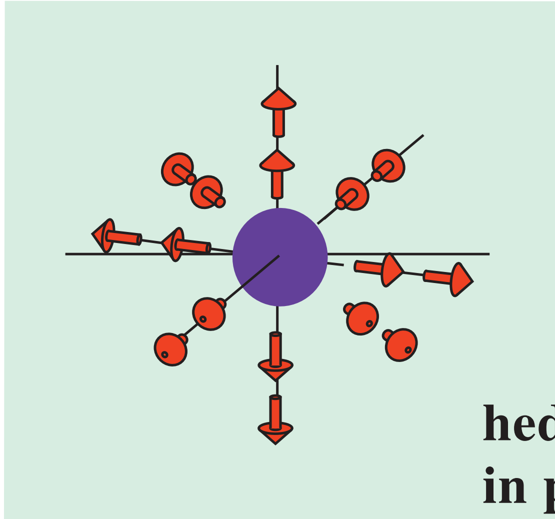
G. Larkins, Y. Vlasov, K. Holland, arXiv:1307.0581

From Weyl point to quantum Hall topological insulators

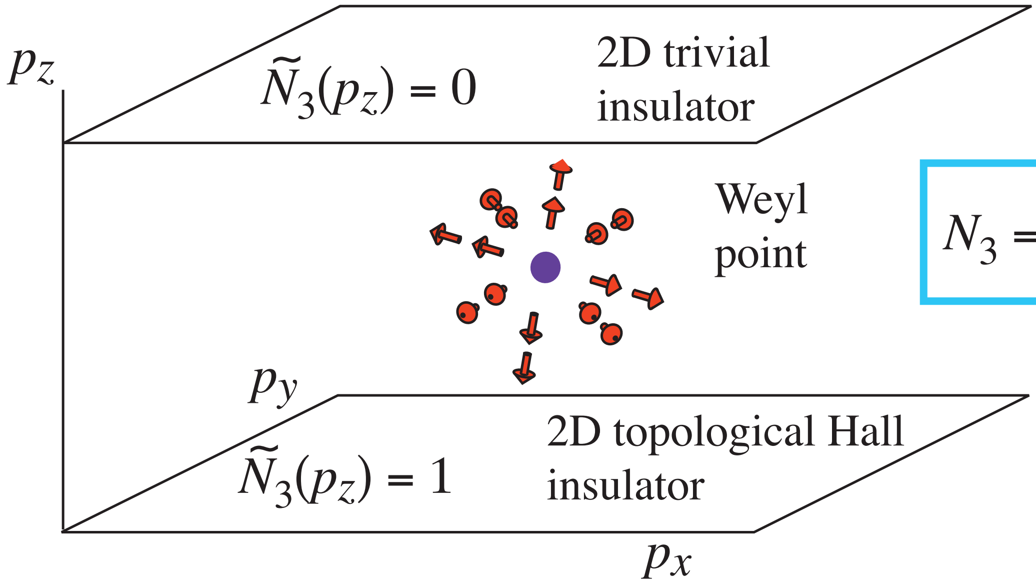
$$N_3 = \frac{1}{8\pi}\, e_{ijk} \int dS^k \; \hat{\mathbf{g}} \cdot \left(\nabla_{p_i} \hat{\mathbf{g}} \times \nabla_{p_j} \hat{\mathbf{g}}\right)$$

over 2D surface S
in 3D momentum space

top. invariant for Weyl point in 3+1 system



hedgehog in p-space



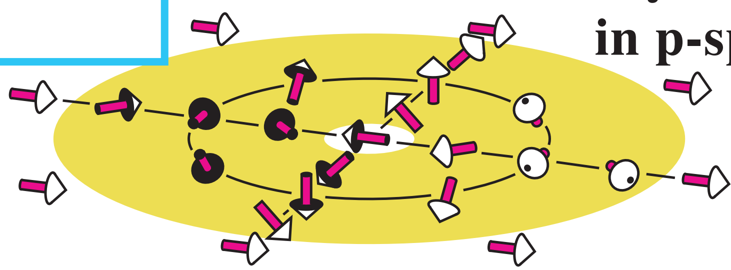
$$N_3 = \tilde{N}_3(p_z < p_0) - \tilde{N}_3(p_z > p_0)$$

at each p_z one has 2D insulator or fully gapped 2D superfluid

$$\tilde{N}_3(p_z) = \frac{1}{8\pi}\, e_{ijk} \int dp_x dp_y \; \hat{\mathbf{g}} \cdot \left(\nabla_{p_x} \hat{\mathbf{g}} \times \nabla_{p_y} \hat{\mathbf{g}}\right)$$

over the whole 2D momentum space
or over 2D Brillouin zone

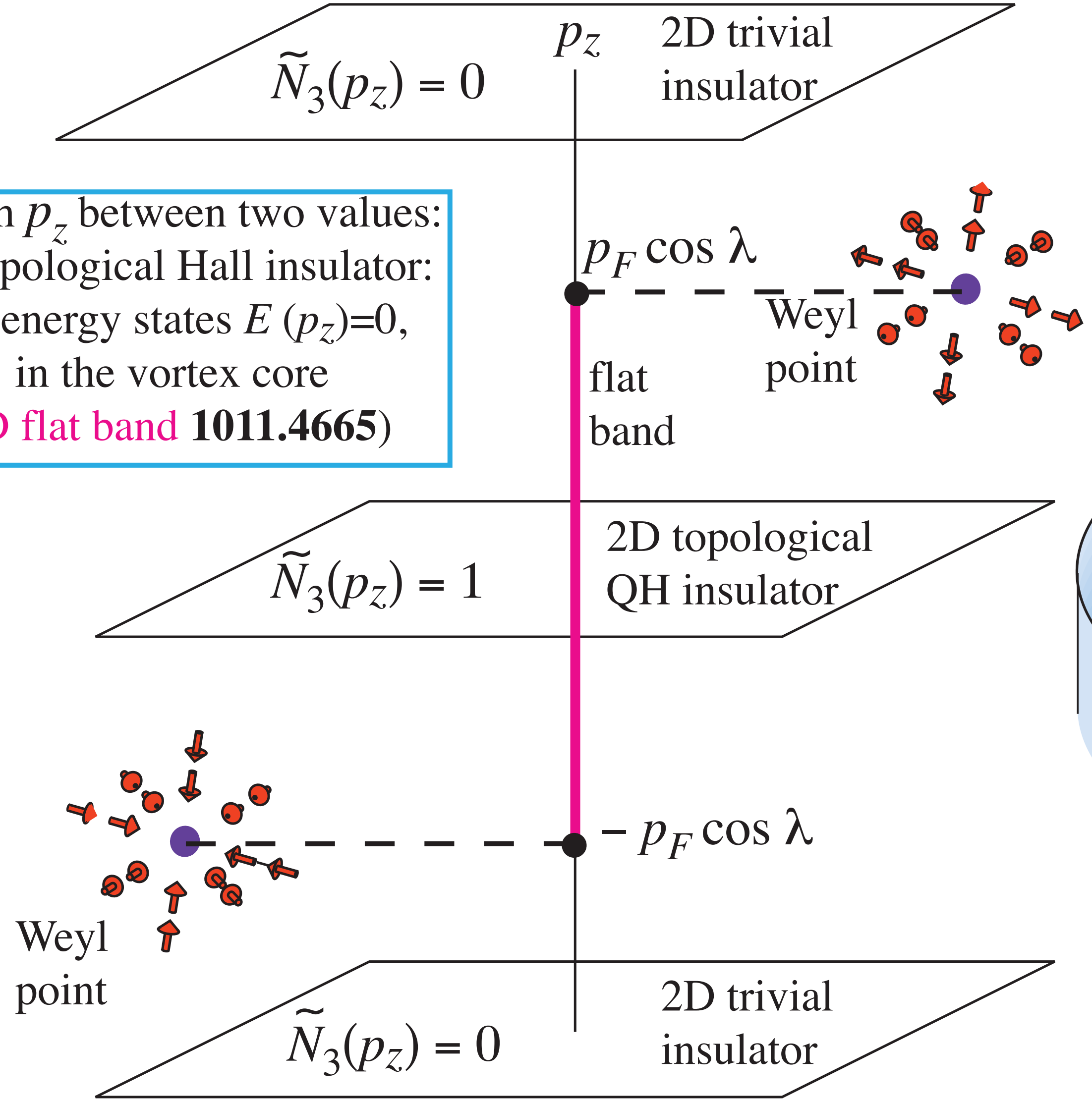
top. invariant for fully gapped 2+1 system



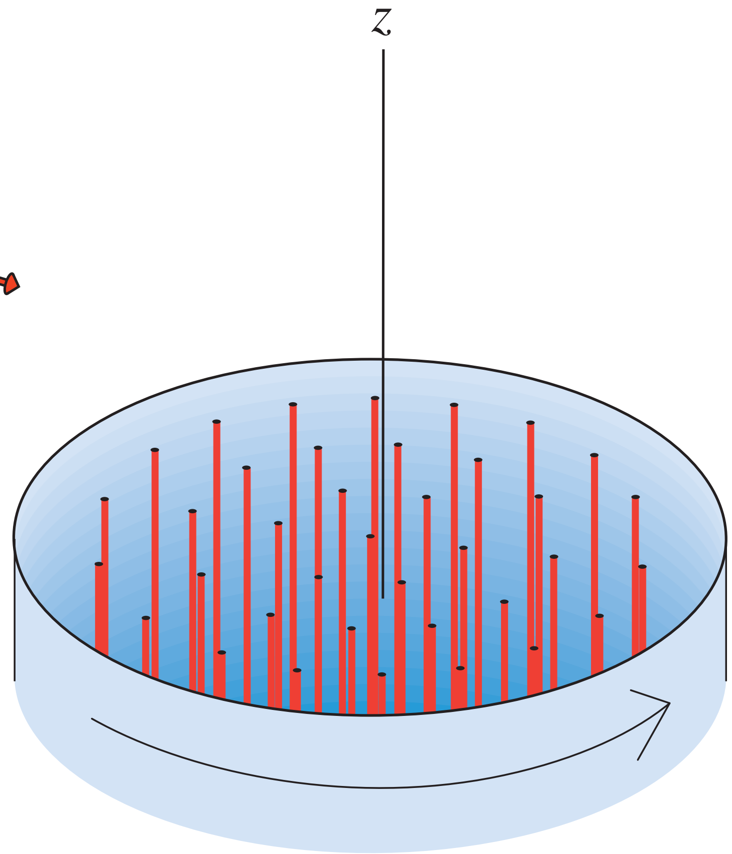
skyrmion in p-space

3D matter with Weyl points:
Topologically protected flat band in vortex core

at each p_z between two values:
2D topological Hall insulator:
zero energy states $E\,(p_z)$=0,
in the vortex core
(1D flat band **1011.4665**)



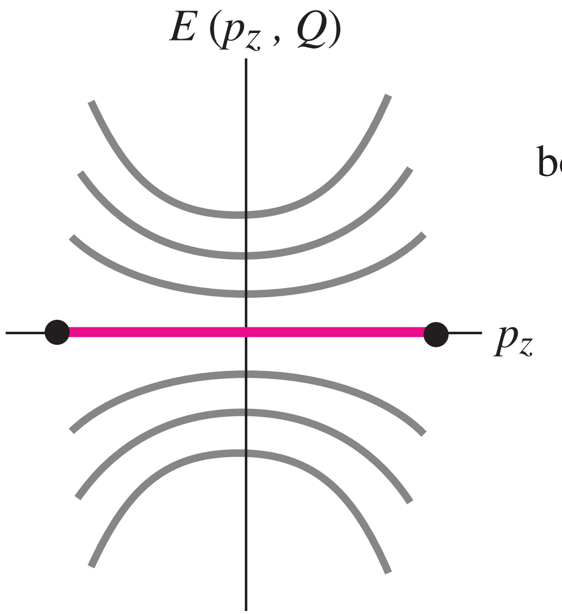
vortices in **r**-space



$$\tilde N_3(p_z) = \frac{1}{4\pi^2}\,\mathbf{tr}\int dp_x\, dp_y d\omega\, \mathbf{G}\,\nabla_\omega\, \mathbf{G}^{-1}\,\mathbf{G}\,\nabla_{p_x}\mathbf{G}^{-1}\mathbf{G}\,\nabla_{p_y}\,\mathbf{G}^{-1}$$

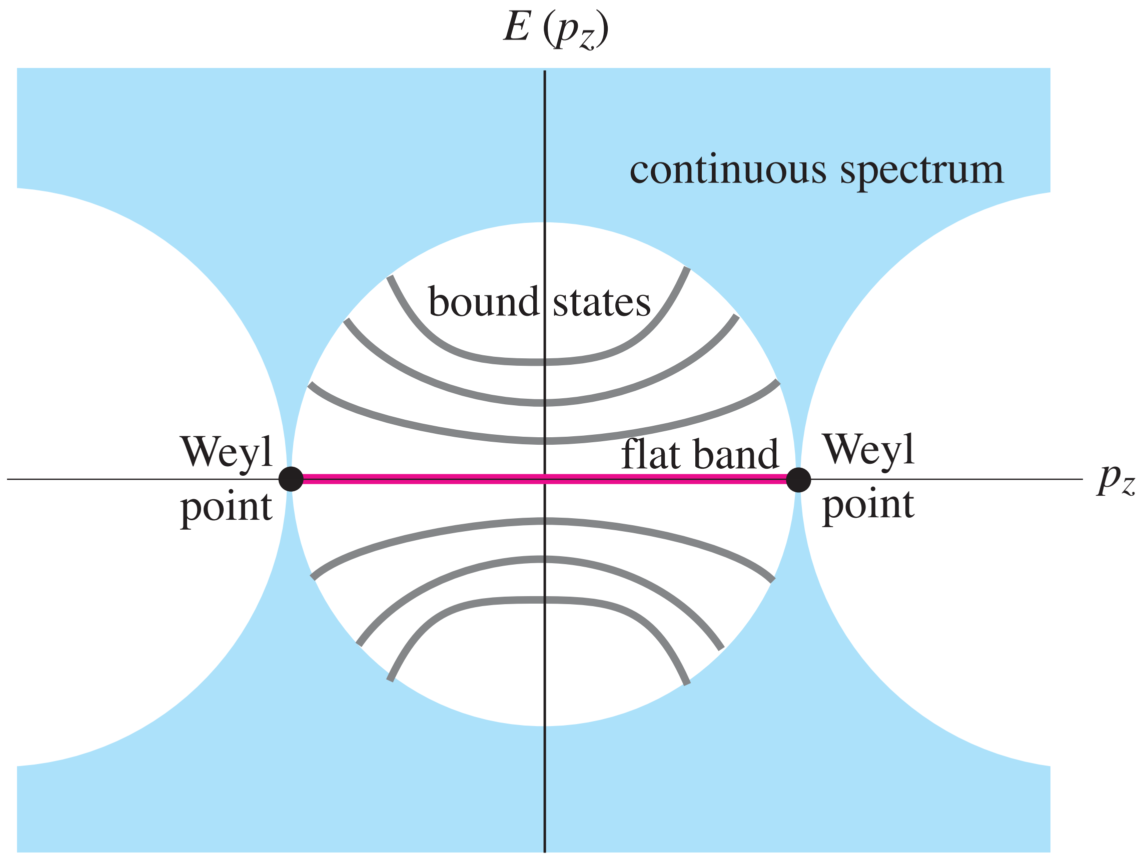
GV & Yakovenko (1989)

Topologically protected flat band in vortex core of superfluids with Weyl points



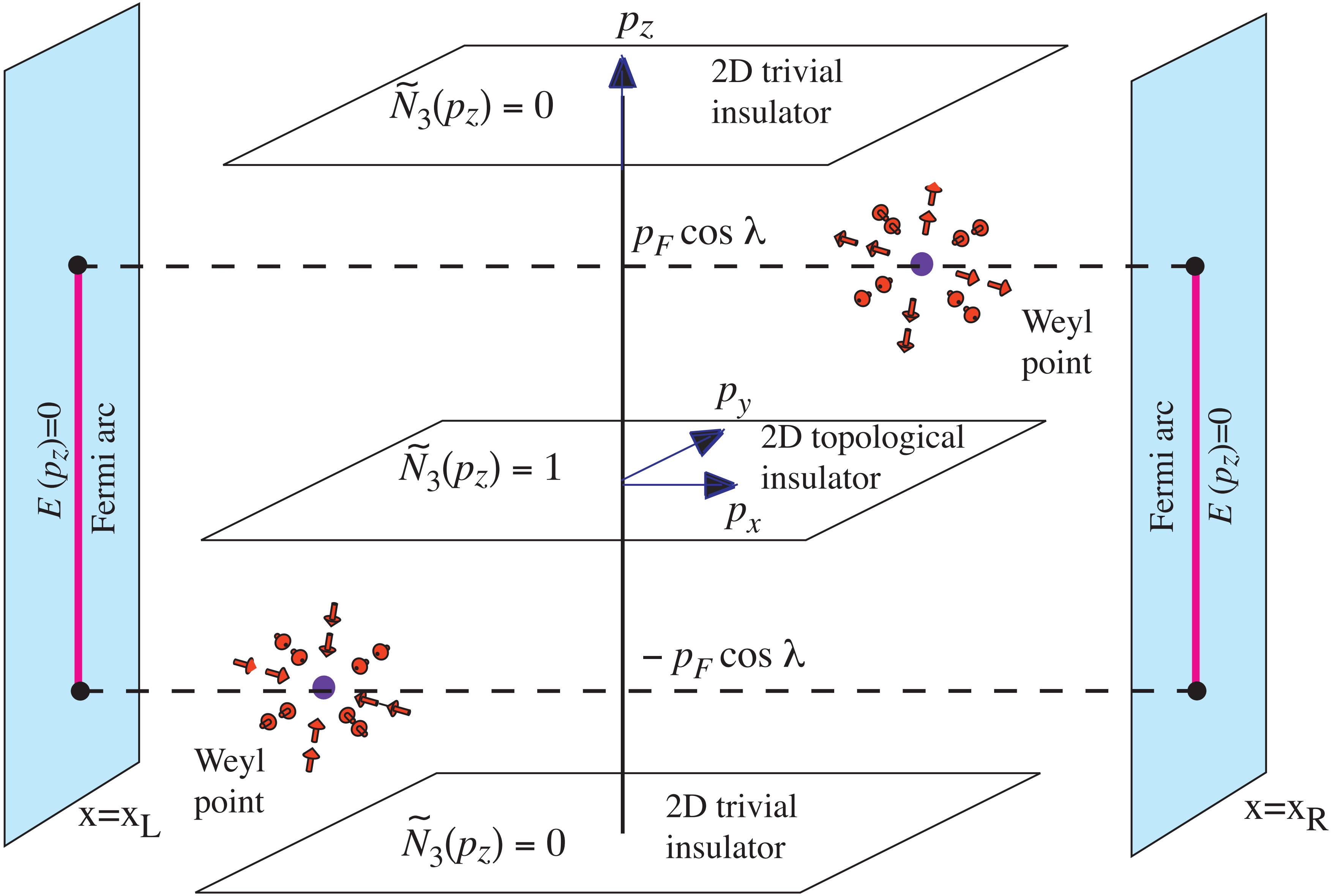
flat band
in spectrum of fermions
bound to core of 3He-A vortex
(Kopnin-Salomaa 1991)

flat band of bound states
terminates on zeroes
of continuous spectrum
(i.e on Weyl points)

