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Truncated CFT Approach to Matrix Product States For Non-Abelian FQH States

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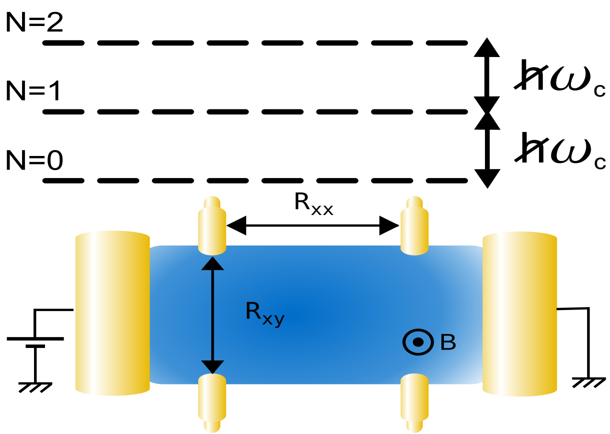
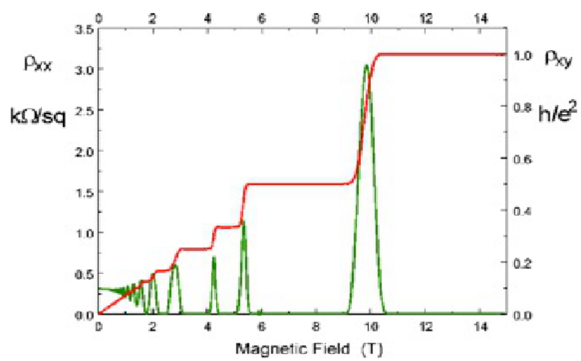
Quantum Hall Problem

Landau levels (spinless case)



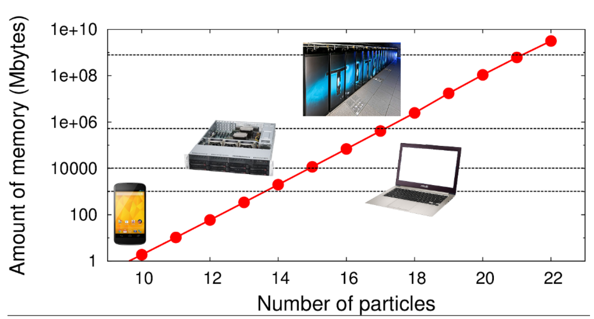


Integer QHE

- Cyclotron frequency : $\omega_c = \frac{eB}{m}$
- Lowest Landau level : orbitals $\langle z|\, m\rangle = z^m \exp\left(-|z|^2/4l^2\right)$

Fractional QHE

- Partial filling + interaction $\rightarrow$ FQHE
- FQHE is a hard N-body problem
- Gapped bulk, massless edge, anyons, ...



The most powerful computer in the world can't store more than 21 particles !

Bulk-Edge Correspondence

Fractional Quantum Hall Trial States have identical form as CFT corelators in the bulk (why? – not known - just an observation)

$$\Psi_{\rm L}(\{z_i\}) = \prod_{i<j}(z_i - z_j)^M e^{-\sum_i \frac{|z_i|^2}{4l}}$$

$$\lim_{z_\infty \to \infty} z_\infty^{MN^2} \langle V_{\rm el,1} V_{\rm el,2} \cdots V_{{\rm el},N} : e^{-i\sqrt{M}N\varphi}(z_\infty) : \rangle \qquad V_{\rm el}(z_i) = : e^{i\sqrt{M}\varphi}(z_i) :$$

This CFT is widely believed to be the same one as on the edge of the sample.

This however cannot work for any state described by a non-unitary CFT (Gapless bulk?)

What is the bulk-edge correspondence for non-unitary states?

Limitations of exact diagonalizations

ED provide the decomposition of any eigenstate $|\Psi\rangle$ on a convenient occupation basis $|m_0, ..., m_{N_\Phi}\rangle$

$$|\Psi\rangle = \sum_{\{m_i\}} c_{\{m_i\}} |m_0, ..., m_{N_\Phi}\rangle$$

Exponential growth of Hilbert space: 2^N coefficients needed

Any state can be written as

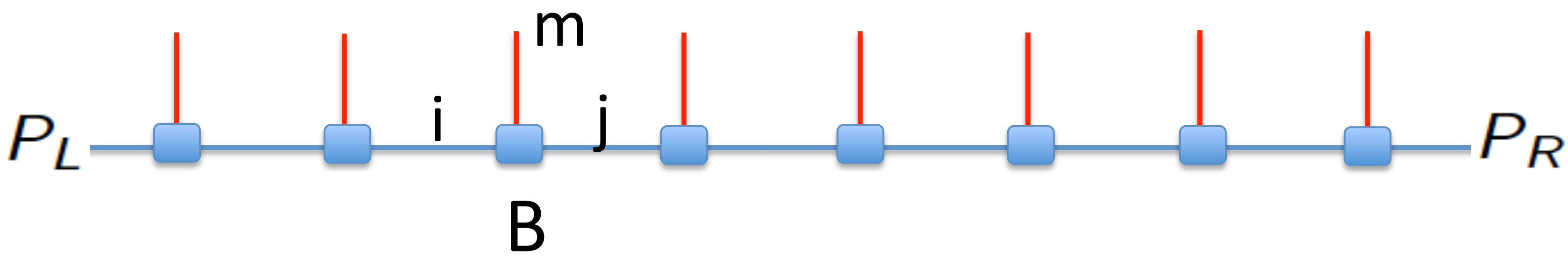
$$|\Psi\rangle = \sum_{\{m_i\}} P_L B^{[m_0]} ... B^{[m_{N_\Phi}]} P_R \; |m_0, ..., m_{N_\Phi}\rangle$$

where the $B^{[m]}$ are matrices and P_L, P_R projectors on given matrix elements.
In some cases :

- The size of the B matrices can be much smaller than the Hilbert space (one dimensional systems)
- The size of the B matrices still grows exponentially but a controlled truncation can be applied

Matrix Product States for Fractional Quantum Hall

m=0,1 are the unoccupied/occupied electron Hilbert space. i, j are the bond dimensions of the MPS



Zaletel and Mong (arXiv :1208.4862) : MPS for Laughlin and MR

$$|\Psi\rangle = \sum_{\{m_i\}} \mathrm{Tr}\left(B^{m_1} B^{m_2} \cdots B^{m_n}\right) |m_1 \cdots m_n\rangle$$

where the MPS matrices B^m can be computed analytically (underlying CFT is non interacting)

Zalatel and Mong used the well-known orbital basis as 1D basis for the 2D FQH problem computed entanglement spectrum for Laughlin, Moore-Read

What Can Matrix Product States In The FQH Gain Us?

1. Build GS and qh Wavefunctions not achievable by ED: for example, superconfomal N=1 wavefunctions or other Virasoro minimal models. unitary vs nonunitary.

2. Test screening properties of plasmas, conjectures about many-component plasmas

3. Compute Braiding

4. Compute Entanglement Properties (spectrum, topological entanglement entropy, etc)

5. Compute Correlation Functions

6. Flows, unitary vs nonunitary

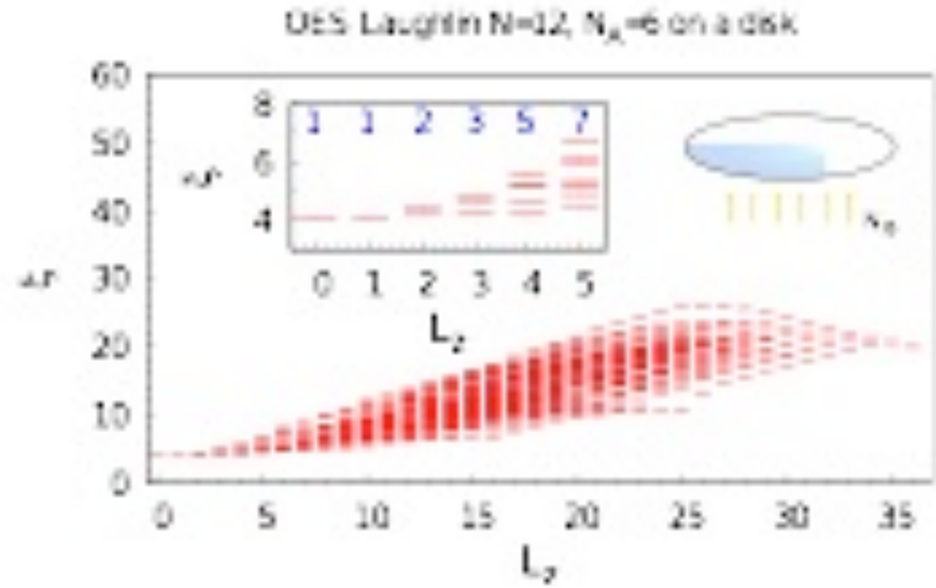
7. MPS representations directly from the differential operator fixing the state, integrability

Where does the MPS structure come from?