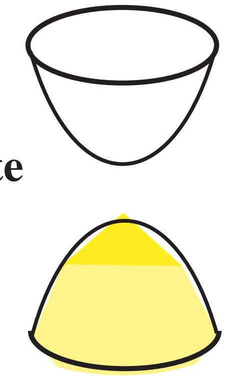


3He-A with Weyl points:
Topologically protected
Dirac valley (Fermi arc) on surface

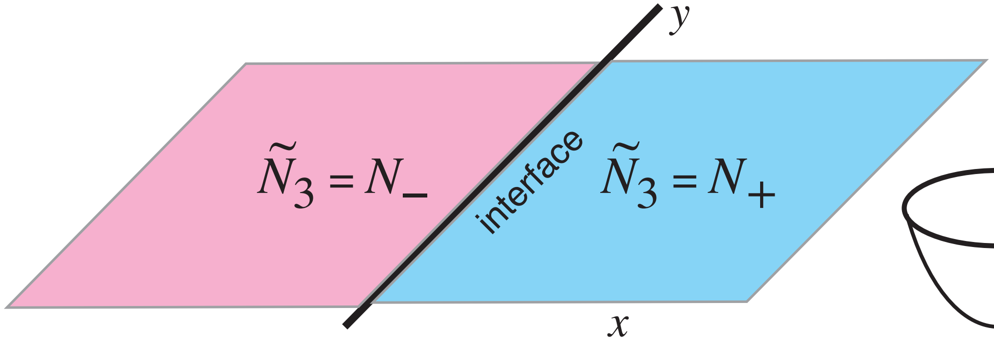
for each $|p_z| < p_F \cos\lambda$
one has 2D topological Hall insulator with
zero energy edge states $E(p_z)=0$
(Dirac valley **PRB 094510** or Fermi arc **PRB 205101**)

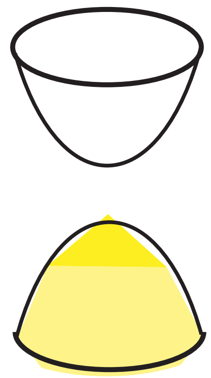


Edge states at interface between effective two 2+1 topological insulators & Fermi arc

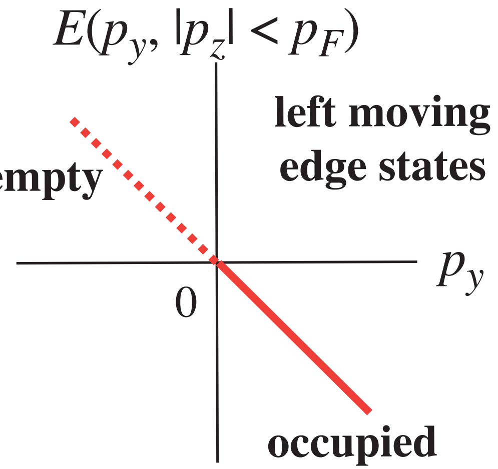


gapped state





gapped state



Index theorem:
number of fermion zero modes
at interface:

$$\nu = N_+ - N_-$$

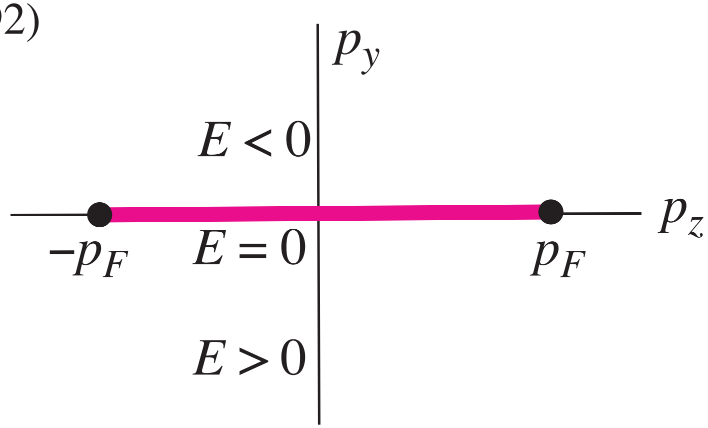
GV JETP Lett. **55**, 368 (1992)

on the edge of insulator with

$$\tilde{N}_3 = 1$$

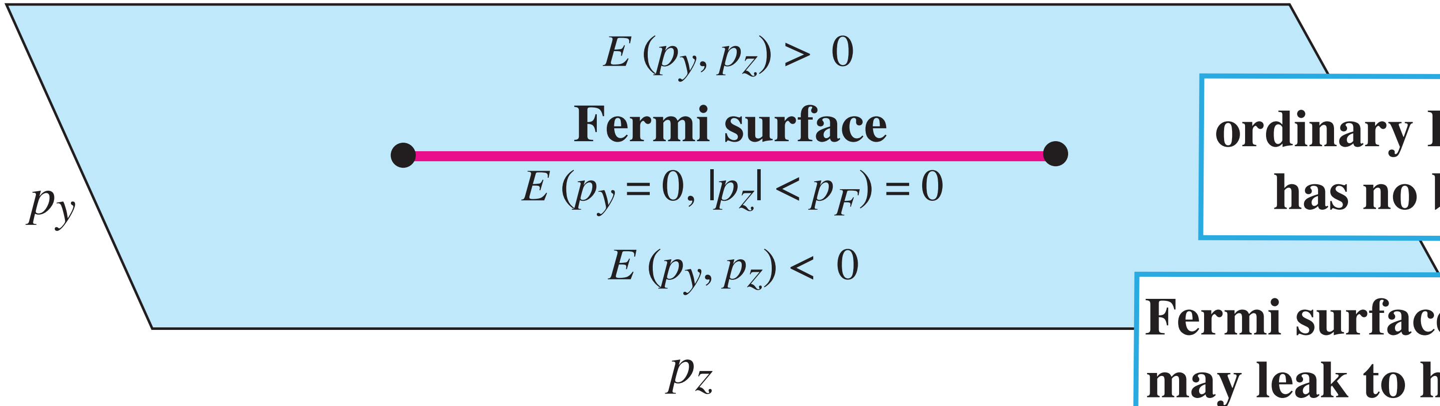
one fermion zero mode

$$\nu = 1$$



Fermi arc in 2D:
Fermi surface which terminates
on two points:
projections of Weyl points

Fermi arc:
Fermi surface separates positive and negative energies, but has boundaries

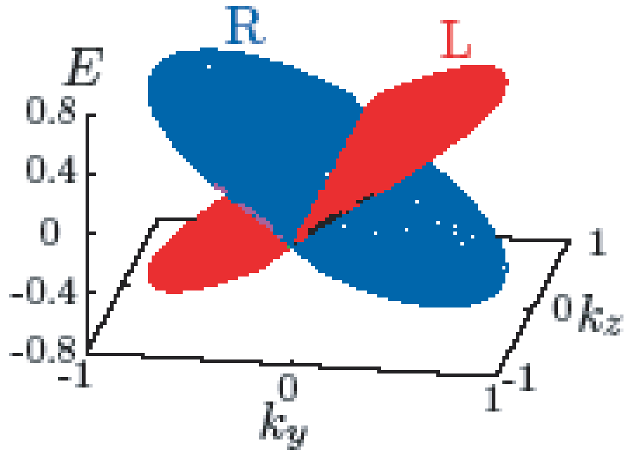


ordinary Fermi surface has no boundaries

Fermi surface of bound states may leak to higher dimension

Fermi surface of localized states is terminated by projections of Weyl points when localized states merge with continuous spectrum

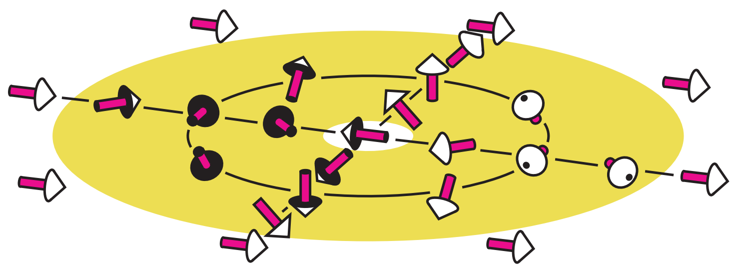
L spectrum of edge states on left wall



Tsutsumi, et al **PRB 094510**

R spectrum of edge states on right wall

5. Fully gapped topological matter



skyrmions in p-space

emergent relativistic fermions as edge states

3+1vacuum with massless fermions

dimensional reduction

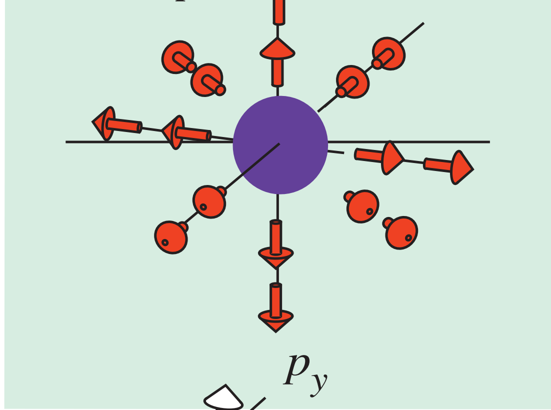
Fully gapped 2+1 vacuum

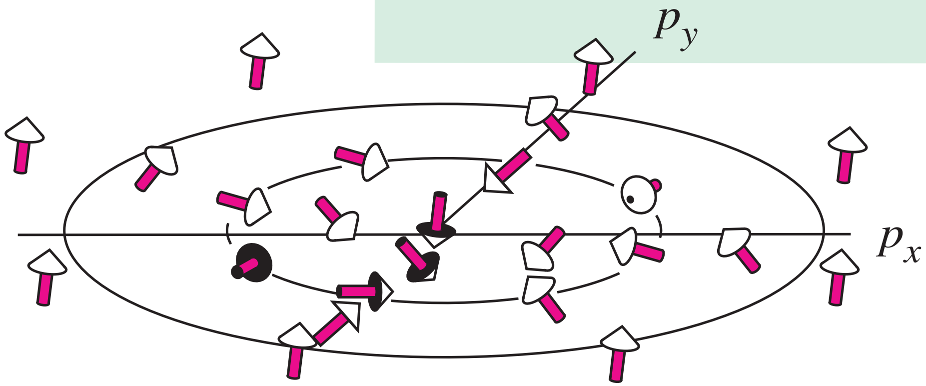
dimensional reduction

gapless 1+1 edge states

$$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \; \hat{\mathbf{g}} \cdot \left(\nabla_{p_i} \hat{\mathbf{g}} \times \nabla_{p_j} \hat{\mathbf{g}} \right)$$

around Fermi point





$$\tilde{N}_3 = \frac{1}{8\pi} e_{ijk} \int dp_x dp_y \;\; \hat{\mathbf{g}} \cdot \left(\nabla_{p_x} \hat{\mathbf{g}} \times \nabla_{p_y} \hat{\mathbf{g}} \right)$$

over the whole 2D momentum space
or over 2D Brillouin zone

Fully gapped 4+1 vacuum gives 3+1 relativistic fermions (Kaplan, arXiv:1112.0302)

topological insulators & gapped superconductors in 2+1

topological insulator =
bulk insulator
with topologically protected
gapless states on the boundary

topological gapped superconductor =
superconductor with gap in bulk
but with topologically protected
gapless states on the boundary

p-wave 2D superconductor (Sr_2RuO_4 ?), ^{3}He-A thin film,
CdTe/HgTe/Cd insulator quantum well, planar phase film

who protects gapless states?

generic example:

$$\mathrm{H} = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} \qquad p^2 = p_x^{\,2} + p_y^{\,2}$$

How to extract useful information on energy states from this Hamiltonian
without solving equation

$$\mathrm{H}\psi = E\psi$$

Topological invariant in momentum space

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} \qquad H = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \tau \cdot \mathbf{g}(\mathbf{p})$$

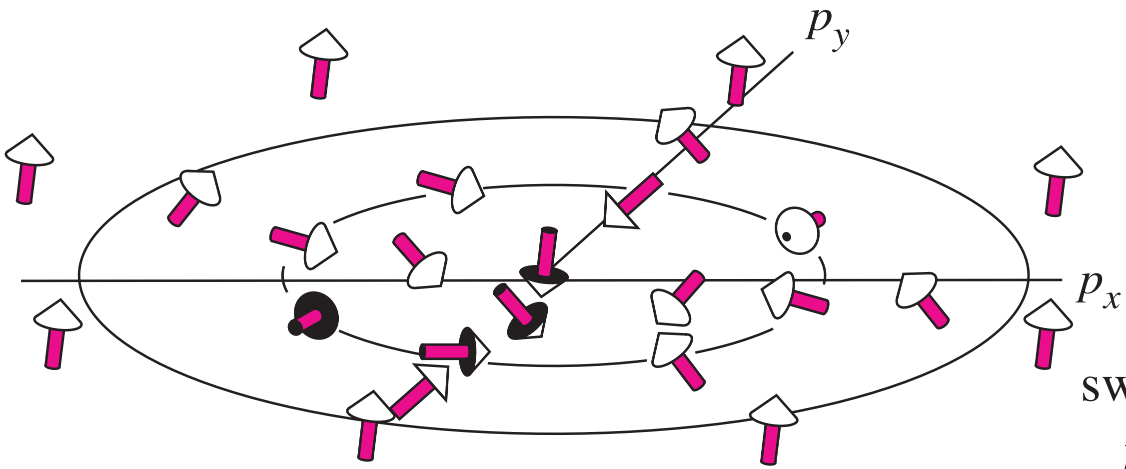
$$p^2 = p_x^2 + p_y^2$$

fully gapped 2D state at $\mu \neq 0$

$$\boxed{\tilde{N}_3 = \frac{1}{8\pi}\, e_{ijk} \int dp_x dp_y \;\; \hat{\mathbf{g}} \cdot (\nabla_{p_x} \hat{\mathbf{g}} \times \nabla_{p_y} \hat{\mathbf{g}})}$$

GV, JETP **67**, 1804 (1988)

Skyrmion (coreless vortex) in momentum space at $\mu > 0$

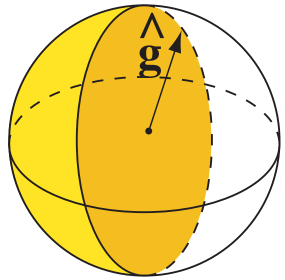


unit vector

$\hat{\mathbf{g}}(p_x, p_y)$

sweeps unit sphere

$\tilde{N}_3 \; (\mu > 0) = 1$

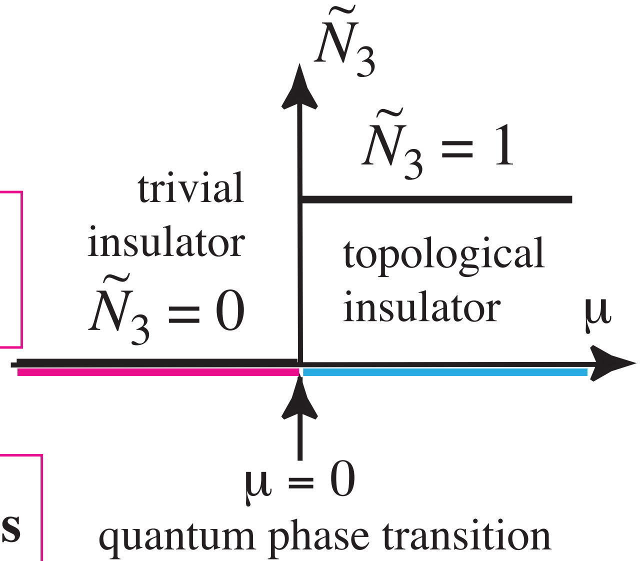


quantum phase transition: from topological to non-topologicval insulator/superconductor

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \tau \cdot \mathbf{g}(\mathbf{p})$$

Topological invariant in momentum space

$$\boxed{N_3 = \frac{1}{8\pi}\, e_{ijk} \int dp_x dp_y \;\; \hat{\mathbf{g}} \cdot (\nabla_{p_x} \hat{\mathbf{g}} \times \nabla_{p_y} \hat{\mathbf{g}})}$$



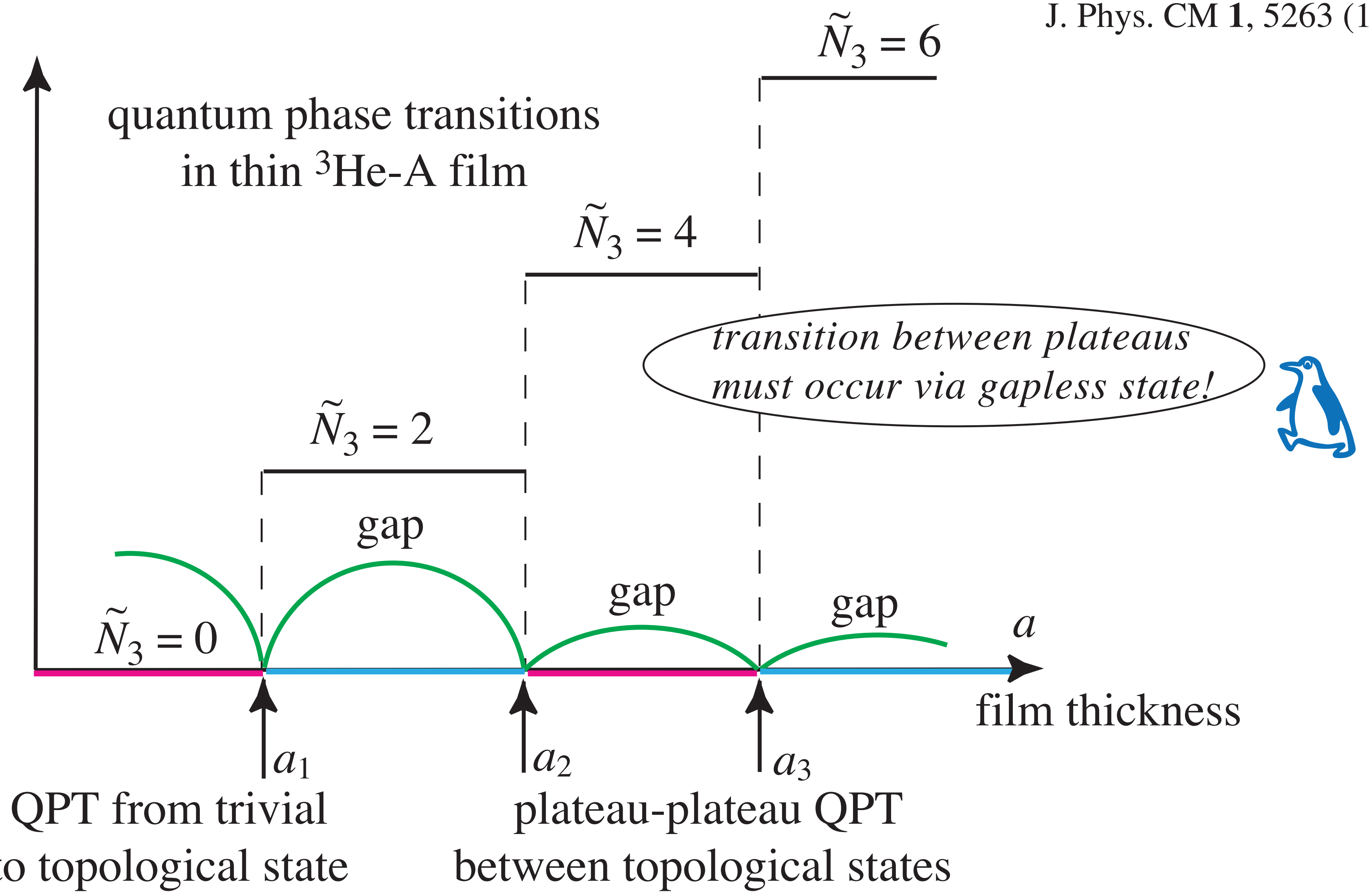
intermediate state at $\mu = 0$ must be gapless

$\Delta\tilde{N}_3 \neq 0$ is origin of fermion zero modes at the interface between states with different $\tilde{N}_3$

p-space invariant in terms of Green's function & topological QPT

$$\tilde{N}_3 = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \,\mathbf{tr} \int d^2p\, d\omega\, \mathbf{G}\nabla^{\mu}\mathbf{G}^{-1}\mathbf{G}\nabla^{\nu}\mathbf{G}^{-1}\mathbf{G}\nabla^{\lambda}\mathbf{G}^{-1}$$

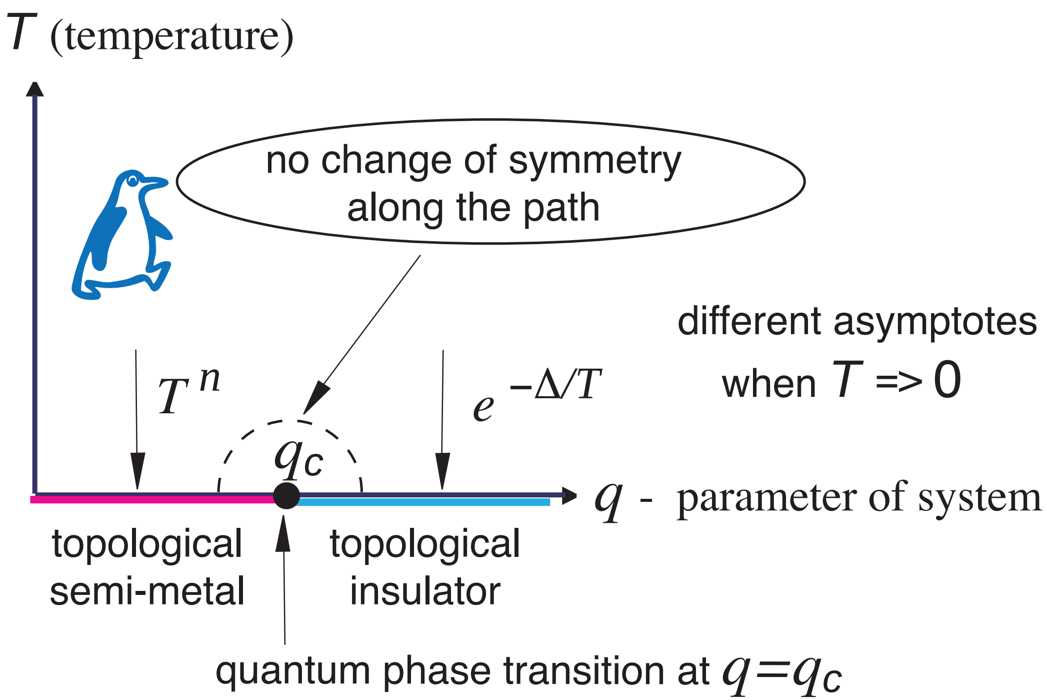
GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)



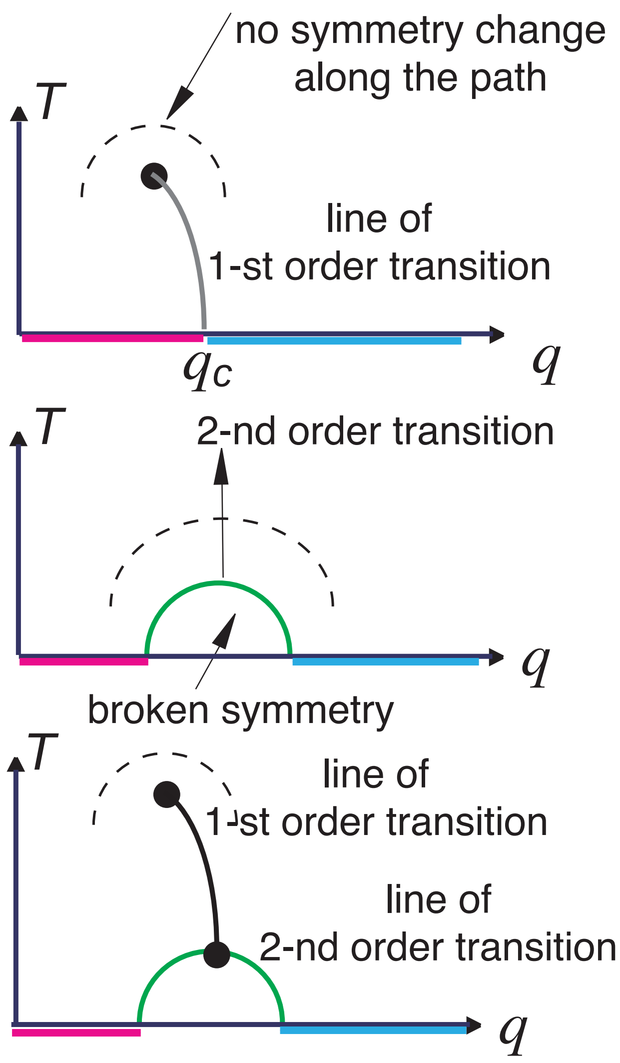
topological quantum phase transitions

transitions between **ground states (vacua)** of the **same symmetry**,
but **different topology** in **momentum space**

example: QPT between gapless & gapped matter

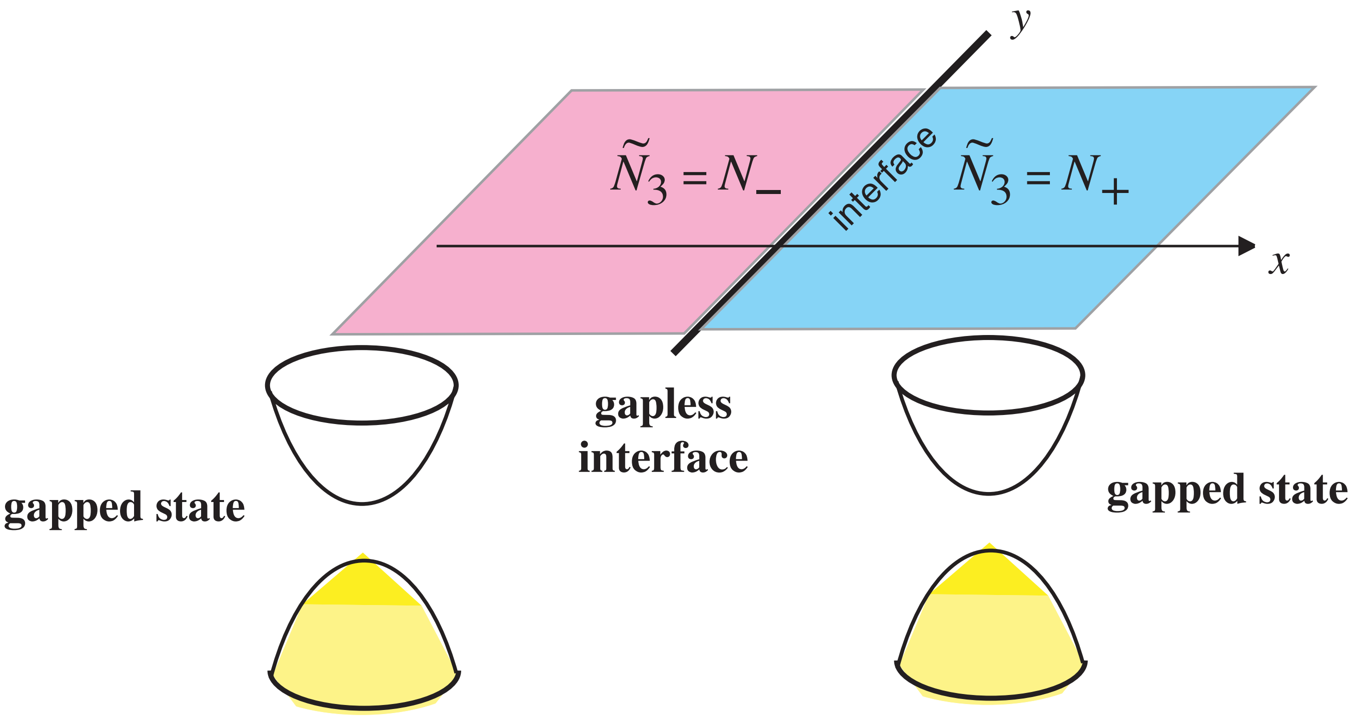


QPT interrupted
by thermodynamic transitions



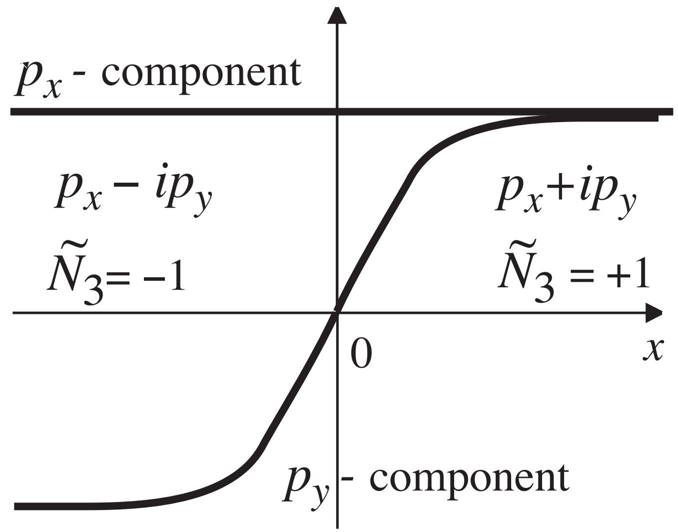
other topological QPT:
Lifshitz transition,
transtion between topological and nontopological superfluids,
plateau transitions,
confinement-deconfinement transition, ...

interface between two 2+1 topological insulators or gapped superfluids

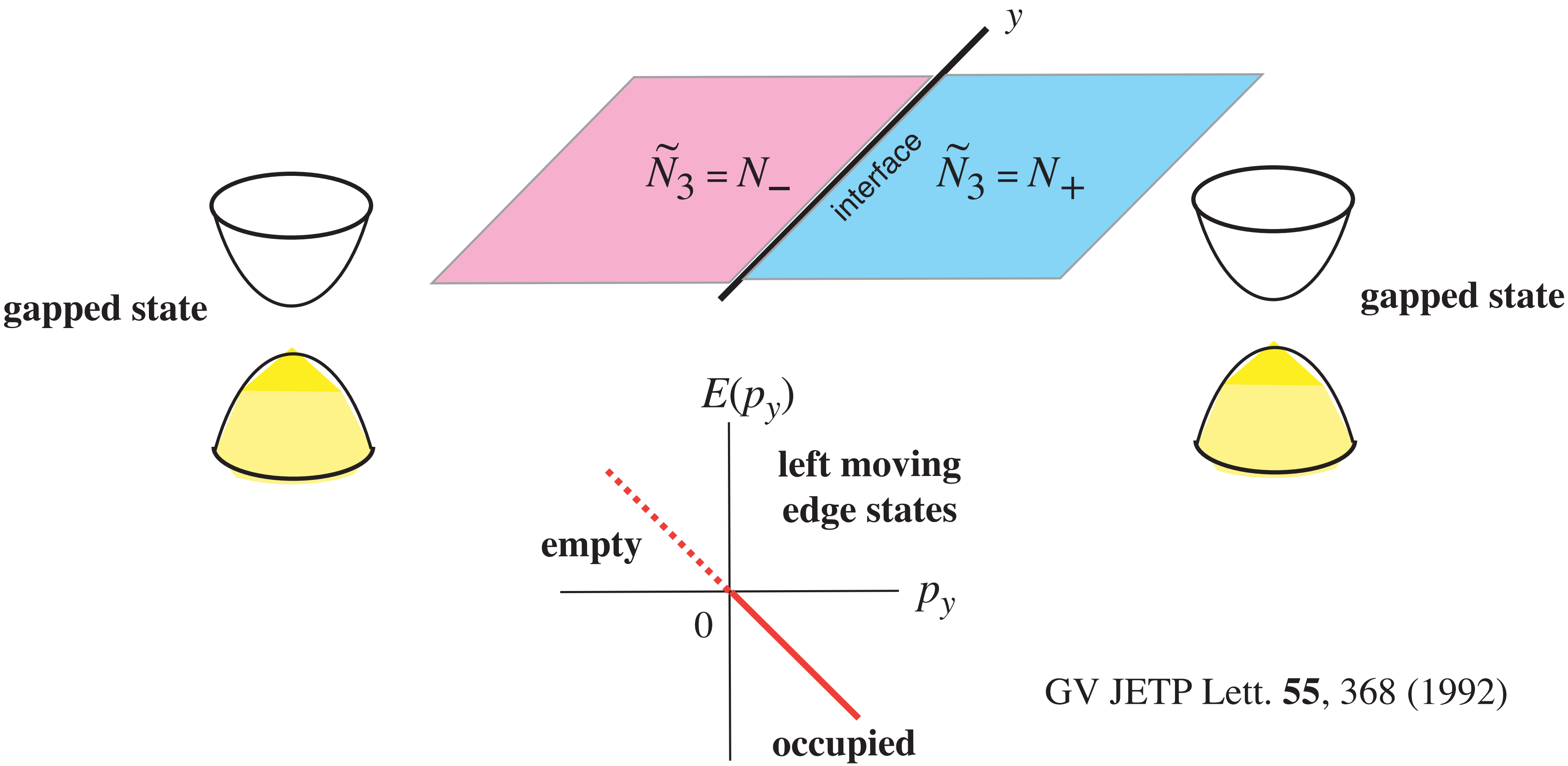


* domain wall in 2D chiral superconductors:

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + i\, p_y \tanh x) \\ c(p_x - i\, p_y \tanh x) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$



Edge states at interface between two 2+1 topological insulators or gapped superfluids

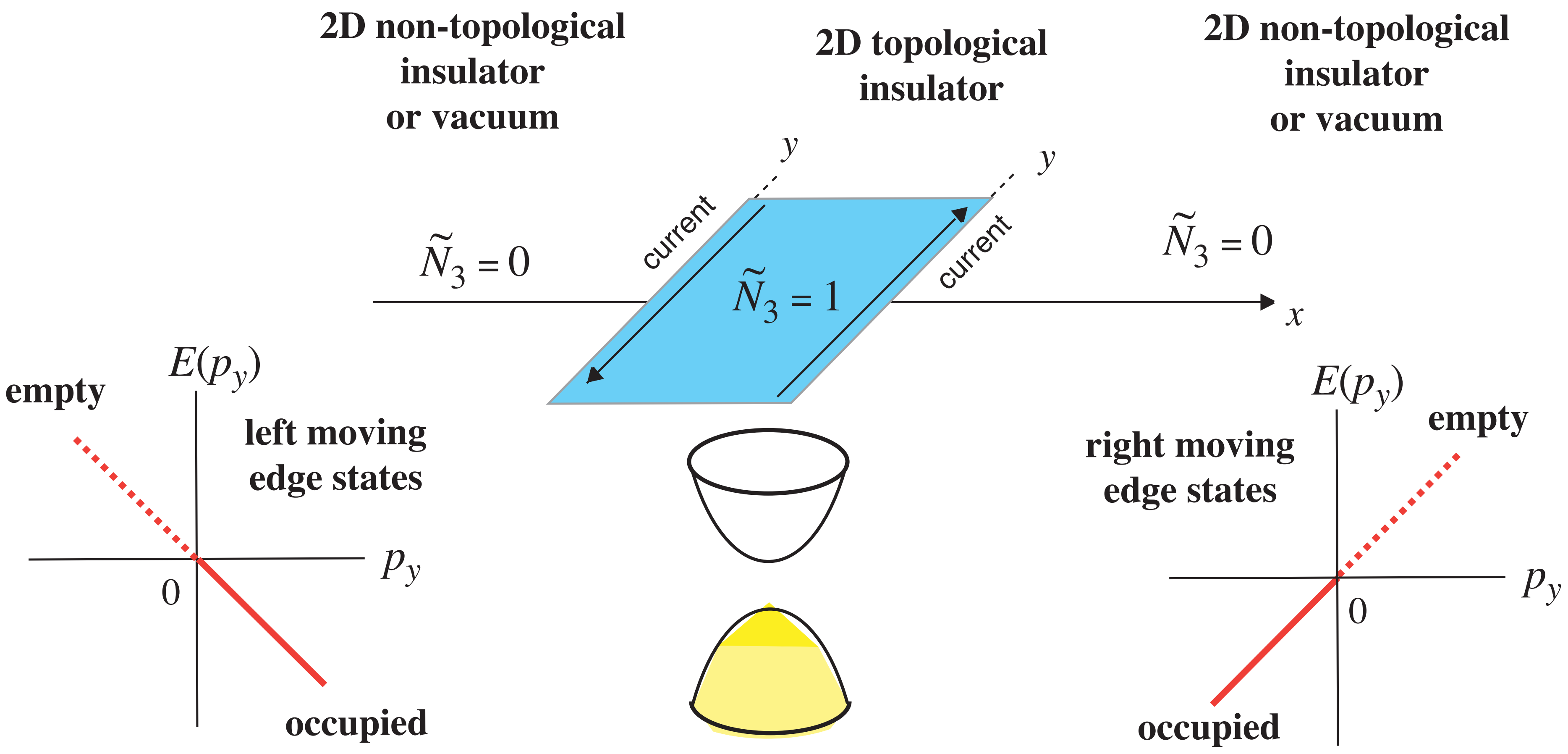


GV JETP Lett. **55**, 368 (1992)