FQH trial wave-function from CFT

(2d) Conformal field theory : field theory of scale invariant (massless) systems

- **Infinite but graded Hilbert space :**
  - $1 : |0\rangle$
  - $1 : a_{-1}\,|0\rangle$
  - $2 : a_{-1}^2\,|0\rangle,\ a_{-2}\,|0\rangle$
  - $3 : a_{-1}^3\,|0\rangle,\ a_{-2}a_{-1}|0\rangle,\ a_{-3}\,|0\rangle$
  - $5 : a_{-1}^4\,|0\rangle,\ a_{-2}a_{-1}^2\,|0\rangle,\ a_{-2}^2\,|0\rangle,$
    $a_{-3}a_{-1}\,|0\rangle,\ a_{-4}\,|0\rangle$
  - $7 : \cdots$



- **Operators** $V(z) = \sum_n z^n V_n$
- **Correlation functions** $\langle u|V_1(z_1)\cdots V_n(z_n)|v\rangle$

Trial wave function $\Psi(z_1,\cdots,z_n) = \langle u|V(z_1)\cdots V(z_n)|v\rangle$

Matrix Product State for FQH trial wave-function

$\Psi(z_1, \cdots, z_n) = \langle u | V(z_1) \cdots V(z_n) | v \rangle$ is a MPS in disguise !

Dubail, Read, Rezayi (2012)

$$|\Psi\rangle = \sum_{\{m_i\}} \left(\langle u | \, B^{m_1} B^{m_2} \cdots B^{m_n} \, | v \rangle \right) | m_1 \cdots m_n \rangle$$

+ numerical checks for non-interacting CFTs (Laughlin and Moore-Read)

Zalatel and Mong (2012)

$$B^0 = e^{-i\sqrt{\nu}\varphi_0}, \qquad B^1 = V_0 \, e^{-i\sqrt{\nu}\varphi_0}$$

- φ_0 is the bosonic zero mode (B_0 shifts the electric charge by $1/q$)
- V_0 is the matrix of the electron operator $V(z) = \sum_n z^n V_n$

$$\langle \beta | V_0 | \alpha \rangle = \delta_{\Delta_\beta, \Delta_\alpha} \, \langle \beta | V(1) | \alpha \rangle$$

MPS ingredients

We have obtained the MPS representation for a large series of non-abelian FQH states whose underlying CFT is interacting
(Estienne, Papic, Regnault, BAB, 2012)

What is required for a numerical implementation ?

- build the basis $|\alpha\rangle$ (auxiliary space)
- compute the matrix elements $\langle\beta|B^m|\alpha\rangle$
- meaningful truncation scheme

Building the Hilbert Space

The auxiliary space $|\alpha\rangle$:
CFT Hilbert space

$$\mathrm{U}(1)\otimes\mathrm{CFT}_n$$

$$V(z) = :e^{i\sqrt{q}\varphi(z)}: \otimes \Phi(z)$$

$\Phi(z)$ lives in the so-called neutral conformal field theory CFT_n

The U(1) Hilbert space

The CFT factorizes $\mathcal{H} = \mathcal{H}_{\text{neutral}} \otimes \mathcal{H}_{U(1)}$

as a neutral CFT times a U(1) chiral free boson.

$$\varphi(w) = \varphi_0 - i a_0 \log(w) + i \sum_{n \neq 0} \frac{1}{n} a_n w^{-n}$$

Primary states $|Q\rangle$ are defined by their U(1) charge Q

$$a_0 |Q\rangle = Q|Q\rangle, \qquad a_n|Q\rangle = 0 \text{ for } n > 0$$

The Hilbert space is simply a Fock space

Descendants are obtained with the lowering operators $a_n^\dagger = a_{-n}$, $n > 0$

$$|Q, \mu\rangle = \prod_{i=1}^{n} a_{-\mu_i} |Q\rangle, \qquad a_0|Q,\mu\rangle = Q|Q,\mu\rangle$$

with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0$

The neutral Hilbert space

L_n (modes of the stress-energy tensor) obey the **Virasoro algebra**

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

Primary fields $|\Delta\rangle$ are annihilated by the positive modes

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \qquad L_n|\Delta\rangle = 0 \qquad n > 0$$

Descendant states : lowering operators $L_n^\dagger = L_{-n}$, $n > 0$

$$|\Delta, \lambda\rangle = L_{-\lambda_1} L_{-\lambda_2} \cdots L_{-\lambda_n}|\Delta\rangle$$

Two issues :

- these states are not orthogonal
- they might not even be independant !

$\Rightarrow$ **No closed formula, has to be implemented numerically.**

Building up an orthonormal basis

The level of a descendant $|\Delta, \lambda\rangle$ is just the size of the partition $|\lambda| = \sum_j \lambda_j$.

e.g. at level 2 we have two states : $L_{-1}^2|\Delta\rangle$, $L_{-2}|\Delta\rangle$.

At each level we compute the overlap matrix between descendants $\langle\Delta, \lambda'|\Delta, \lambda\rangle$.

e.g. at level 2 we have $\begin{pmatrix} 4\Delta(2\Delta+1) & 6\Delta \\ 6\Delta & 4\Delta + \frac{c}{2} \end{pmatrix}$

- if positive definite, Gramm-Schmidt
- if states with **vanishing** norm (null-states), they have to be discarded

Or build the Gram matrix to have nonzero determinant from very beginning

From CFT the number of null-vectors is known : check for numerics.

How to compute the MPS matrix elements:

The U(1) part

CFT factorization : $V(z) = \Phi(z) \otimes : e^{i\sqrt{q}\varphi(z)} :$

where $\Phi(z)$ is a primary field in the neutral CFT.

The matrix elements of the vertex operator

$$\langle Q', \mu' | : e^{i\beta\varphi(1)} : | Q, \mu \rangle = \delta_{Q', Q+\beta} \, A_{\mu', \mu}$$

can be easily computed through the commutation relation

$$\left[a_m \, , \, e^{i\beta\varphi(z)} \right] = \beta \, e^{i\beta\varphi(z)}$$

$$A_{\mu',\mu} = \prod_{j\geq 1} \sum_{r=0}^{m'_j} \sum_{s=0}^{m_j} \frac{(-1)^s}{r!s!} \left(\frac{\beta}{\sqrt{j}} \right)^{r+s} \delta_{m'_j+s, m_j+r} \frac{\sqrt{(m_j+r-s)! m_j!}}{(m_j-s)!}$$

Works for quasiholes too.

The neutral part

Matrix element for an arbitrary primary field $\Phi^{(h)}(z)$ are of the form

$$\langle \Delta', \lambda' | \Phi^{(h)}(1) | \Delta, \lambda \rangle$$

They can be computed (in principle) using

$$[L_m - L_0, \Phi^{(h)}(1)] = m\, h\, \Phi^{(h)}(1)$$

where h is the conformal dimension of $\Phi^{(h)}(z)$.

$\Rightarrow$ **But no analytical closed formula, has to be implemented numerically.**

For more complicated CFTs (parafermions, W algebras, $N=1$ susy, S_3...), much more involved... but it can be done. For $k=3$ Jack states :

The underlying algebra is the $\mathcal{W}_3$ algebra :

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$
$$[L_n, W_m] = (2n-m)\,W_{n+m}$$
$$[W_n, W_m] = \frac{16}{22+5c}(n-m)\Lambda_{n+m} + \frac{c}{360}n(n^2-1)(n^2-4)\delta_{n+m,0}$$
$$+(n-m)\left[\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right] L_{n+m}$$

The matrix elements can be computed using :

$$\langle\alpha'|\,[W_n, \Psi(1)]\,|\alpha\rangle = C_n(\Delta, \Delta_{\alpha'}, \Delta_\alpha)\langle\alpha'|\Psi(1)W_n|\alpha\rangle$$
$$-\frac{6\omega(\Delta+1)}{\Delta(5\Delta+1)}\sum_{m\geq 1}\left[\langle\alpha'|L_{-m}\Psi(1)|\alpha\rangle + \langle\alpha'|\Psi(1)L_m|\alpha\rangle\right]$$

Truncation of the auxiliary CFT basis

The auxiliary space (i.e. the CFT Hilbert space) basis is of the form

$$|Q, \mu\rangle \otimes |\Delta, \lambda\rangle$$

The natural cut-off is the level $|\lambda| + |\mu| \leq P$.