Index theorem:
number of fermion zero modes
at interface:

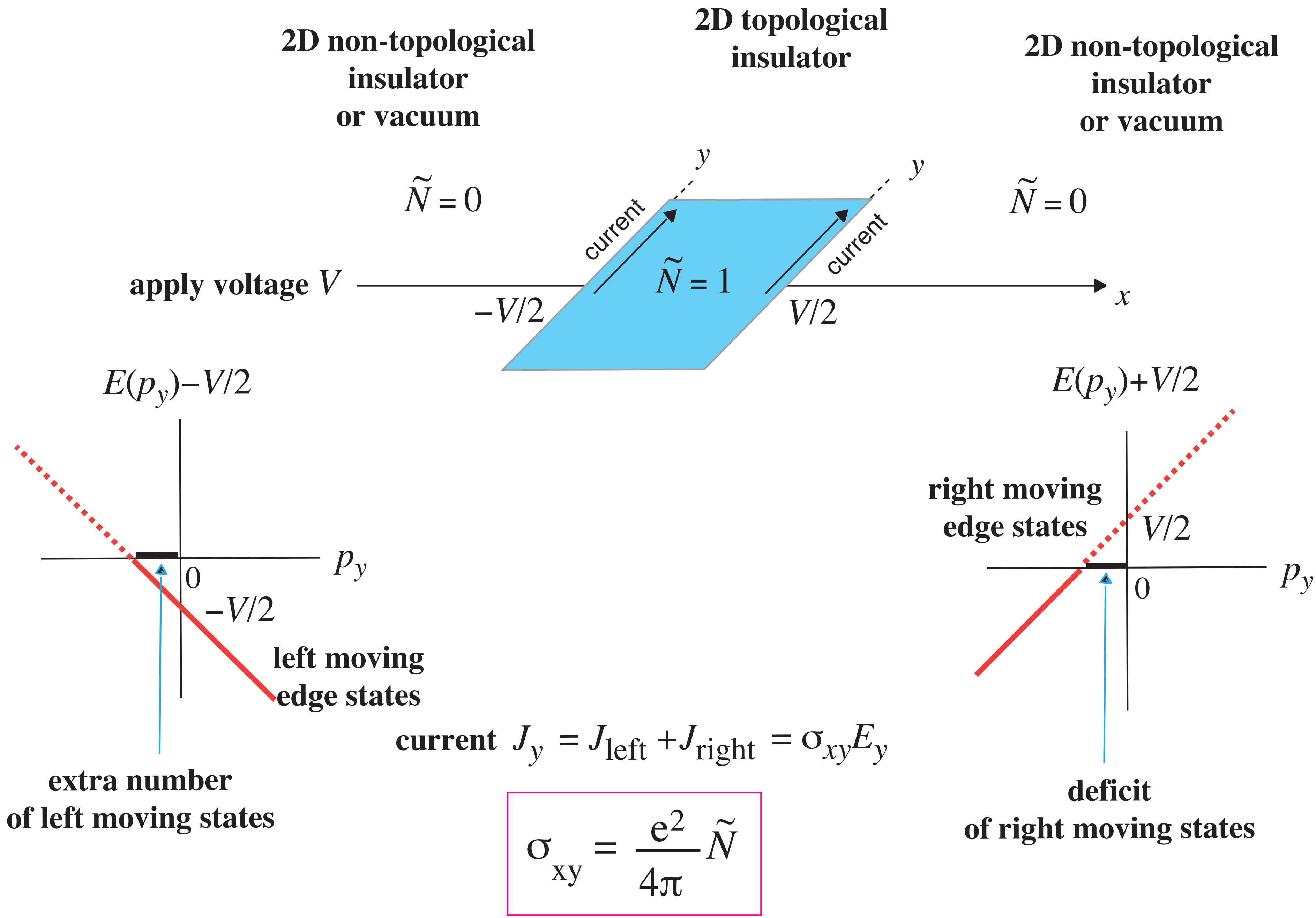
$$\nu = N_+ - N_-$$

Edge states and currents



current $J_y = J_{\mathrm{left}} + J_{\mathrm{right}} = 0$

Edge states & intrinsic QHE: topological invariant determines Hall quantization



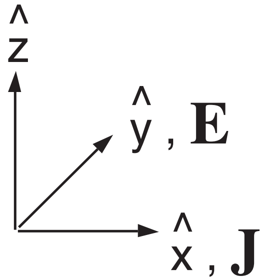
Intrinsic quantum Hall effect & momentum-space invariant

$$\boxed{S_{CS} = \frac{e^2}{16\pi}\,\tilde{N}_3\, e^{\mu\nu\lambda} \int d^2x\, dt\, A_\mu\, F_{\nu\lambda}}$$

p-space invariant   **r**-space invariant

A_μ - electromagnetic field

electric current $J_x = \delta S_{CS} / \delta A_x = \dfrac{e^2}{4\pi}\tilde{N}_3 E_y$

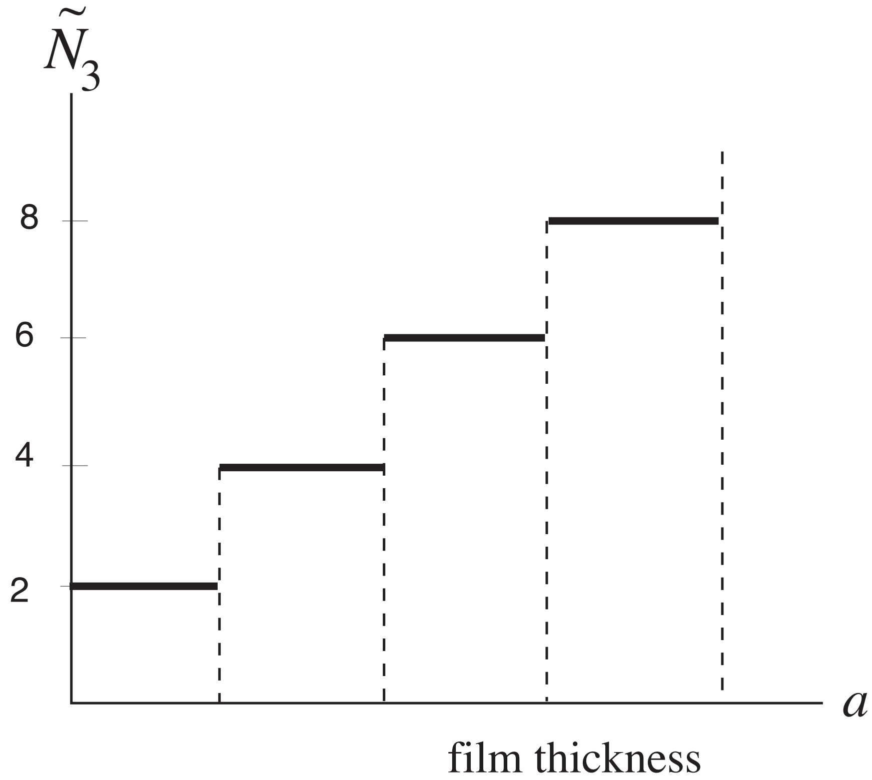


quantized intrinsic Hall conductivity
(without external magnetic field)

$$\boxed{\sigma_{xy} = \frac{e^2}{4\pi}\tilde{N}_3}$$

GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

film of topological quantum liquid



general Chern-Simons terms & momentum-space invariant

(interplay of ***r***-space and ***p***-space topologies)

$$\boxed{S_{\mathrm{CS}} = \frac{1}{16\pi}\,\tilde{N}_{3\mathrm{IJ}}\, e^{\mu\nu\lambda} \int d^2x\, dt\, A_\mu^{\ \mathrm{I}} F_{\nu\lambda}^{\ \mathrm{J}}}$$

***r*-space invariant**

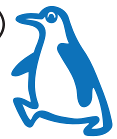
***p*-space invariant protected by symmetry**

dimensional reduction of chiral anomaly in 3+1

$$\boxed{\tilde{N}_{3\mathrm{IJ}} = \frac{1}{24\pi^2}\, e_{\mu\nu\lambda}\, \mathbf{tr}\left[\int d^2p\, d\omega\, \mathrm{K_I}\, \mathrm{K_J}\, \mathbf{G}\nabla^\mu \mathbf{G}^{-1}\, \mathbf{G}\nabla^\nu \mathbf{G}^{-1} \mathbf{G}\nabla^\lambda \mathbf{G}^{-1}\right]}$$

$\mathrm{K_I}$ - charge interacting with gauge field $A_\mu^{\ \mathrm{I}}$

gauge fields can be
real, artificial or auxiliary



$\mathrm{K}=e$ for electromagnetic field A_μ

$\mathrm{K}=\hat{\sigma}_z$ for effective spin-rotation field $A_\mu^{\ z}$ $\quad (A_0^{\ z} = \gamma H^{\,z})$

$$id/dt - \gamma\hat{\sigma}\cdot\mathbf{H} = id/dt - \hat{\sigma}\cdot\mathbf{A}_0$$

applied Pauli magnetic field plays the role of components of effective SU(2) gauge field $A_\mu^{\ \mathrm{i}}$

Intrinsic spin-current quantum Hall effect & momentum-space invariant

spin current $J_x^z = \dfrac{1}{4\pi}\left(\gamma N_{ss}\, dH^z/dy + N_{se}\, E_y\right)$

spin-spin QHE

spin-charge QHE

2D singlet superconductor:

$$\boxed{\sigma_{xy}^{spin/spin} = \frac{N_{ss}}{4\pi}}$$

s-wave: $N_{ss} = 0$

$p_x + ip_y$: $N_{ss} = 2$

$d_{xx\text{-}yy} + id_{xy}$: $N_{ss} = 4$

film of planar phase of superfluid ^{3}He

$$\boxed{\sigma_{xy}^{spin/charge} = \frac{N_{se}}{4\pi}}$$

GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

spin quantum Hall effect: planar phase film of 3He & 2D topological insulator

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + i\,p_y\sigma_z) \\ c(p_x - i\,p_y\sigma_z) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$\boxed{\tilde{N}_3 = \frac{1}{24\pi^2} e_{\mu\nu\lambda}\,\mathbf{tr}\left[\int d^2p\, d\omega\, \mathbf{G}\nabla^\mu \mathbf{G}^{-1}\, \mathbf{G}\nabla^\nu \mathbf{G}^{-1} \mathbf{G}\nabla^\lambda \mathbf{G}^{-1}\right] = 0}$$

$$\boxed{\tilde{N}_{se} = \frac{1}{24\pi^2} e_{\mu\nu\lambda}\,\mathbf{tr}\left[\int d^2p\, d\omega\, \sigma_z\, \mathbf{G}\nabla^\mu \mathbf{G}^{-1}\, \mathbf{G}\nabla^\nu \mathbf{G}^{-1} \mathbf{G}\nabla^\lambda \mathbf{G}^{-1}\right]}$$

$$\tilde{N}_3^+ = +1 \qquad \tilde{N}_3^- = -1$$

$$\tilde{N}_3 = \tilde{N}_3^+ + \tilde{N}_3^- = 0 \qquad \tilde{N}_{se} = \tilde{N}_3^+ - \tilde{N}_3^- = 2$$

spin quantum Hall effect

spin current $J_x^z = \frac{1}{4\pi} N_{se} E_y$

spin-charge QHE

$$\boxed{\sigma_{xy}^{\text{spin/charge}} = \frac{N_{se}}{4\pi}}$$

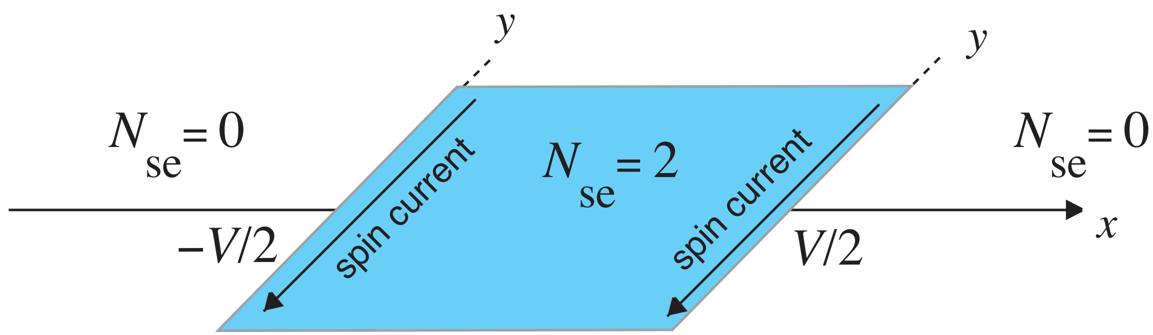
$N_{se} = 2$

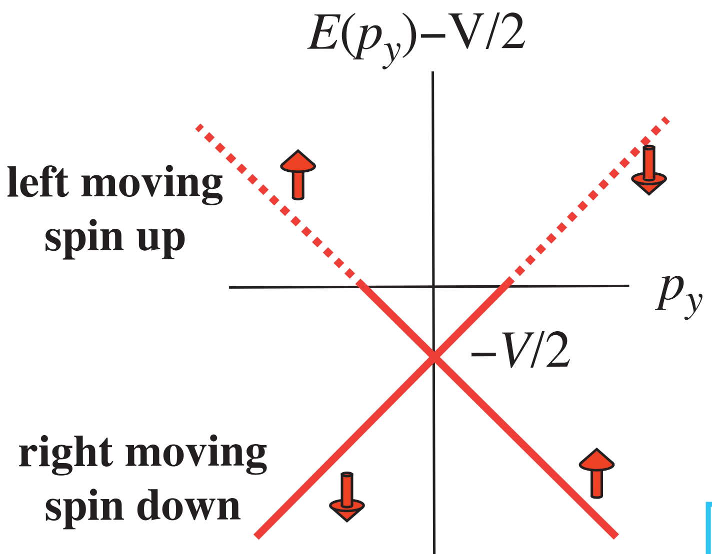
GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

Intrinsic spin-current quantum Hall effect & edge state

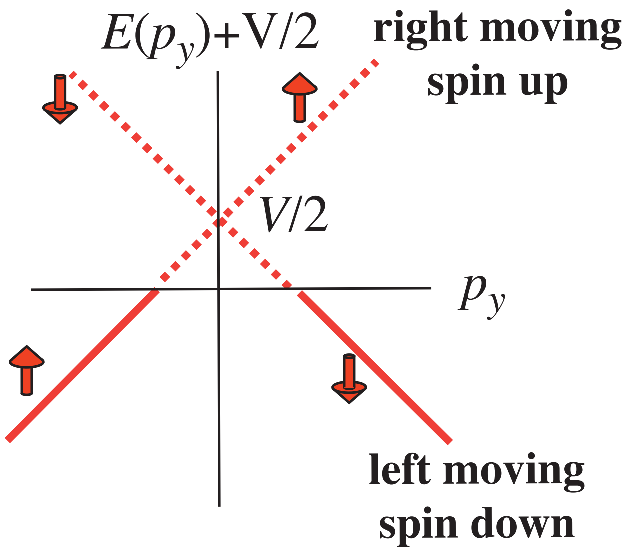
spin current $J_x^{\,z} = \dfrac{1}{4\pi}\left(\gamma N_{ss}\, dH^z/dy + N_{se} E_y\right)$

$\boxed{\text{spin-charge QHE}}$





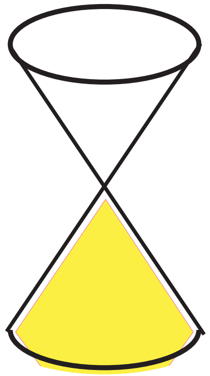
$$\sigma_{xy}^{\text{spin/charge}} = \frac{N_{se}}{4\pi}$$



electric current is zero
spin current is nonzero

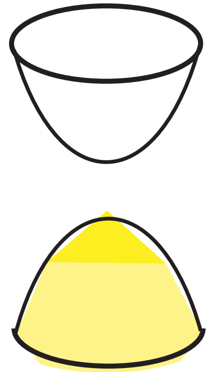
3D topological superfluids / insulators / semiconductors / vacua

gapless topologically
nontrivial vacua



3He-A,
Standard Model
above electroweak transition,
semimetals,
4D graphene
(cryocrystalline vacuum)

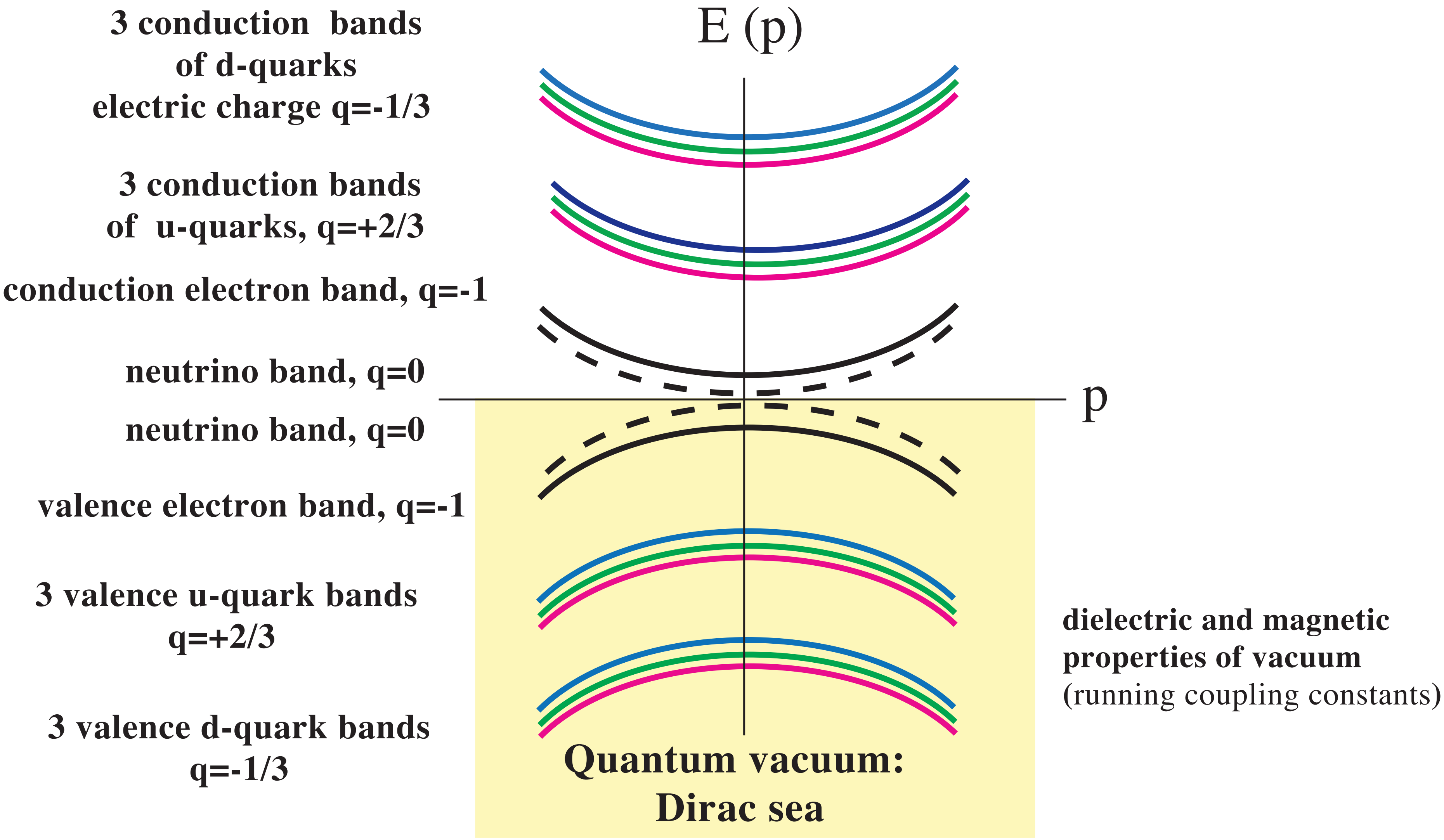
fully gapped topologically
nontrivial vacua



3He-B,
Standard Model
below electroweak transition,
topological insulators,
triplet & singlet
chiral superconductor, ...

Bi_2Te_3

Present vacuum as semiconductor or insulator



electric charge of quantum vacuum

$$Q = \sum_a q_a = N\,[-1 + 3\times(-1/3) + 3\times(+2/3)\,] = 0$$

fully gapped 3+1 topological matter

superfluid ^{3}He-B, topological insulator Bi_2Te_3 , present vacuum of Standard Model

* Standard Model vacuum as topological insulator

Topological invariant protected by symmetry

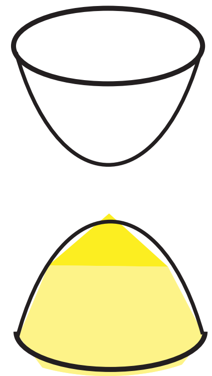
$$N_K = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \, \mathbf{tr} \int dV \, K \, \mathbf{G} \nabla^\mu \mathbf{G}^{-1} \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G} \nabla^\lambda \mathbf{G}^{-1}$$

over 3D momentum space

$\mathbf{G}$ is Green's function at $\omega=0$, K is symmetry operator $\mathbf{G}K = +/- K\mathbf{G}$

Standard Model vacuum: $K=\gamma_5$ $\mathbf{G}\gamma_5 = -\gamma_5\mathbf{G}$

$$\boxed{N_K = 8 n_g}$$



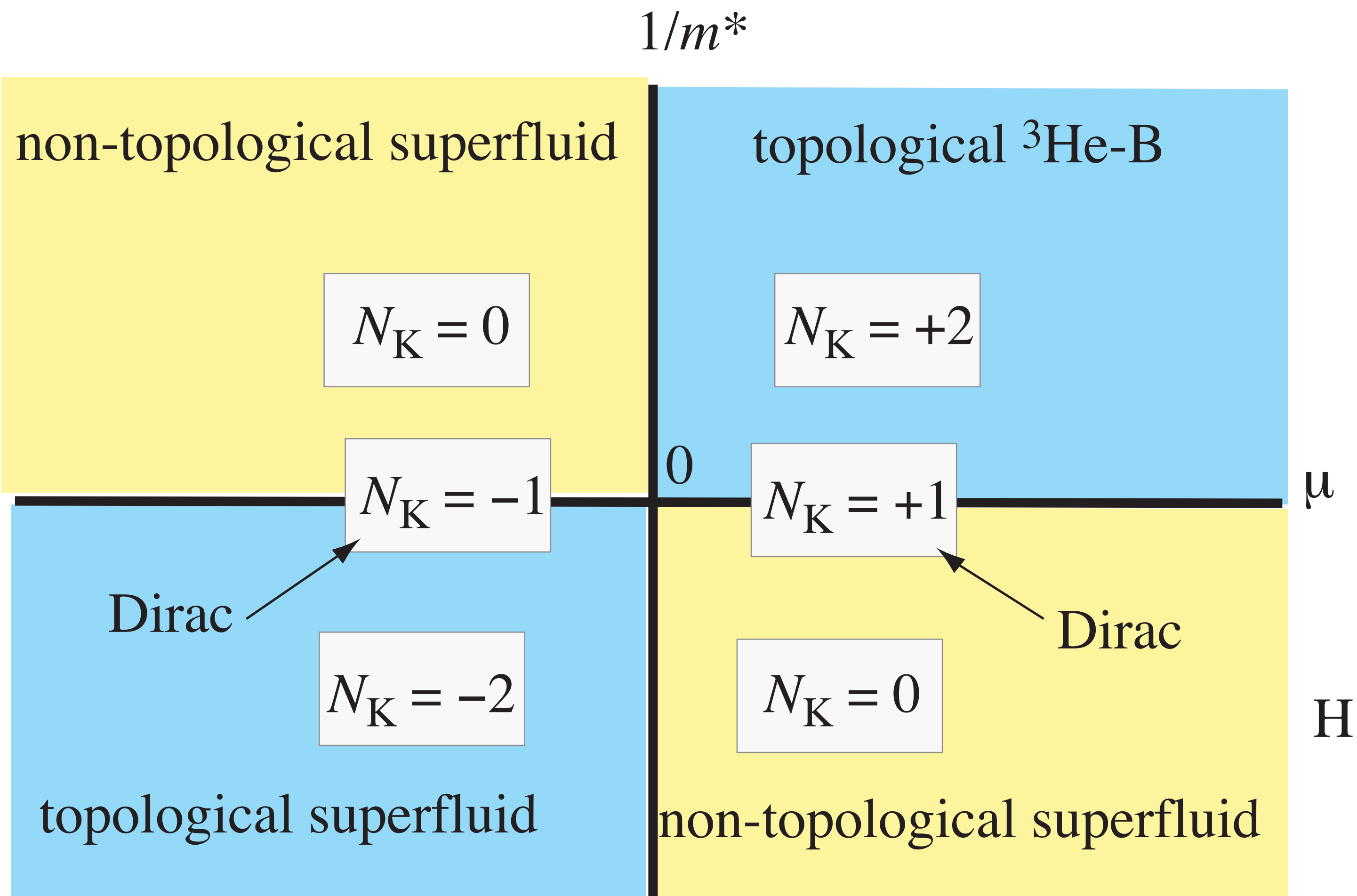
8 massive Dirac particles in one generation

topological superfluid ^{3}He-B

$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu \end{pmatrix} = \left(\frac{p^2}{2m^*} - \mu\right)\tau_3 + c_B \sigma\cdot\mathbf{p}\,\tau_1$$

$$\mathbf{H}\tau_2 = -\tau_2\mathbf{H}$$

$$K = \tau_2$$



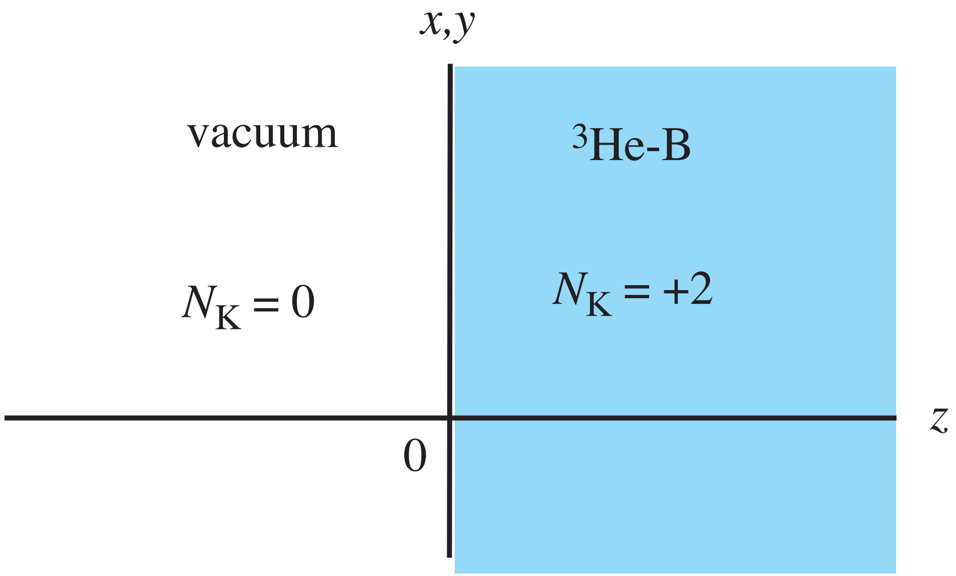
Dirac vacuum

$$1/m^* = 0$$

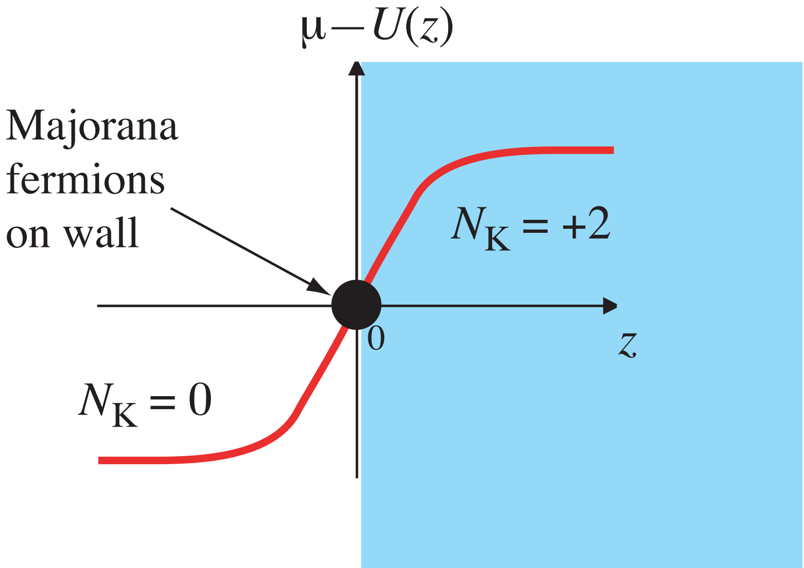
$$H = \begin{pmatrix} -M & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & +M \end{pmatrix}$$

GV JETP Lett. **90**, 587 (2009)

Boundary of 3D gapped topological superfluid



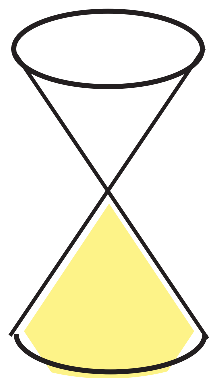
$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu + U(z) & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu - U(z) \end{pmatrix}$$



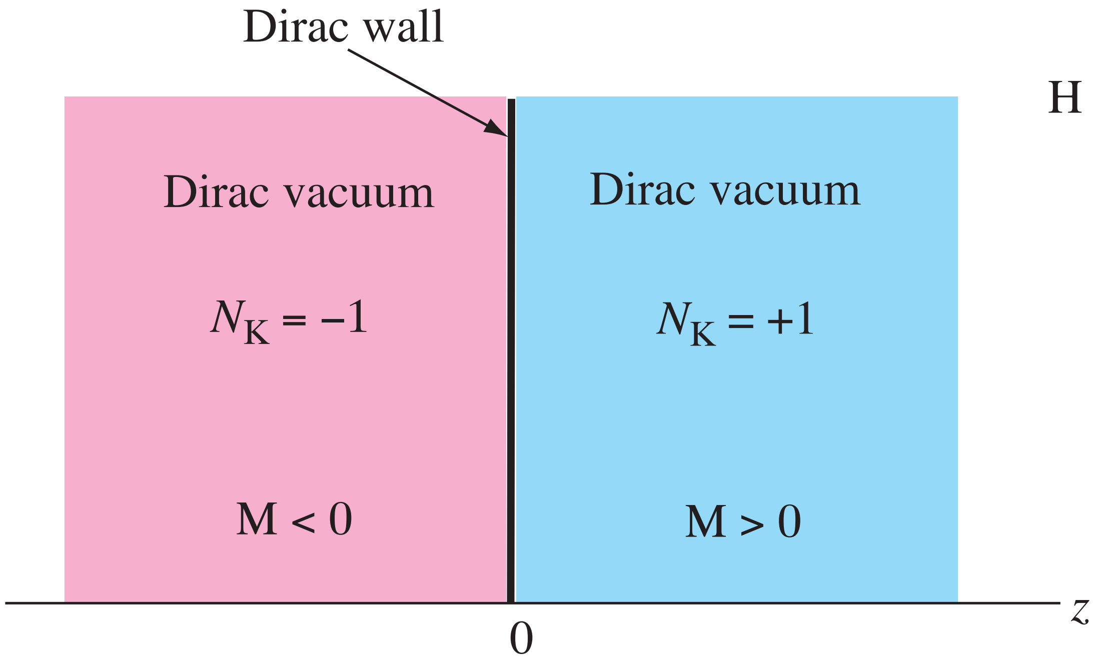
spectrum of Majorana fermion zero modes

$$H_{zm} = c_B \hat{\mathbf{z}} \cdot \sigma \times \mathbf{p} = c_B (\sigma_x p_y - \sigma_y p_x)$$

helical fermions

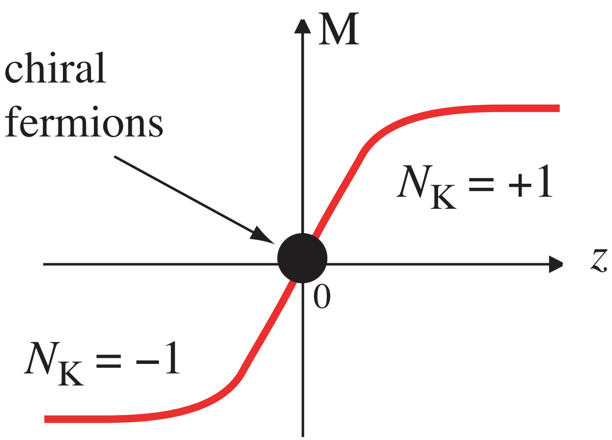


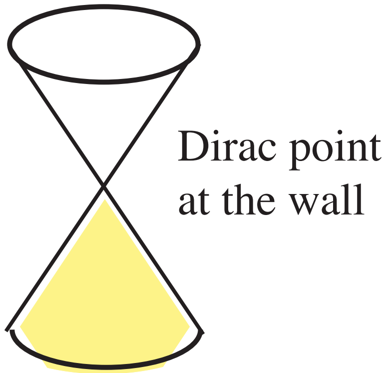
fermion zero modes on Dirac wall

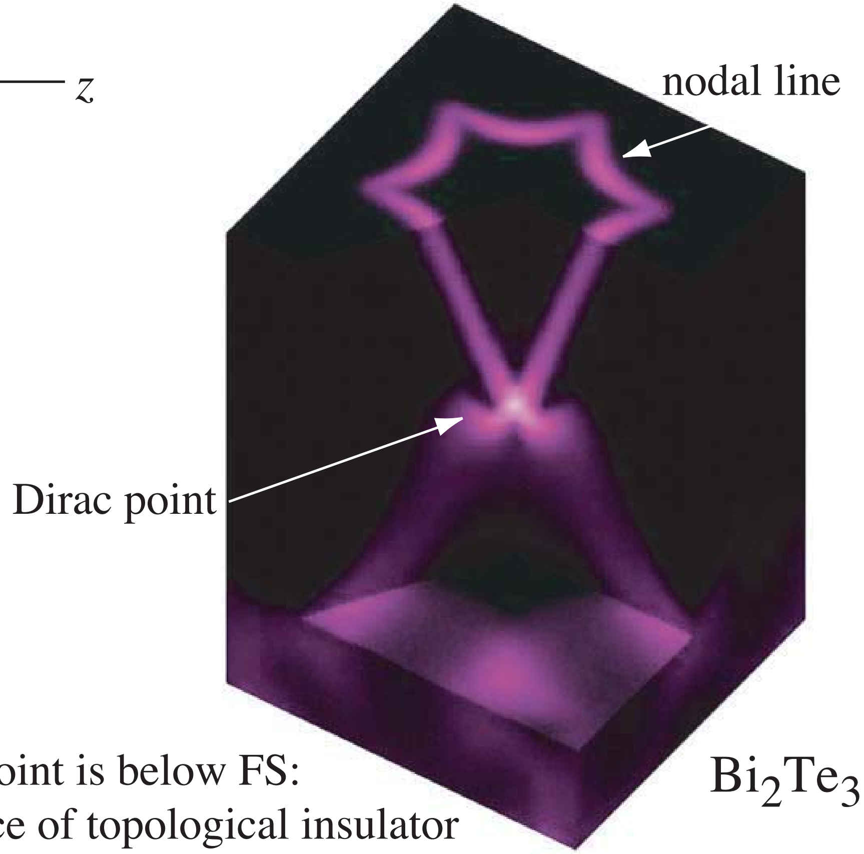


$$H = \begin{pmatrix} -M(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & +M(z) \end{pmatrix}$$

Volkov-Pankratov,
2D massless fermions
in inverted contacts
JETP Lett. **42**, 178 (1985)