- $P = 0$ recovers the thin-torus limit (Jack *root partition*)
- The c_λ are either exact or zero at a given truncation level
- In finite size the truncated MPS becomes exact for P large enough
- But the overlap is extremely good way before this $P(N)$

Just exactly how good is it for different states on different geometries?

"N=1 Superconformal" Wavefunctions

A conformal dimension 3/2 field with Z_2 fusion rules

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

$$\{\Psi_n, \Psi_m\} = \frac{3}{c}L_{n+m} + \frac{1}{2}\left(n^2 - \frac{1}{4}\right)\delta_{n+m,0}$$

$$[L_n, \Psi_m] = \left(\frac{1}{2}n - m\right)\Psi_{n+m}$$

Gives rise to clustered (k,r)=(2,6) wavefunctions (Jackson, Simon, Read, 2013): they die as 6^{th} power of coordinates when 3 particles come together. C= -21/4 is a Jack polynomial M(3,8) (Haldane, BAB, 2008) and C=7/10 is M(4,5).

$$|\Delta, \lambda, \mu\rangle = L_{-\lambda_1} \cdots L_{-\lambda_n} \Psi_{-\mu_1} \cdots \Psi_{-\mu_m} |\Delta\rangle$$

Central charge is a free parameter, and one can probe irrational, non unitary and unitary cases with the same algebra. (of course, for irrational we don't know mps)

We have the MPS for RR Z=3.. and for all the (k,r)=(2,r) Jacks as well as N=1 superconformal states.

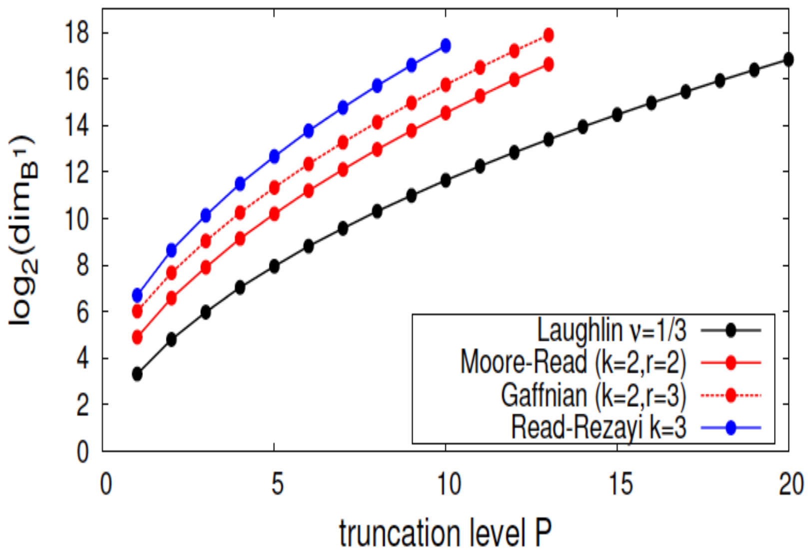
Tricks understood (easier ways of doing RR than the W algebra, dealing with null vectors, computing matrix elements, etc). Can be applied to truncated CFT in other areas

Estienne, Regnault, BAB, to appear

What are the properties of the MPS and what can we numerically do?

Benchmarks for MPS

P=0 approximation is the root partition of the FQH states (the Tao-Thouless state CDW in orbital space)



We obtain ALL the degenerate FQH states, even though we work on the plane. This is because the plane is effectively identical to the cylinder, and the infinite cylinder should contain all the degenerate states.

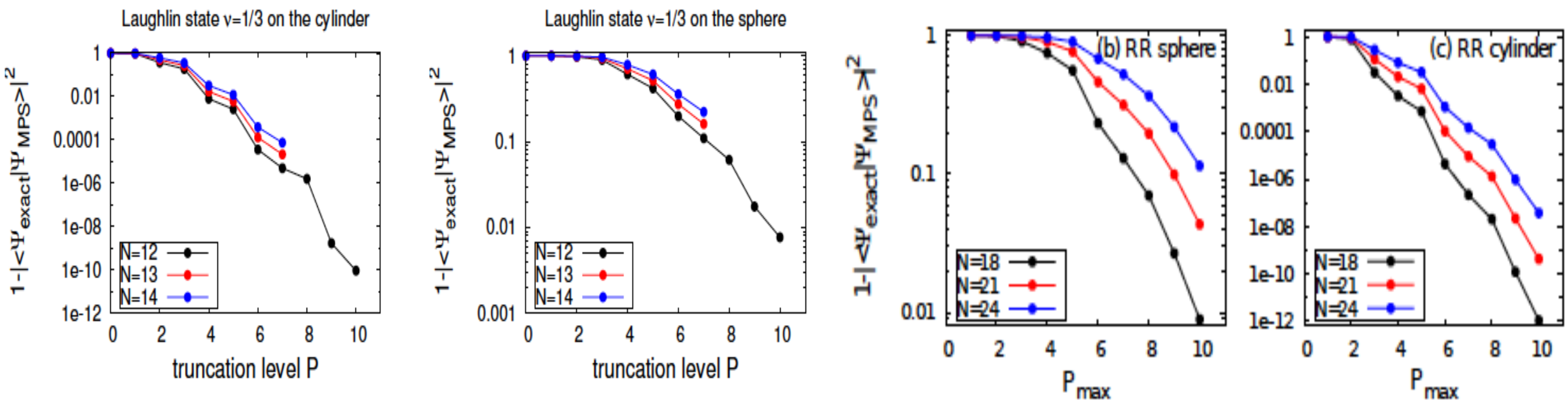
P=25 for Laughlin doable (B matrix 2*10^8 row dimension), but only P=11 for RR state (optimizing maybe goes to P=12).

We have obtained the (k,r) = (2,r) and (k, 2)- Z_k parafermions for cylinder, plane, sphere; for k>3, the method starts having increasing deficiencies due to large number of null vectors. k=3 has another description in terms of Virasoro CFT.

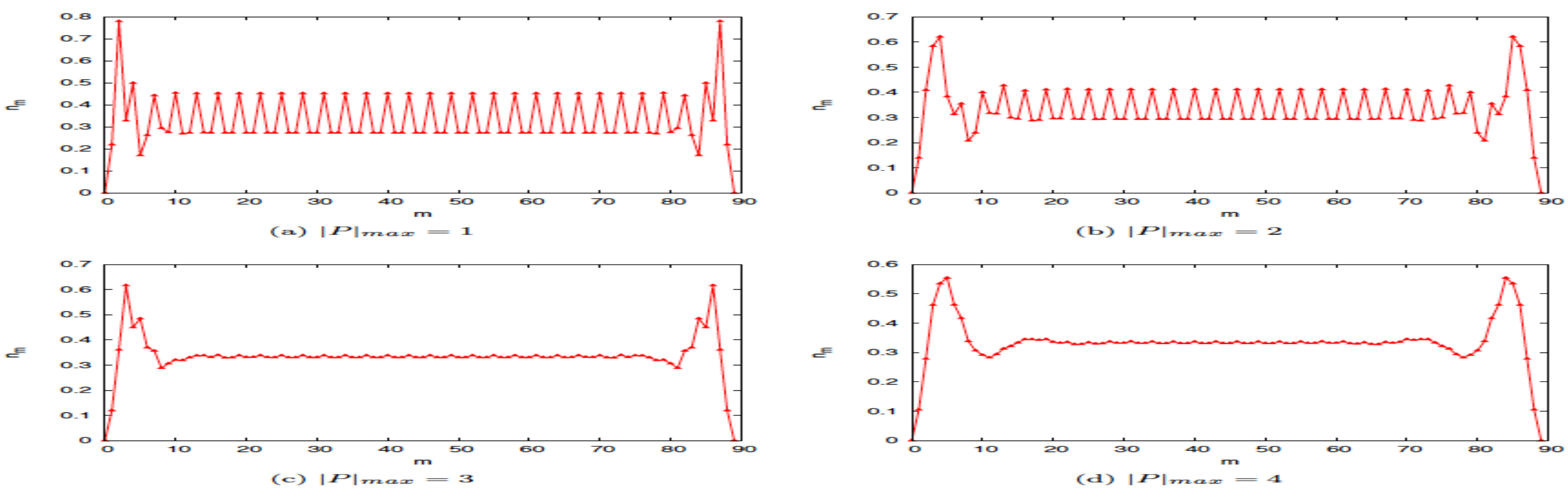
Power of MPS is in approximating upon truncation. For exact state, Jack polynomial machinery is faster!

MPS on Sphere VS Cylinder

Convergence much better on the cylinder than the sphere.



On cylinder, for Laughlin states, 30 particles can be simulated with P=5.



(a) $|P|_{max} = 1$

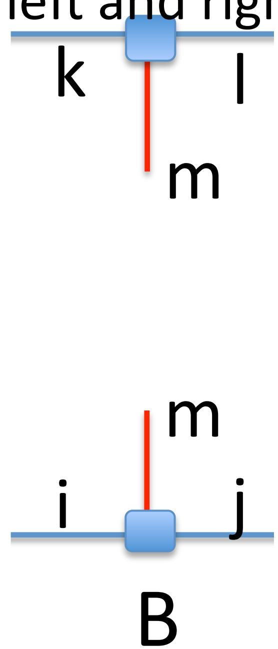
(b) $|P|_{max} = 2$

(c) $|P|_{max} = 3$

(d) $|P|_{max} = 4$

Computing With MPS

The E matrix is an important quantity in the MPS. Its largest eigenvalue gives the norm, the left and right eigenvectors are the projectors in the case of infinite MPS



$$E = \sum_m B^{m\star} \otimes B^m$$

The next eigenvalue bounds the decay of correlations.
Entanglement spectrum (real space, orbital, particle) in infinite limit, etc, computable from E matrix.
What are the E matrix properties for FQH states?

Properties of E matrix; Orbital, Particle and Real Space Spectra In Infinite Limit

The E-matrix, though built on plane, knows the state degeneracy (both abelian and nonabelian). Integrable structure of operators.

Build projector onto q degenerate states per neutral CFT sector:

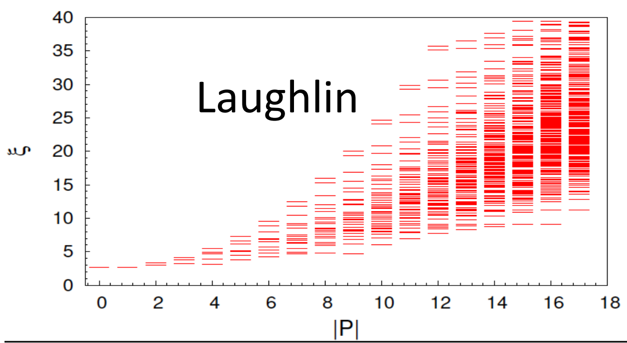
$$P_k,\, k = 0 \ldots q-1$$

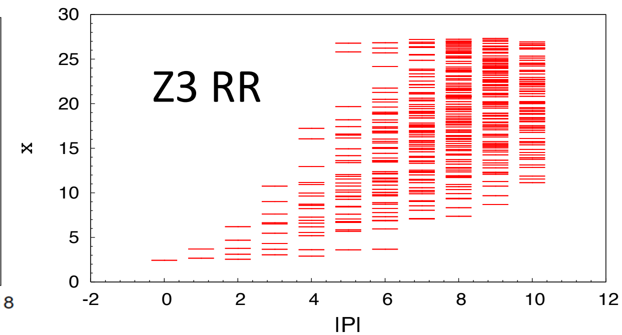
$$B^m P_k = P_{k-1} B^m$$

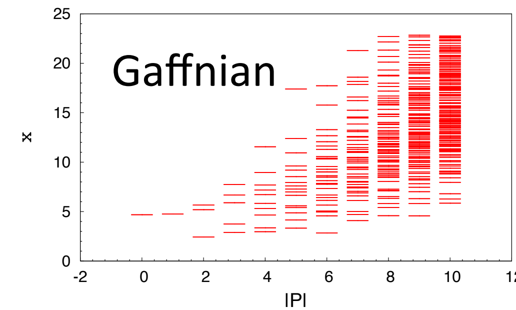
The largest eigenvalue eigenstate of the E matrix contains (depending on matrix elements) the orbital, particle and real space entanglement spectrum in the infinite limit.

$$E_{\alpha\beta} = \sum_{i=0}^{q-1} \lambda_i \psi_\alpha^i \phi_\beta^i$$

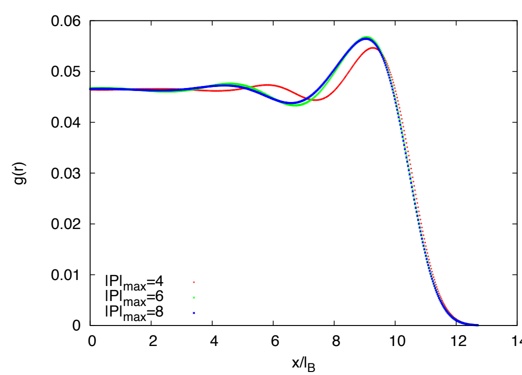
Entanglement Spectra

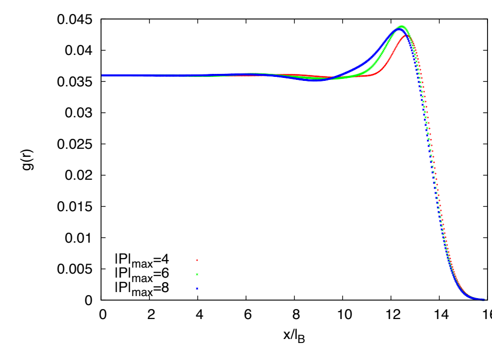






Pair correlation functions are also computable

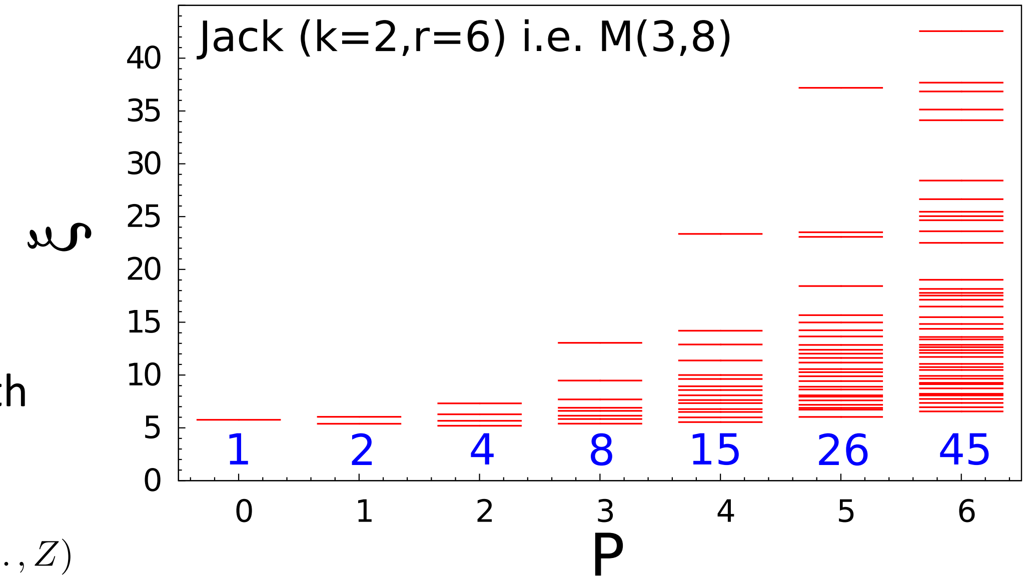




States With Same Clustering

In 2008 and 2009, we introduced a series of symmetric and antisymmetric polynomials, labeled by a a root partition (pattern of zeroes, thin torus limit – Bergholtz, Seidel, Kardhele, Barkeshli, Wen, Ardonne, Lee, and many others) with the property:

$$\prod_{i=1}^{N}(Z-z_i)^r J^{\alpha_{k,r}}_{\lambda^0(k,r)}(\{z\}) = J^{\alpha_{k,r}}_{\lambda^0(k,r)}(\{z\}, Z, \ldots, Z)$$