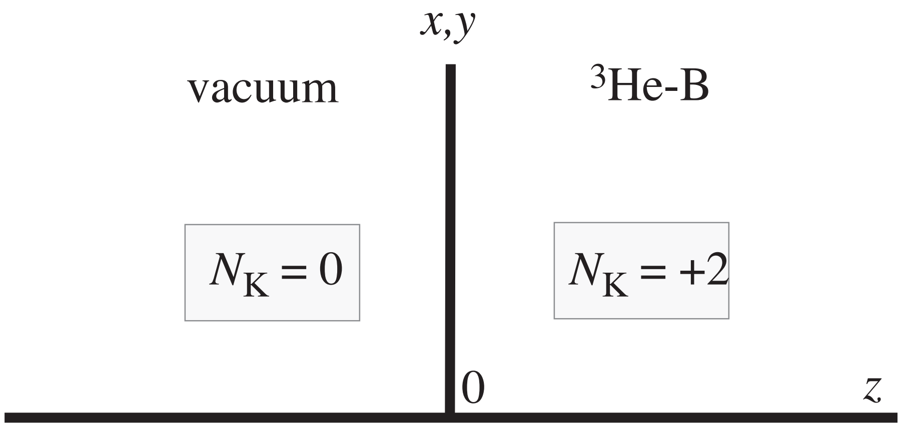


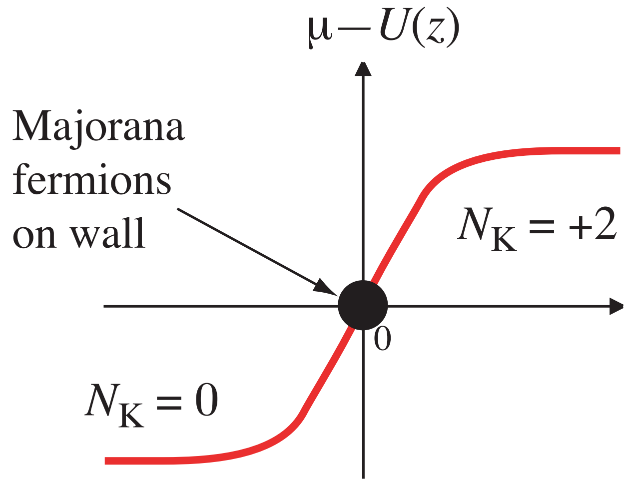
in Bi_2Te_3 Dirac point is below FS:
nodal line on surface of topological insulator

Majorana fermions: edge states on the boundary of 3D gapped topological matter

$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu + U(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu - U(z) \end{pmatrix}$$

* boundary of topological superfluid ^{3}He-B



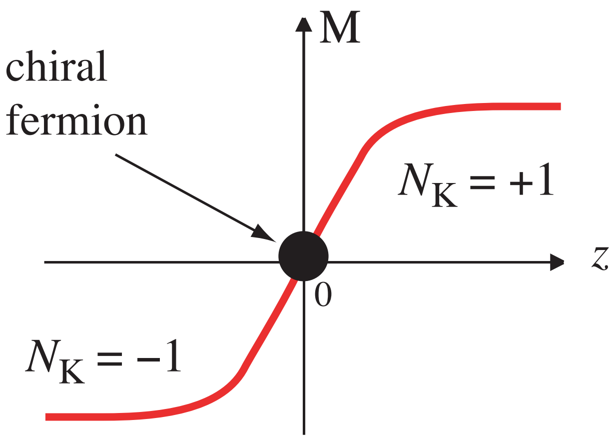


spectrum of fermion zero modes

$$H_{zm} = c\,(\sigma_x p_y - \sigma_y p_x)$$

helical fermions

* Dirac domain wall

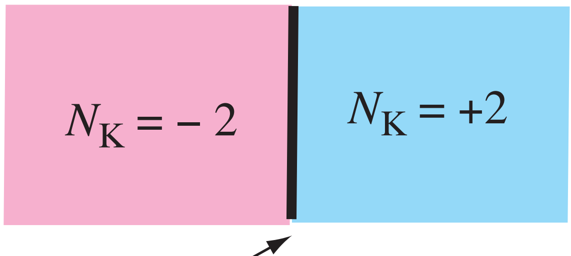


$$H = \begin{pmatrix} -M(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & +M(z) \end{pmatrix}$$

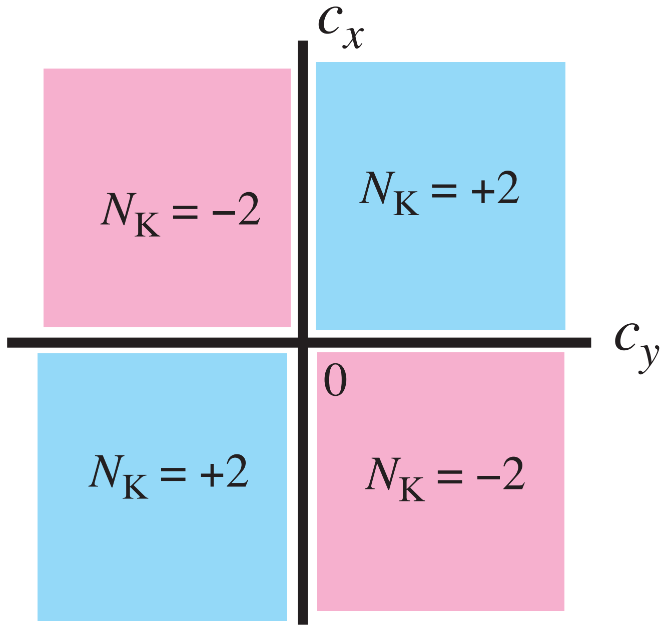
Volkov-Pankratov,
2D massless fermions
in inverted contacts
JETP Lett. **42**, 178 (1985)

Majorana fermions on interface in topological superfluid ^{3}He-B

$$
H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z \\ \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z & -\frac{p^2}{2m^*} + \mu \end{pmatrix}
$$

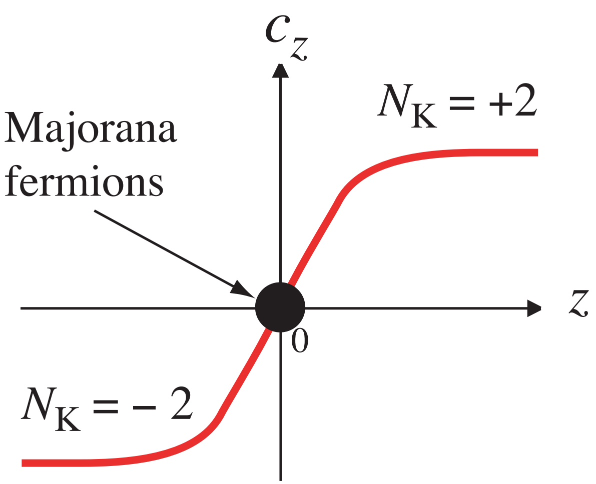


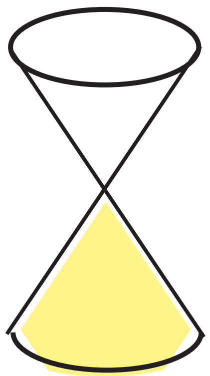
domain wall



phase diagram

one of 3 "speeds of light" changes sign across wall





spectrum of fermion zero modes

$$H_{zm} = c\,(\sigma_x p_y - \sigma_y p_x)$$

Zero energy states in the core of vortices in topological superfluids

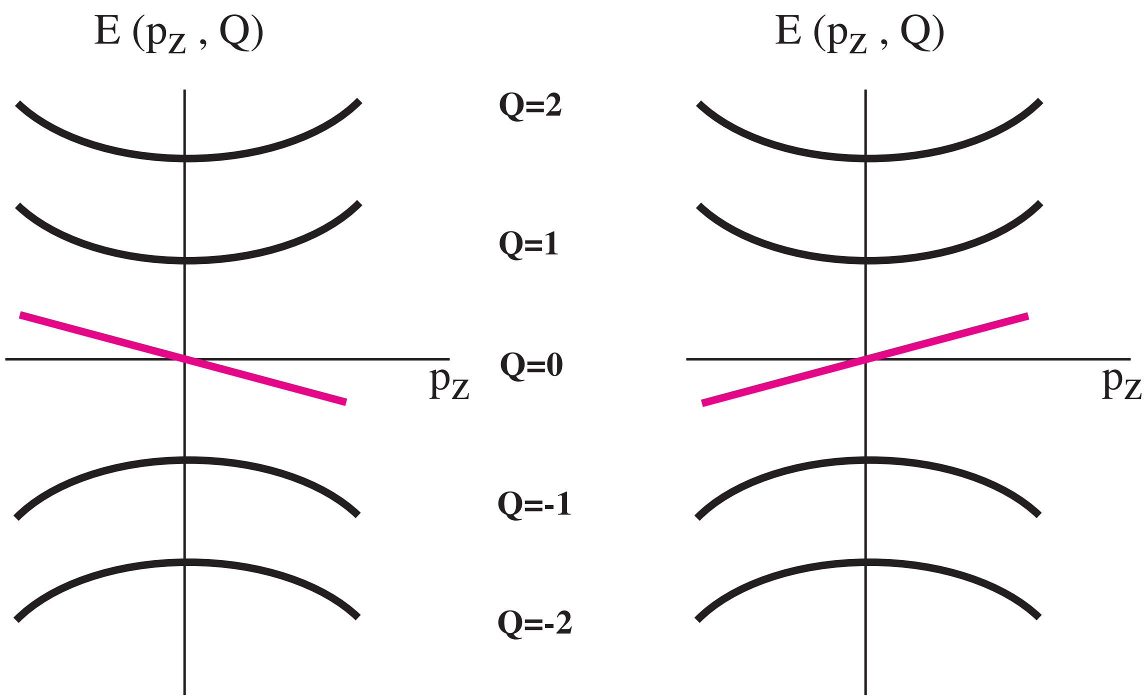
vortices in fully gapped 3+1 system

fermion zero modes in vortex core

Bound states of fermions on cosmic strings and vortices

Spectrum of quarks in core of electroweak cosmic string

quantum numbers: Q - angular momentum & p_z - linear momentum



$E(p_z) = - cp_z$ for d quarks $\quad$ $E(p_z) = cp_z$ for u quark

asymmetric branches cross zero energy

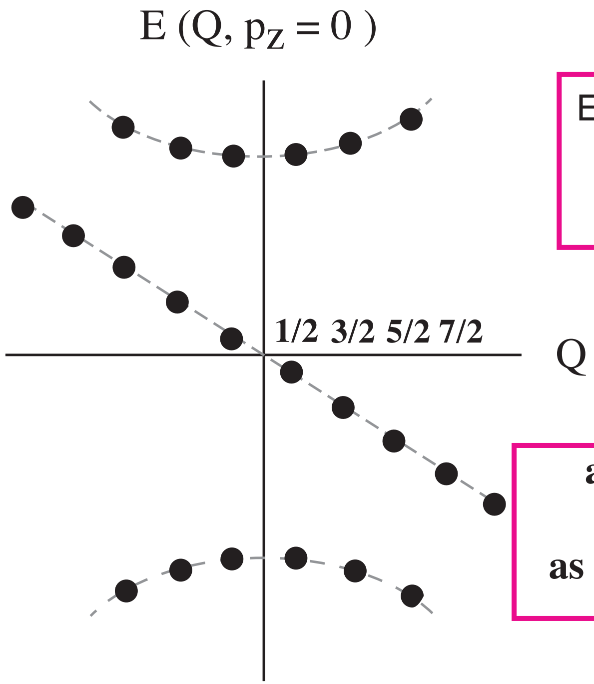
Index theorem: $\boxed{\text{Number of asymmetric branches} = N;\ N \text{ is vortex winding number}}$

Jackiw & Rossi
Nucl. Phys. B**190**, 681 (1981)

Bound states of fermions on vortex in s-wave superconductor

Caroli, de Gennes & J. Matricon, Phys. Lett. **9** (1964) 307

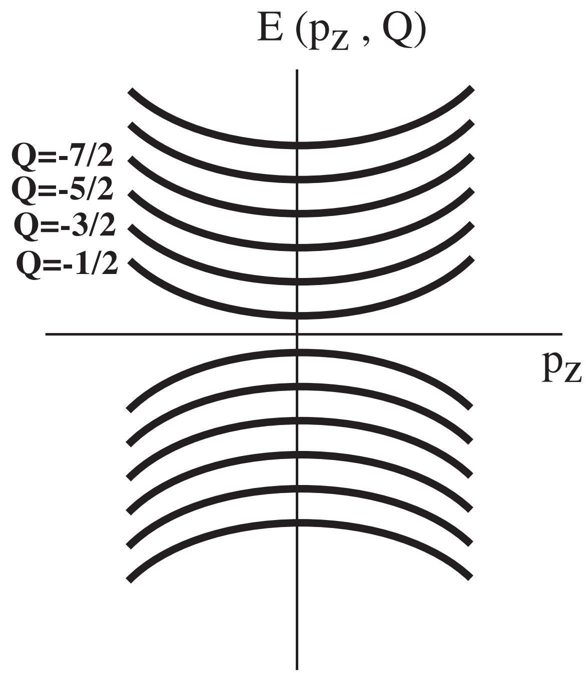
$N_K = 0$



$E(Q,p_z) = -Q\,\omega_0\,(p_z)$

$\omega_0 = \Delta^2/E_F << \Delta$

asymmetric branch as function of Q



Angular momentum Q is half-odd integer in s-wave superconductor

no true fermion zero modes: no asymmetric branch as function of p_Z

Index theorem for approximate fermion zero modes:

Number of asymmetric Q-branches = 2N
N is vortex winding number

GV JETP Lett. **57**, 244 (1993)

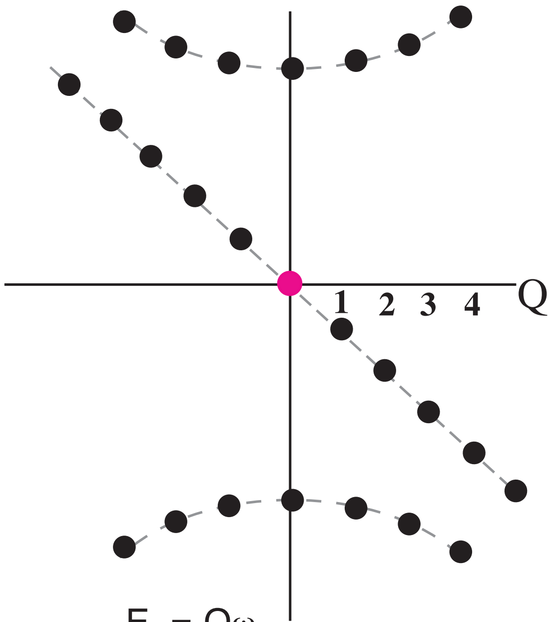
Index theorem for true fermion zero modes?

is the existence of fermion zero modes related to topology in bulk?

fermions zero modes on symmetric vortex in 3He-B

topological ^{3}He-B at $\mu > 0$: $N_K = 2$

$E\,(Q, p_z = 0\,)$

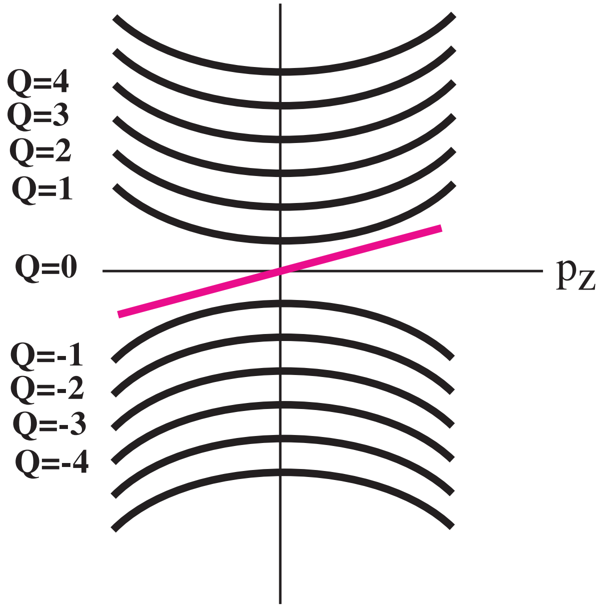


$E_Q = -Q\omega_0$

$\omega_0 = \Delta^2/E_F << \Delta$

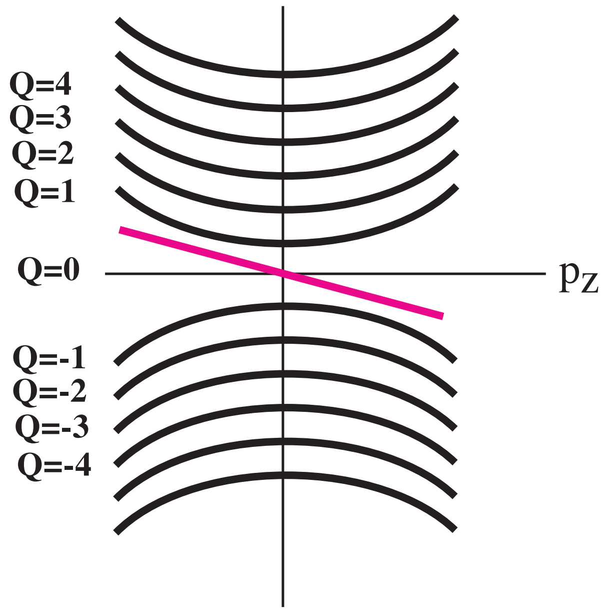
Q is integer
for p-wave superfluid ^{3}He-B

$E\,(p_z\,,\,Q)$



helicity +

$E\,(p_z\,,\,Q)$



helicity –

gapless fermions on Q=0 branch form

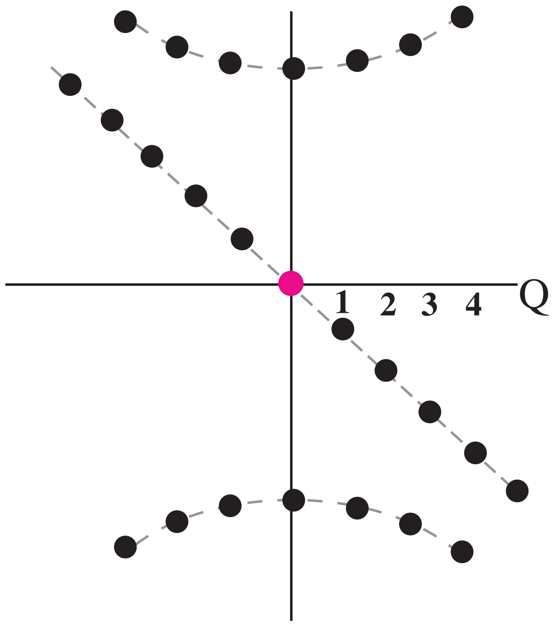
1D Fermi-liquid

Misirpashaev & GV
Fermion zero modes in symmetric vortices in superfluid 3He,
Physica B **210**, 338 (1995)

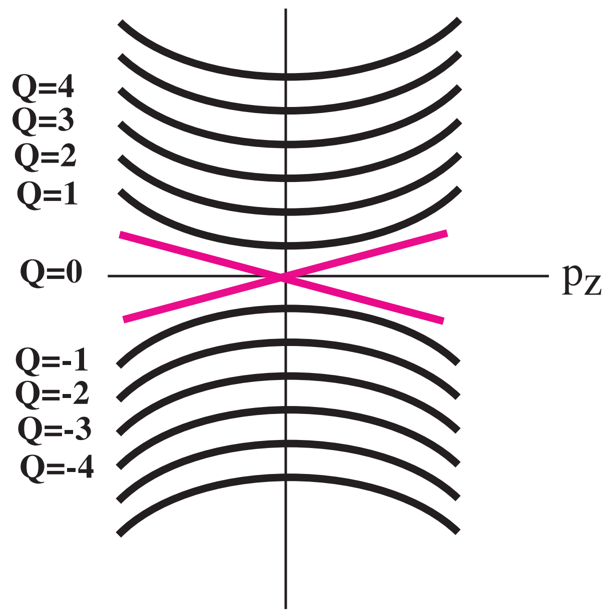
fermions zero modes on symmetric vortex in 3He-B

topological ^{3}He-B at $\mu > 0$: $\boxed{N_K = 2}$

E (Q, $p_z = 0$)