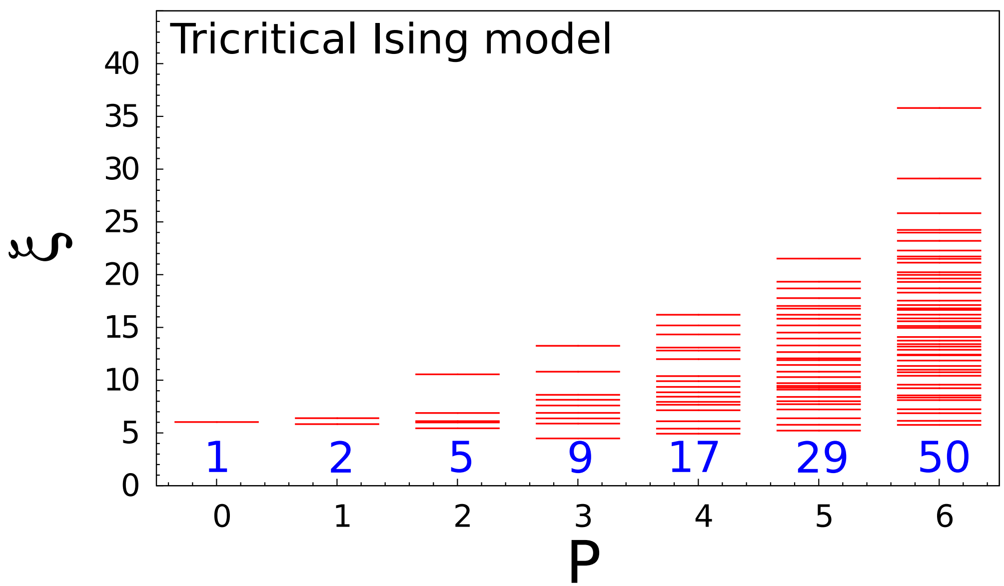


It was soon realized by us and others that:

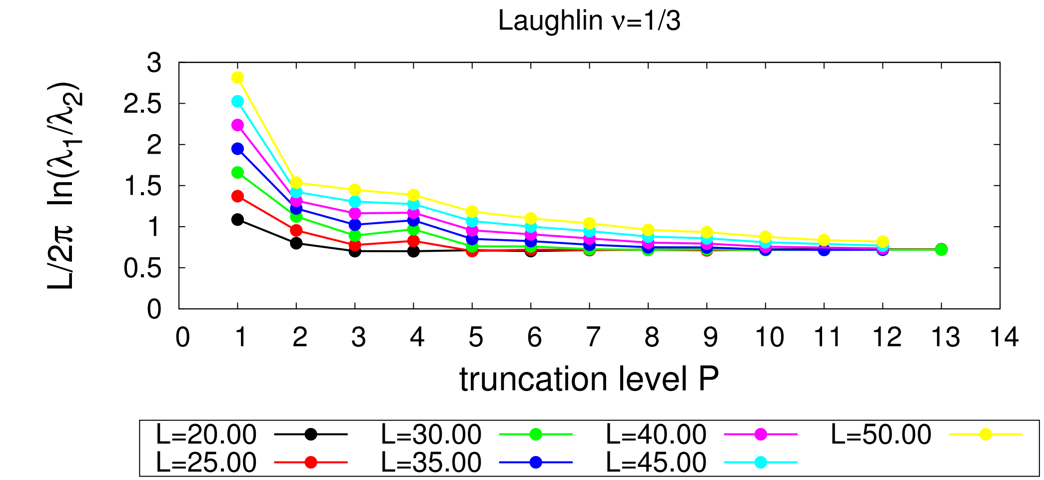
1. many of these states are nonunitary (see Read, 2009)

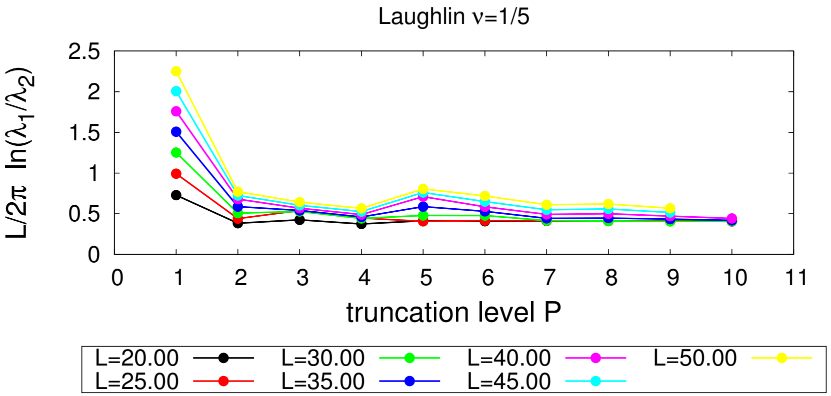
2. many are not uniquely defined by clustering.

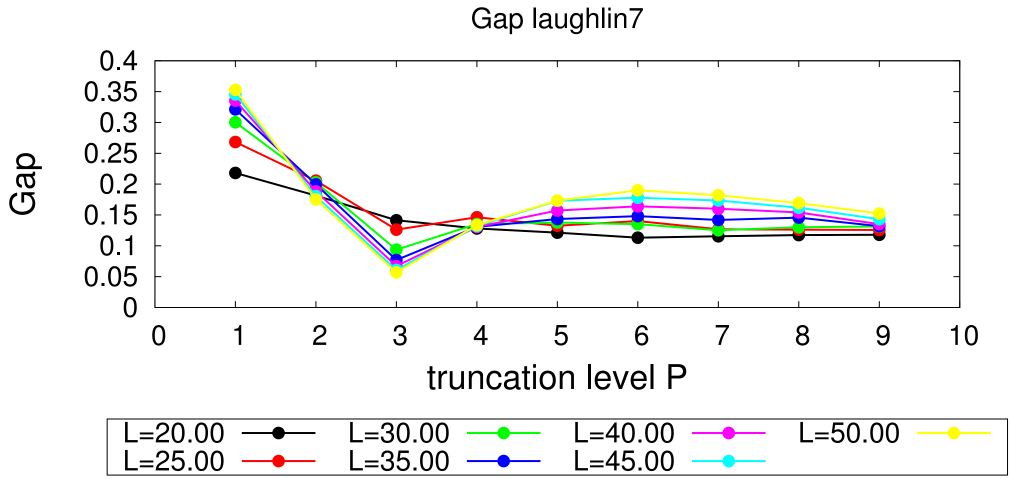
Question: can we distinguish them



Gap Of The E matrix







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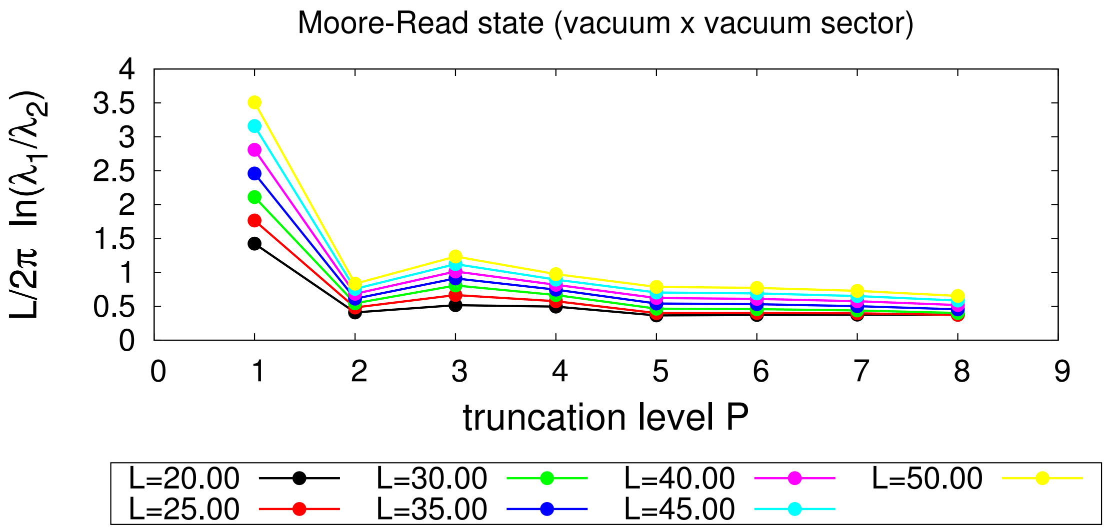
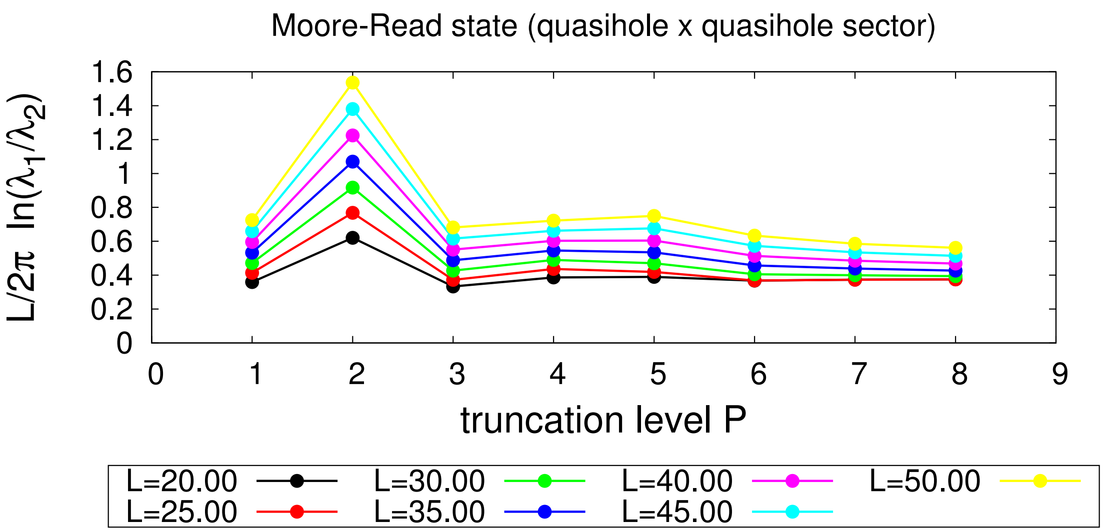
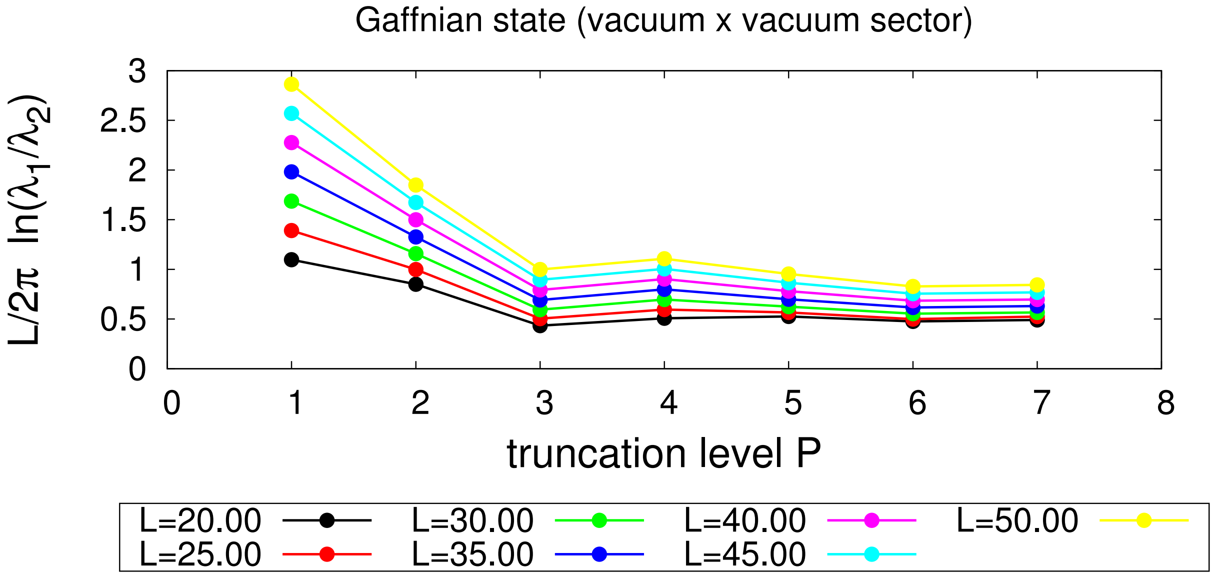
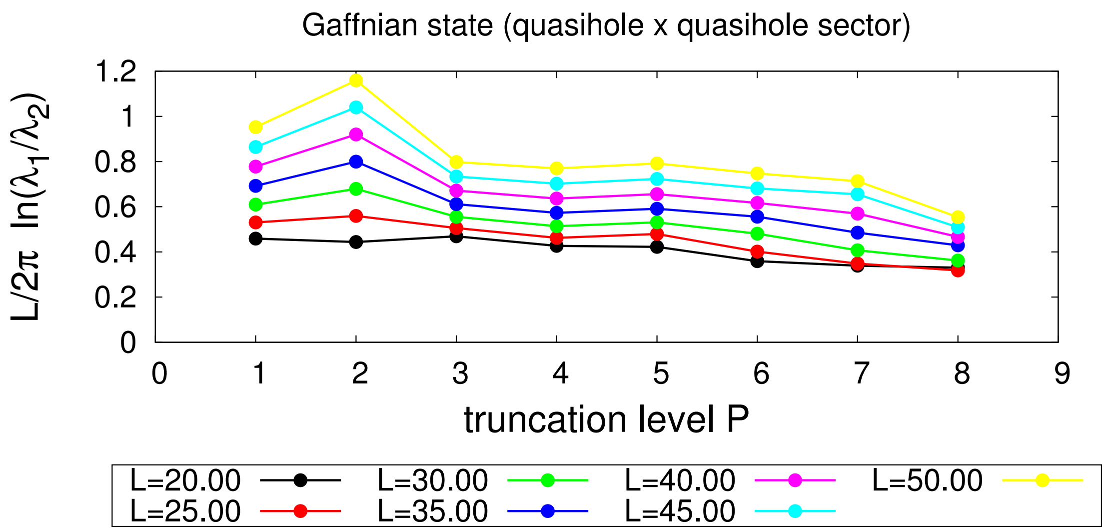