E (p_z , Q)



$E_Q = -Q\omega_0$

$\omega_0 = \Delta^2/E_F << \Delta$

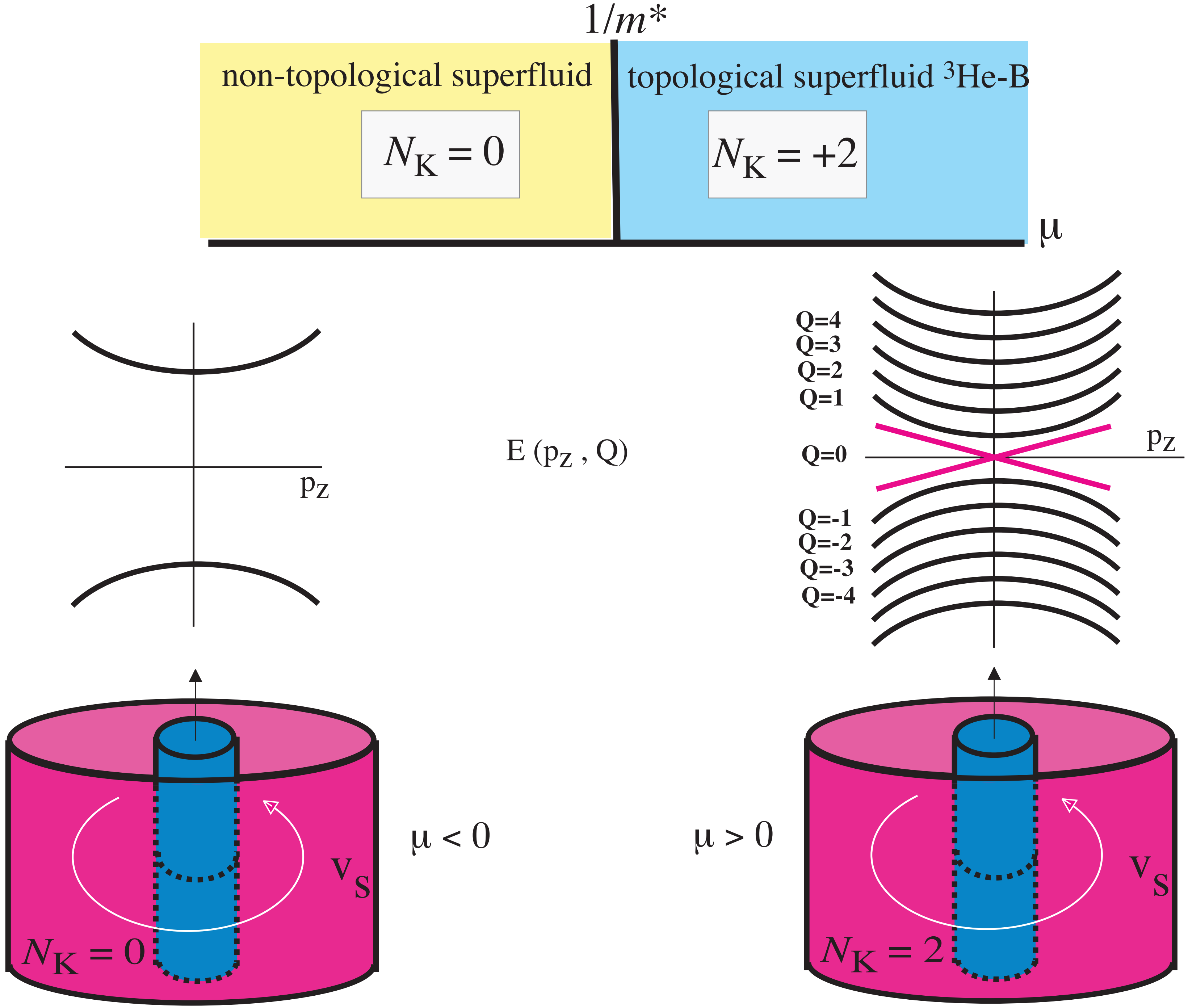
Q is integer
for p-wave superfluid ^{3}He-B

gapless fermions on Q=0 branch form

$\boxed{\text{1D Fermi-liquid}}$

Misirpashaev & GV
Fermion zero modes in symmetric vortices in superfluid 3He,
Physica B **210**, 338 (1995)

topological quantum phase transition in bulk & in vortex core



superfluid ^{3}He-B as non-relativistic limit of relativistic triplet superconductor

$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

relativistic triplet superconductor

$$\Big\downarrow \quad \begin{matrix} cp \ll M \\ \mu \ll M \end{matrix}$$

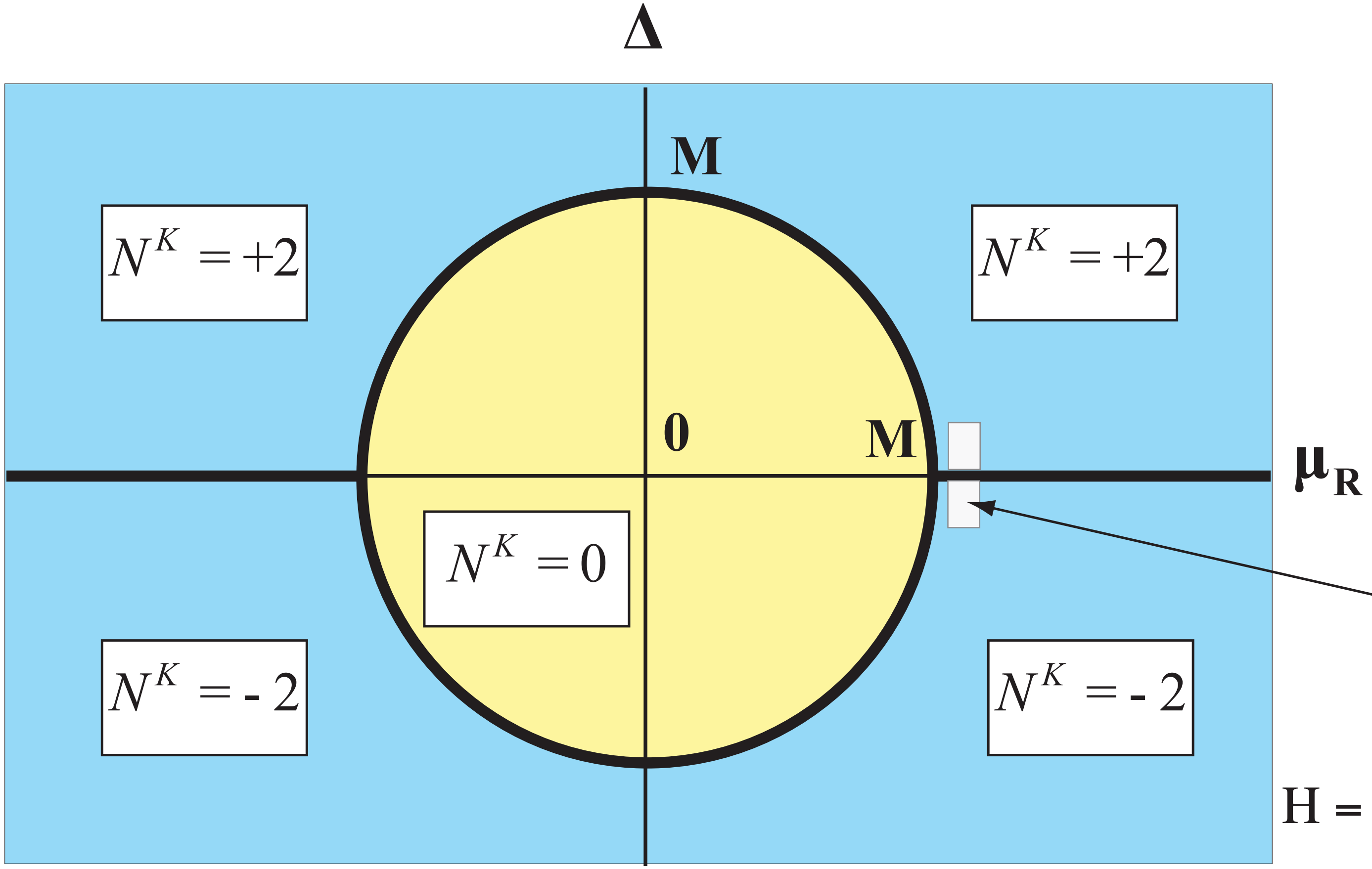
$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c_B\sigma\cdot\mathbf{p} \\ c_B\sigma\cdot\mathbf{p} & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

superfluid ^{3}He-B

$$c_B = c\,\Delta/M \qquad m = M/c^2$$

$$(\mu + M)^2 = \mu_R^2 + \Delta^2$$

phase diagram of topological states of relativistic triplet superconductor

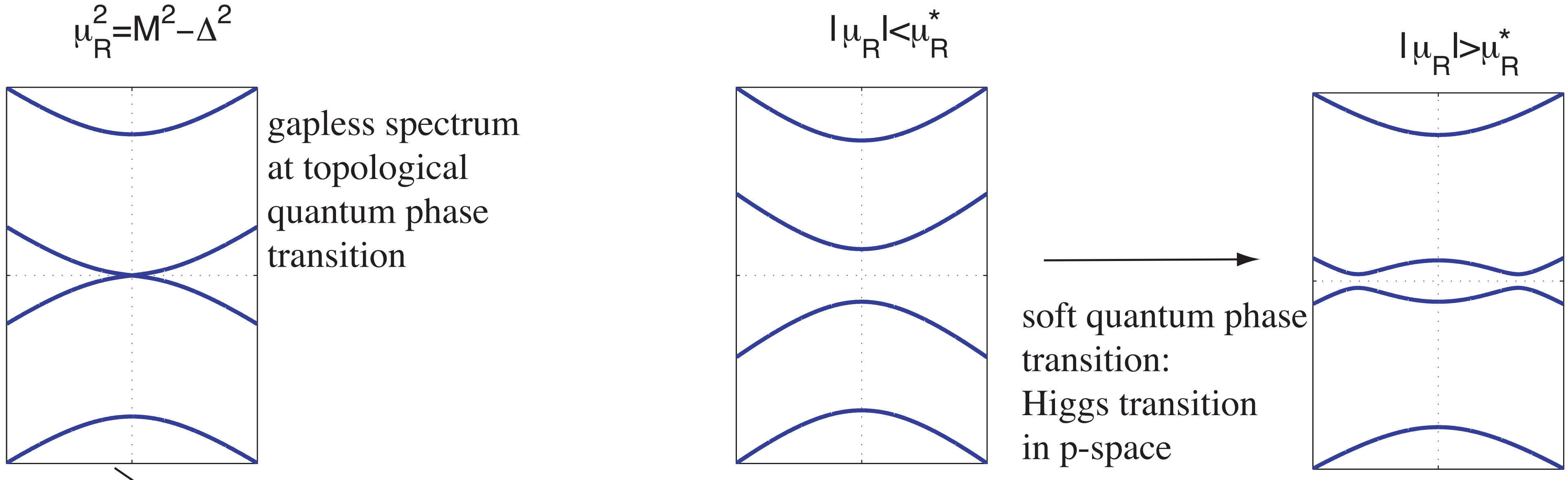


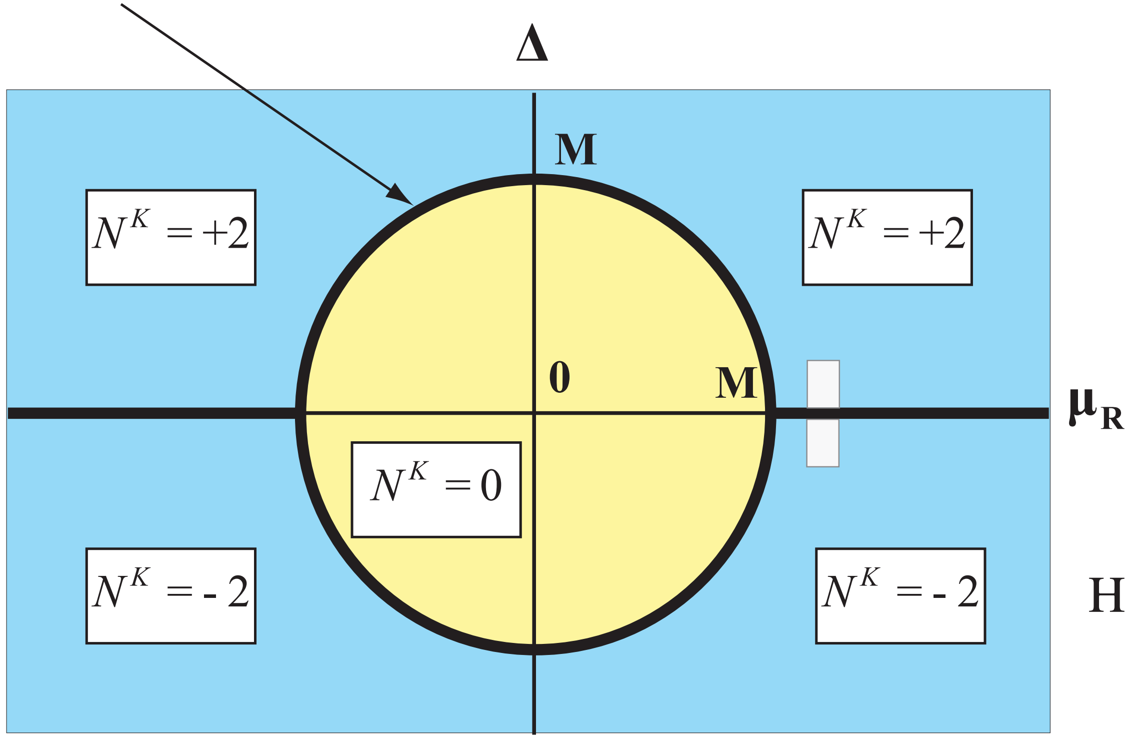
^{3}He-B

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5 \Delta \\ \gamma_5 \Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

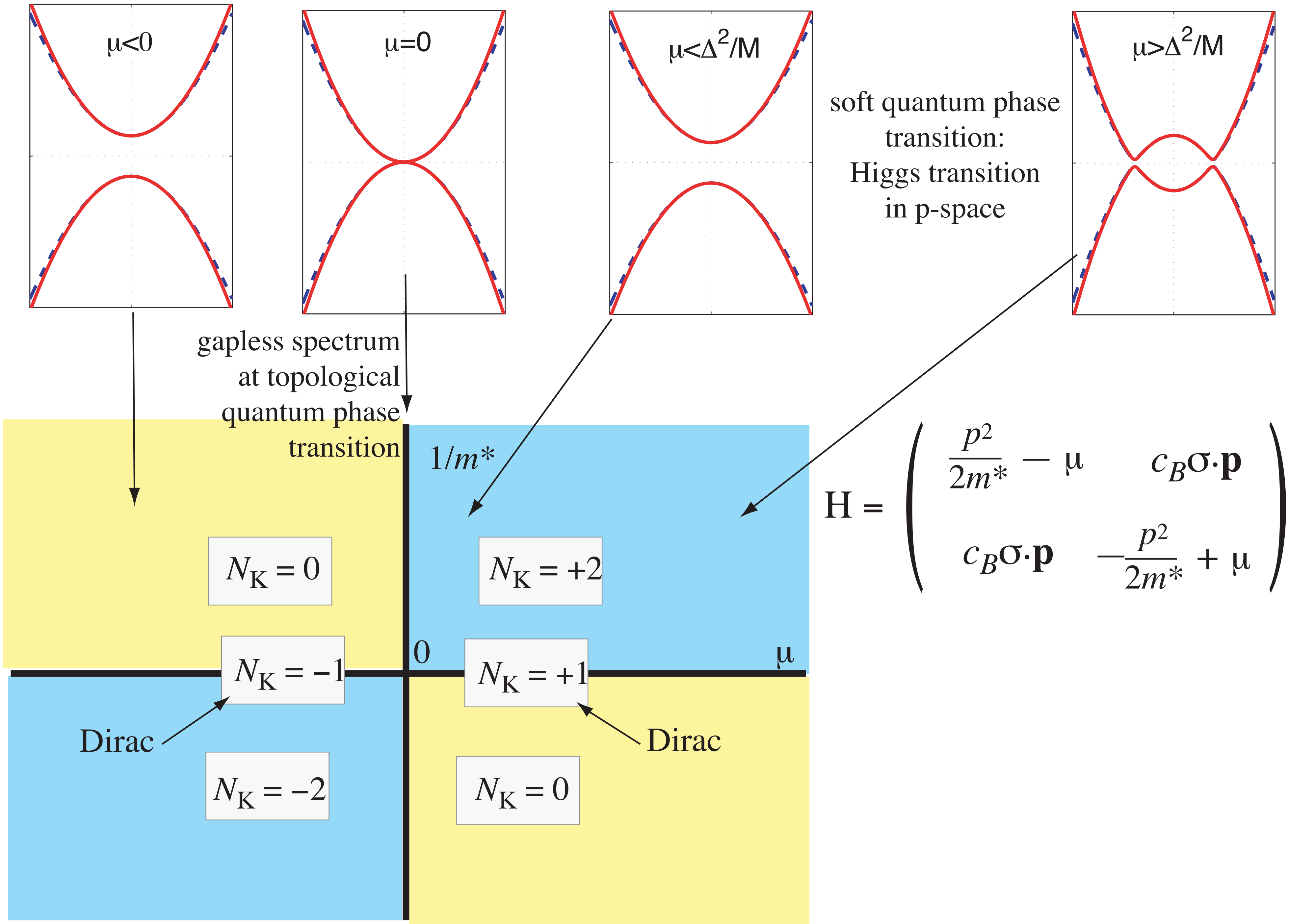
energy spectrum in relativistic triplet superconductor