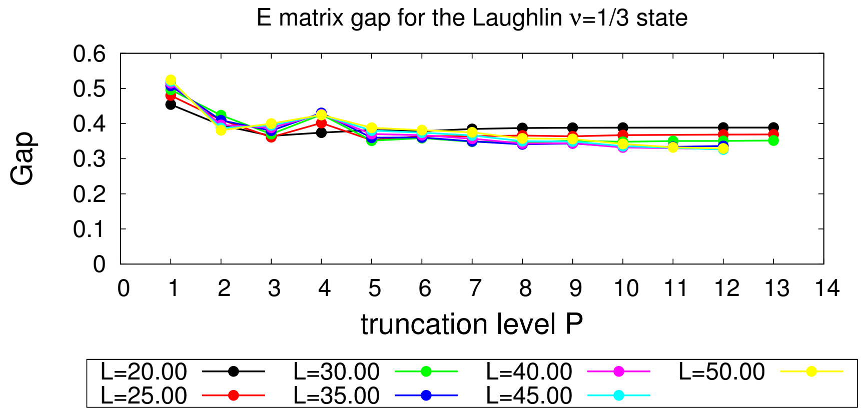


Gap Of The E matrix

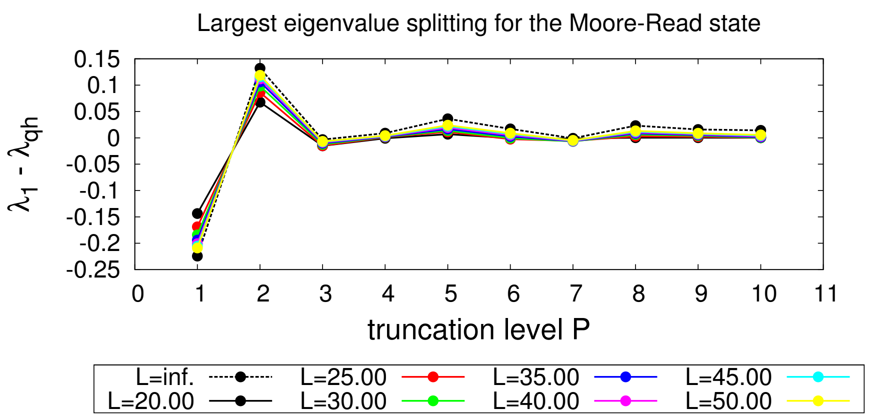


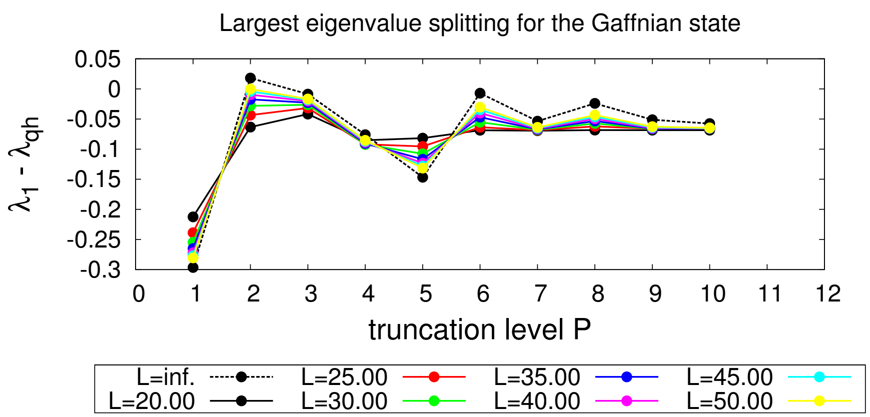


Transfer matrix for the FQH model states



- E matrix for a cylinder of perimeter L
- For the Laughlin $\nu = 1/3$, the gap seems to converge to a finite value
- For states such as the Moore-Read or the Gaffnian, we have to consider several sectors ($\{1, \Psi\}$ and $\{\sigma\}$ for MR)



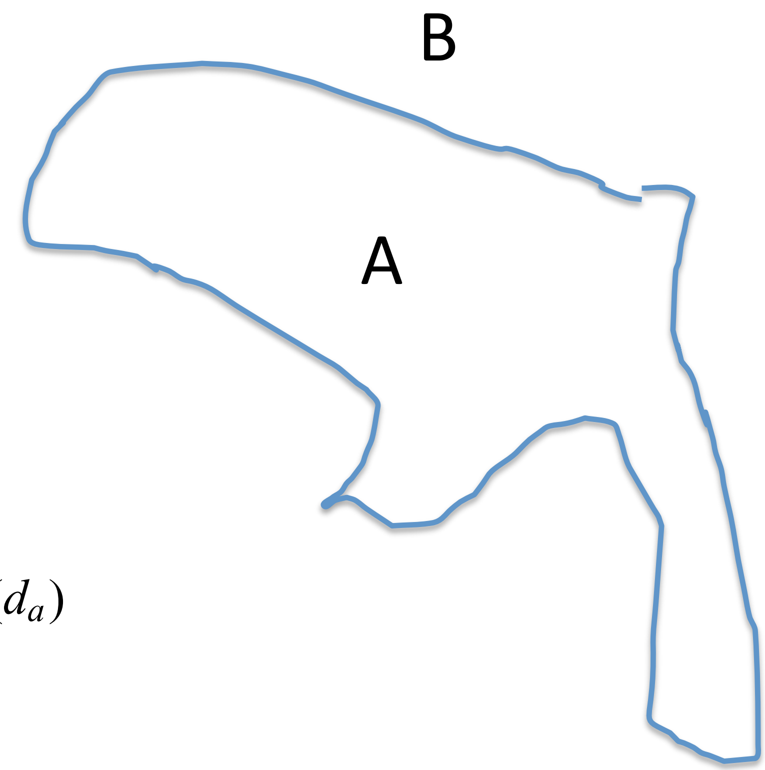


Entanglement Entropies: How Well Do They Behave And What Do the Nonunitary Wfs Show

Topological Entanglement Entropies:

One can obtain some info about the modular S matrix directly from the ground-state wavefunction

$$d_a = \frac{S_a^0}{S_0^0}.$$



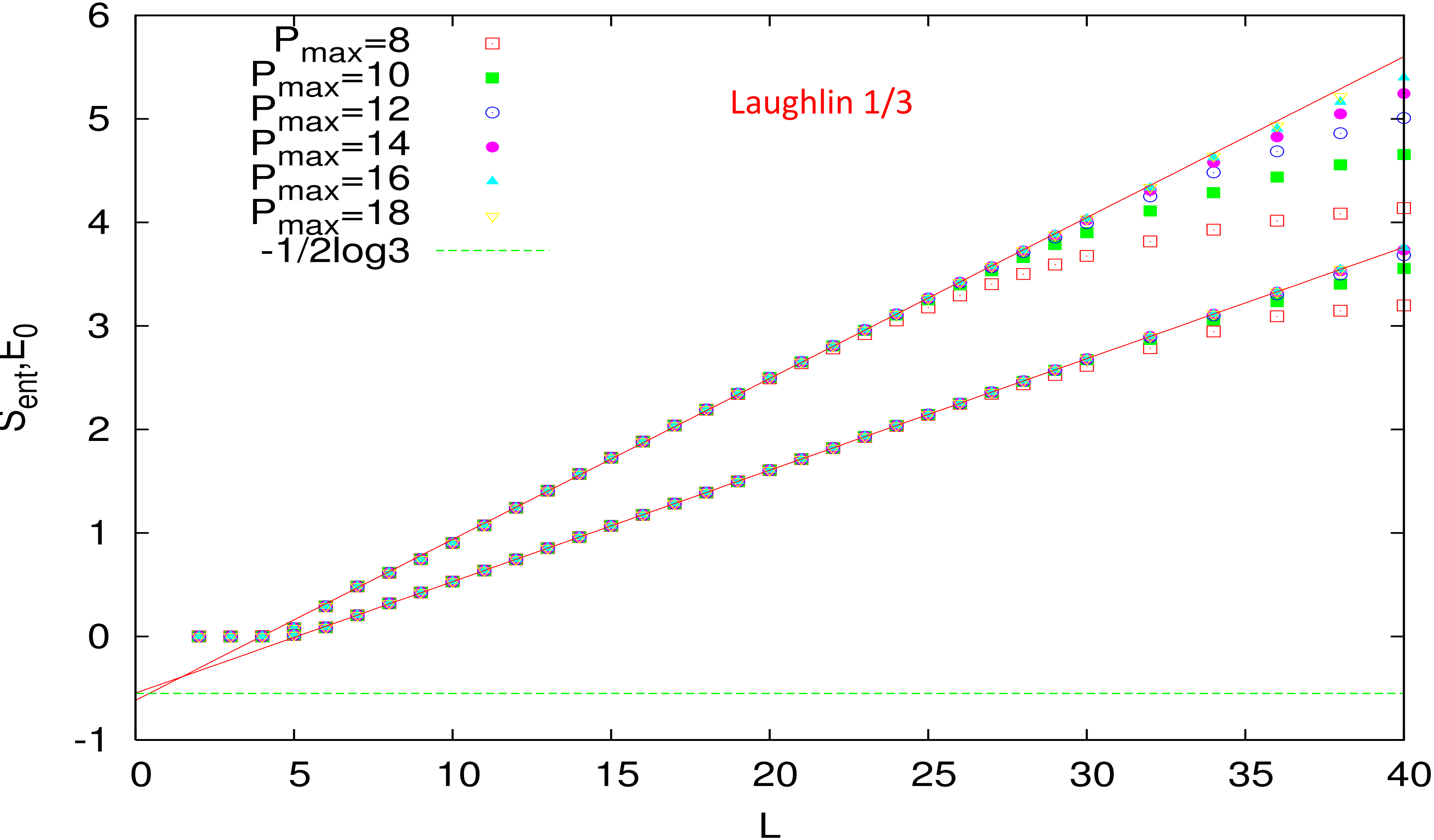
$$\rho_A = \mathrm{tr}_B(|\Psi\rangle\langle\Psi|)$$