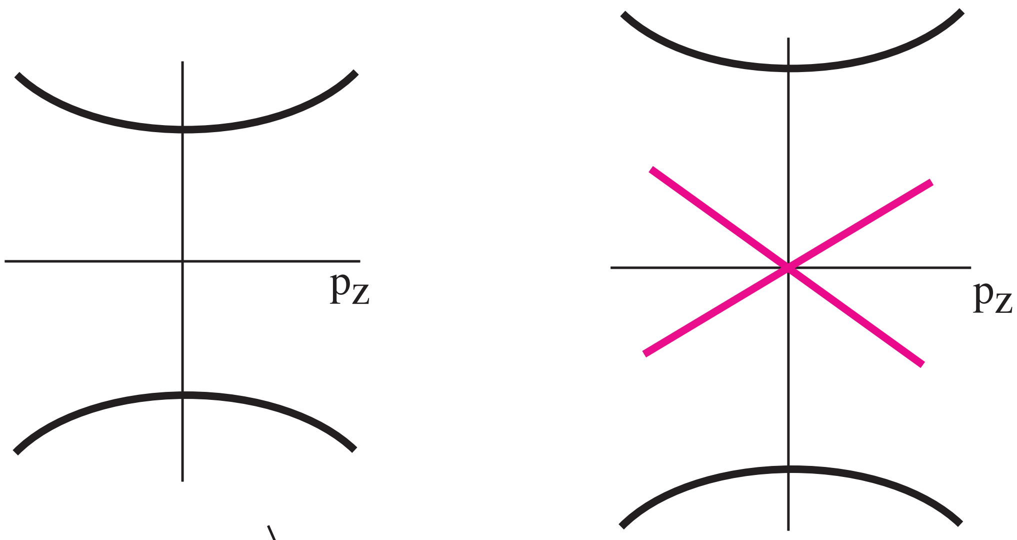
$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

spectrum of non-relativistic ^{3}He-B

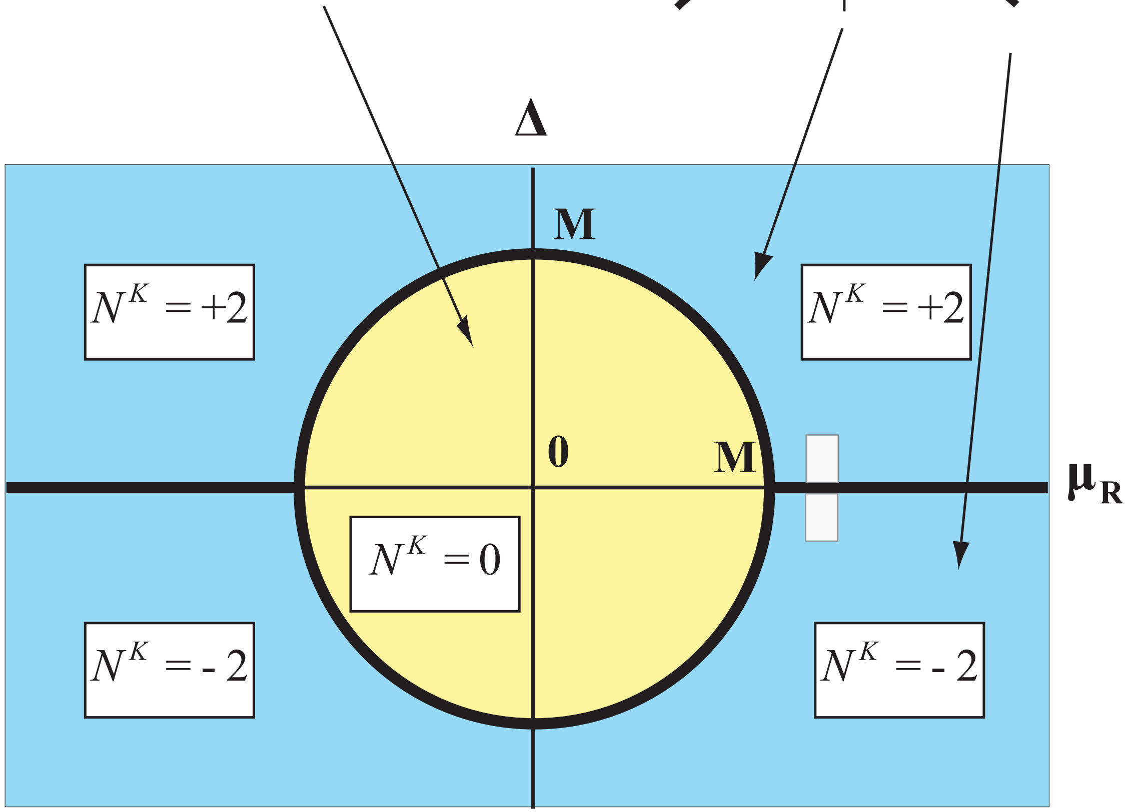


$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu \end{pmatrix}$$

fermion zero modes in relativistic triplet superconductor



$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$



vortices in topological superconductors have fermion zero modes

generalized index theorem ?

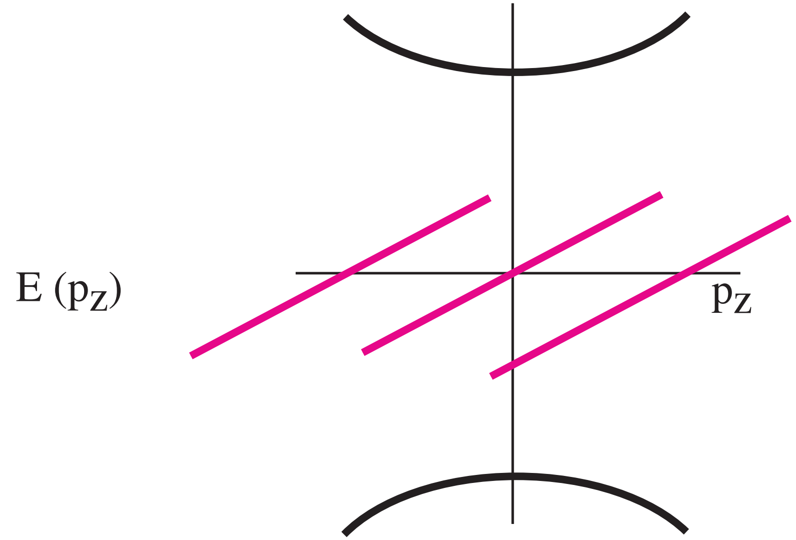
index theorem for fermion zero modes on vortices

(interplay of ***r***-space and ***p***-space topologies)

$$\boxed{N_5 = \frac{1}{4\pi^3 i}\mathbf{tr}\left[\int d^3p\, d\omega\, d\phi\ \mathbf{G}\,\nabla_\omega \mathbf{G}^{-1}\,\mathbf{G}\,\nabla_\phi \mathbf{G}^{-1}\mathbf{G}\,\nabla_{p_x} \mathbf{G}^{-1}\mathbf{G}\,\nabla_{p_y} \mathbf{G}^{-1}\mathbf{G}\,\nabla_{p_z} \mathbf{G}^{-1}\right]}$$

for vortices in Dirac vacuum

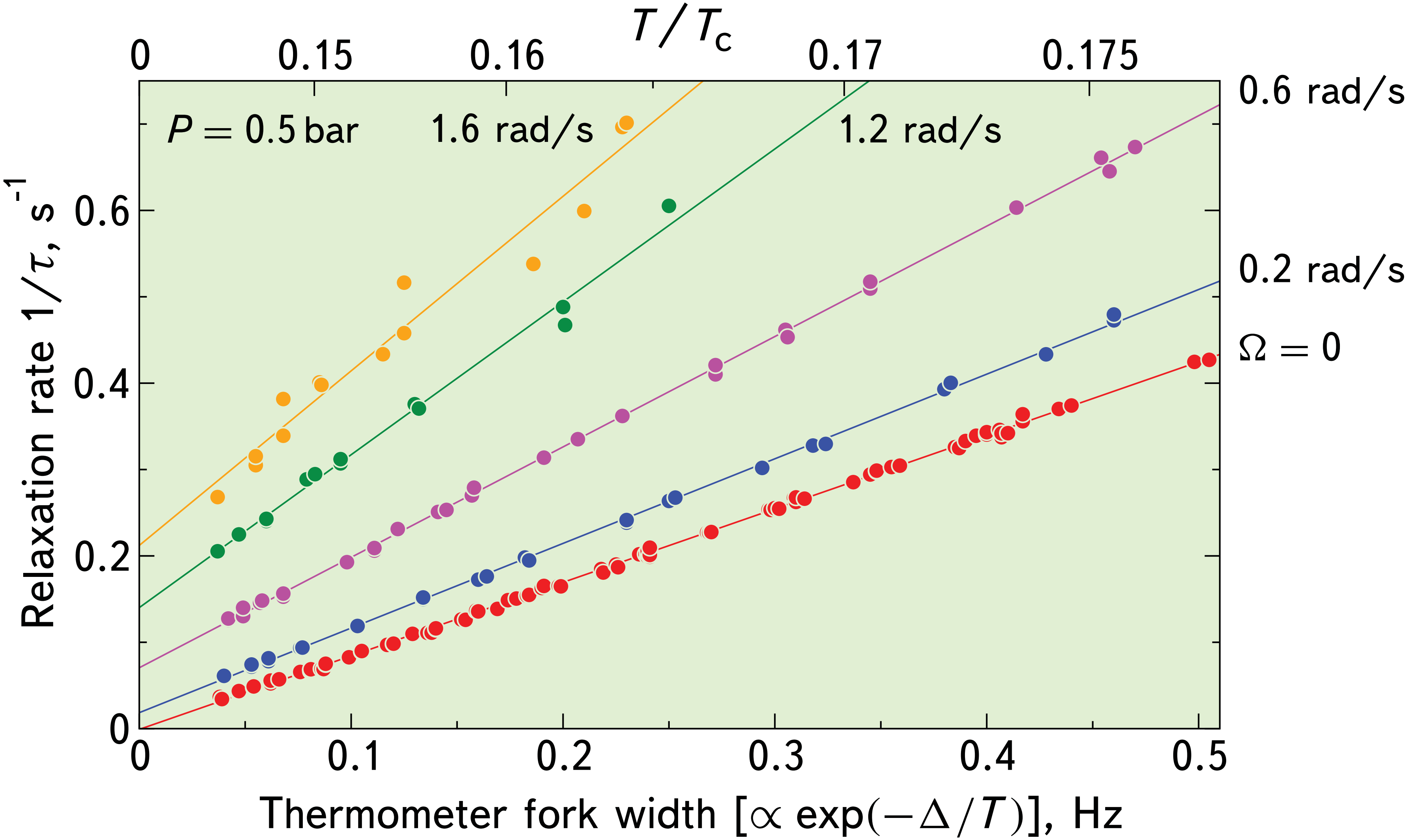
$N_5 = N$ **winding number**



N_5 invariant was introduced by Golterman, Jansen & Kaplan for lattice fermions
Phys. Lett. B **301** (1993) 219

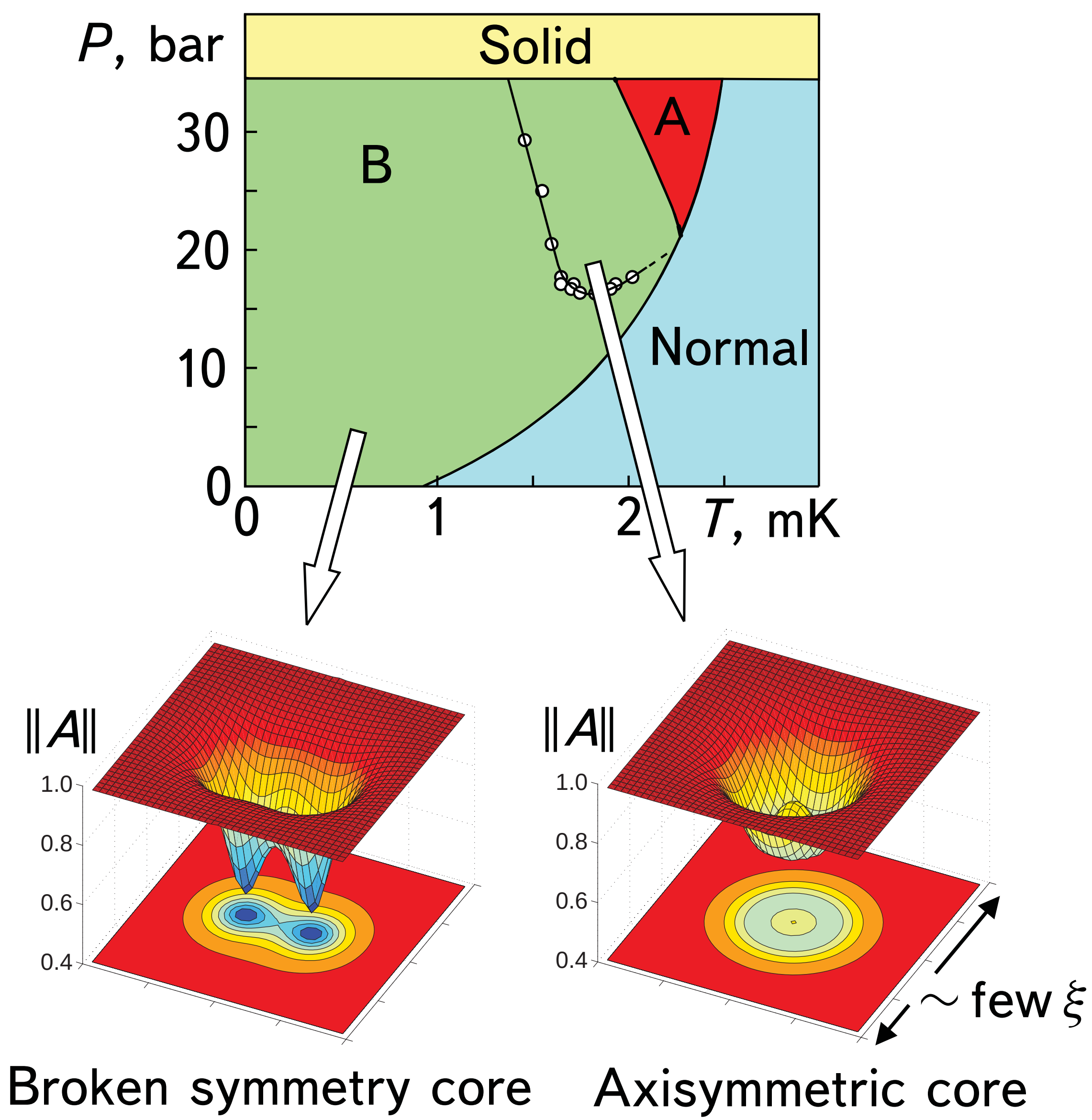
see also M.A. Zubkov & GV
Momentum space topological invariants for the 4D relativistic vacua with mass gap
Nucl. Phys. B **860** (2012) 295

RELAXATION IN THE VORTEX STATE



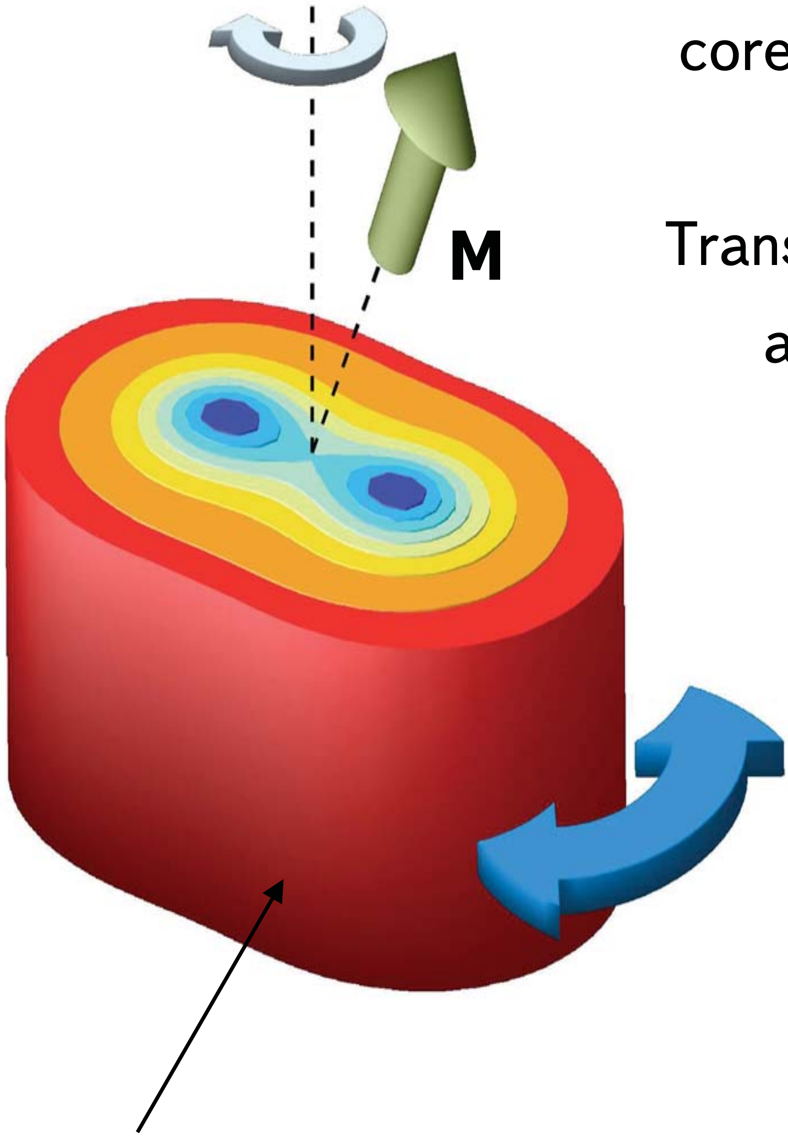
Relaxation rate: $1/\tau = 1/\tau_0(\Omega) + C(\Omega)\,\exp(-\Delta/T)$

BROKEN SYMMETRY OF VORTEX CORES IN ^{3}He-B



Ikkala, Hakonen, Bunkov, Krusius et al 1982-
Salomaa, Volovik, Thuneberg et al

DAMPING OF SPIN PRECESSION VIA VORTEX CORES



Torque from precessing magnetic moment puts vortex core in twisting motion (oscillations / precession)

$\Downarrow$

Transitions between the core-bound fermion states are triggered and the core gets overheated

$\Downarrow$

Dissipation

(Kopnin and Volovik, 1998)

Core of the non-axisymmetric vortex

Conclusion

Momentum-space topology determines:

universality classes of quantum vacua

effective field theories in these quantum vacua

topological quantum phase transitions (Lifshitz, plateau, etc.)

quantization of Hall and spin-Hall conductivity

topological Chern-Simons & Wess-Zumino terms

quantum statistics of topological objects

edge states (bulk-surface correspondence)

fermion zero modes on quantum vortices (bulk-vortex correspondence)

chiral anomaly, chiral magnetic effect, spectral flow force in vortex dynamics ...

exotic fermions: Dirac fermions with quadratic, cubic, quartic ... spectrum,
flat band, Fermi arc, Majorana fermions, etc.

effective gravity, where tetrad $e_a{}^{\mu}$ is more fundamental than metric $g^{\mu\nu}$

new type of gravity (Horava gravity with anisotropic scaling)