$$H_a(\mathcal{N}) \propto d_a^{\mathcal{N}}.$$

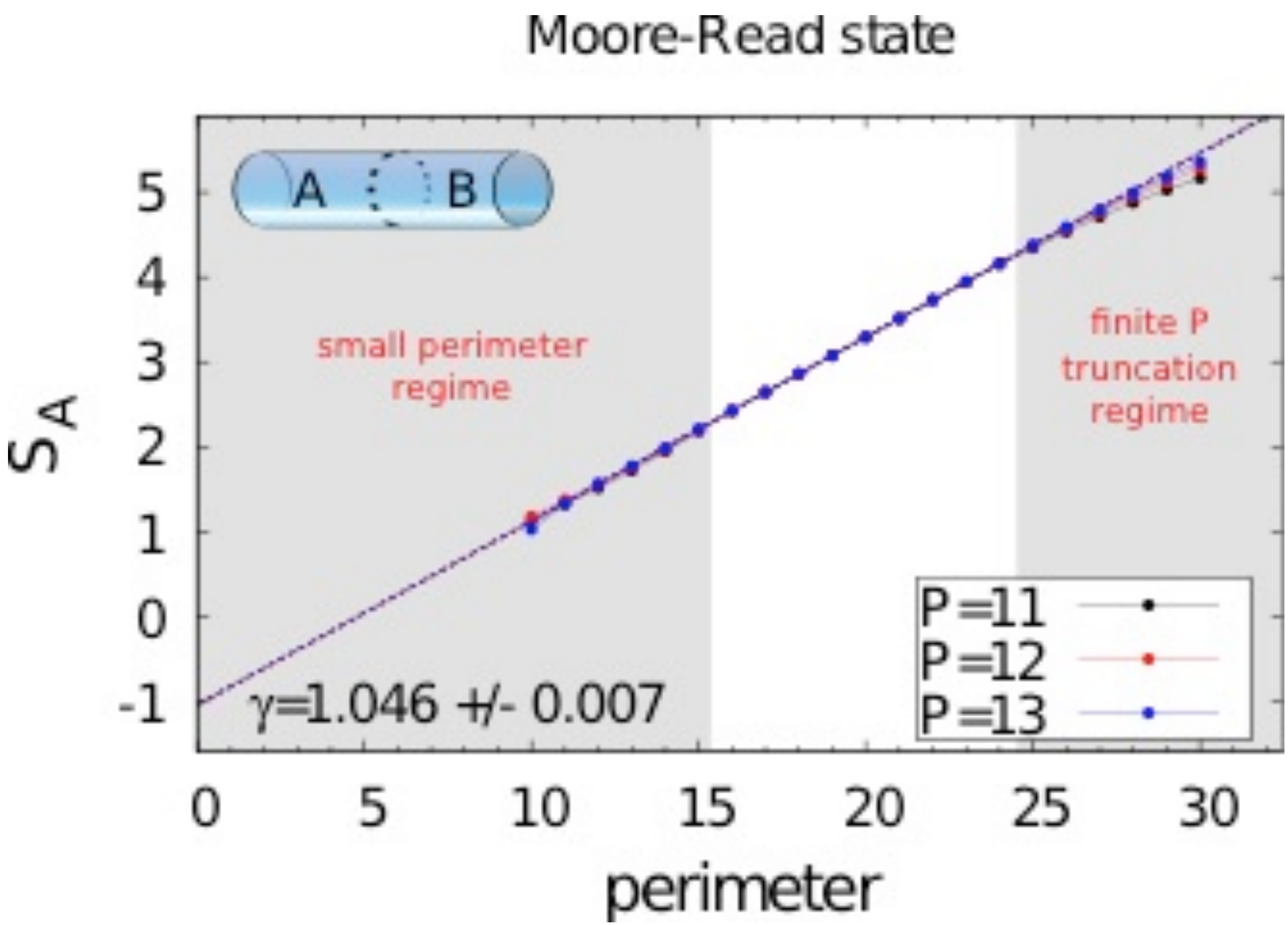
$$\mathcal{S}_A = a\mathcal{L} - \ln\mathcal{D} + \cdots \sum_a n_a \ln(d_a)$$

$$\mathcal{D} = \sqrt{\sum_a d_a^2}.$$

Entanglement Entropy - Laughlin

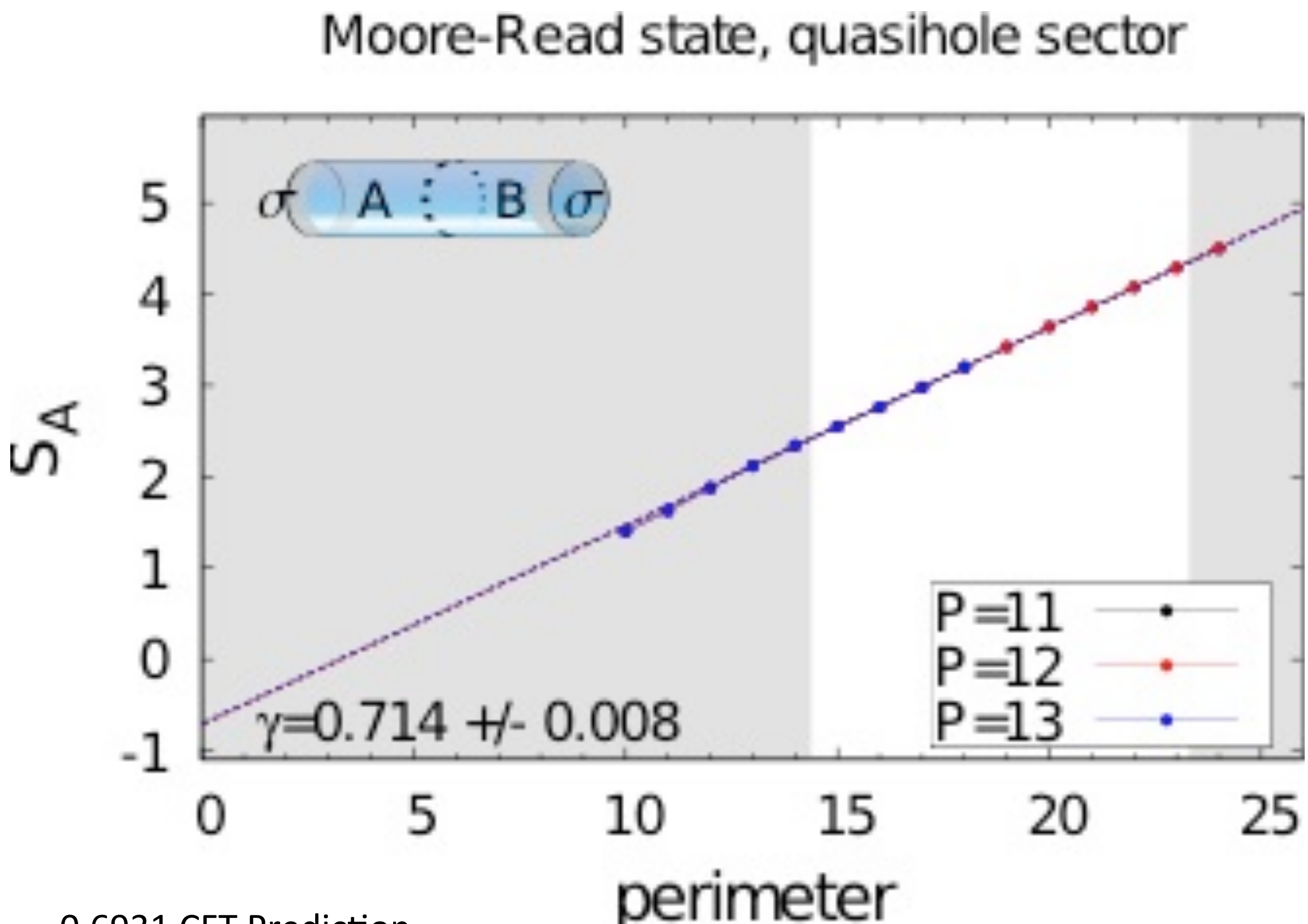


Entanglement Entropy – Moore-Read



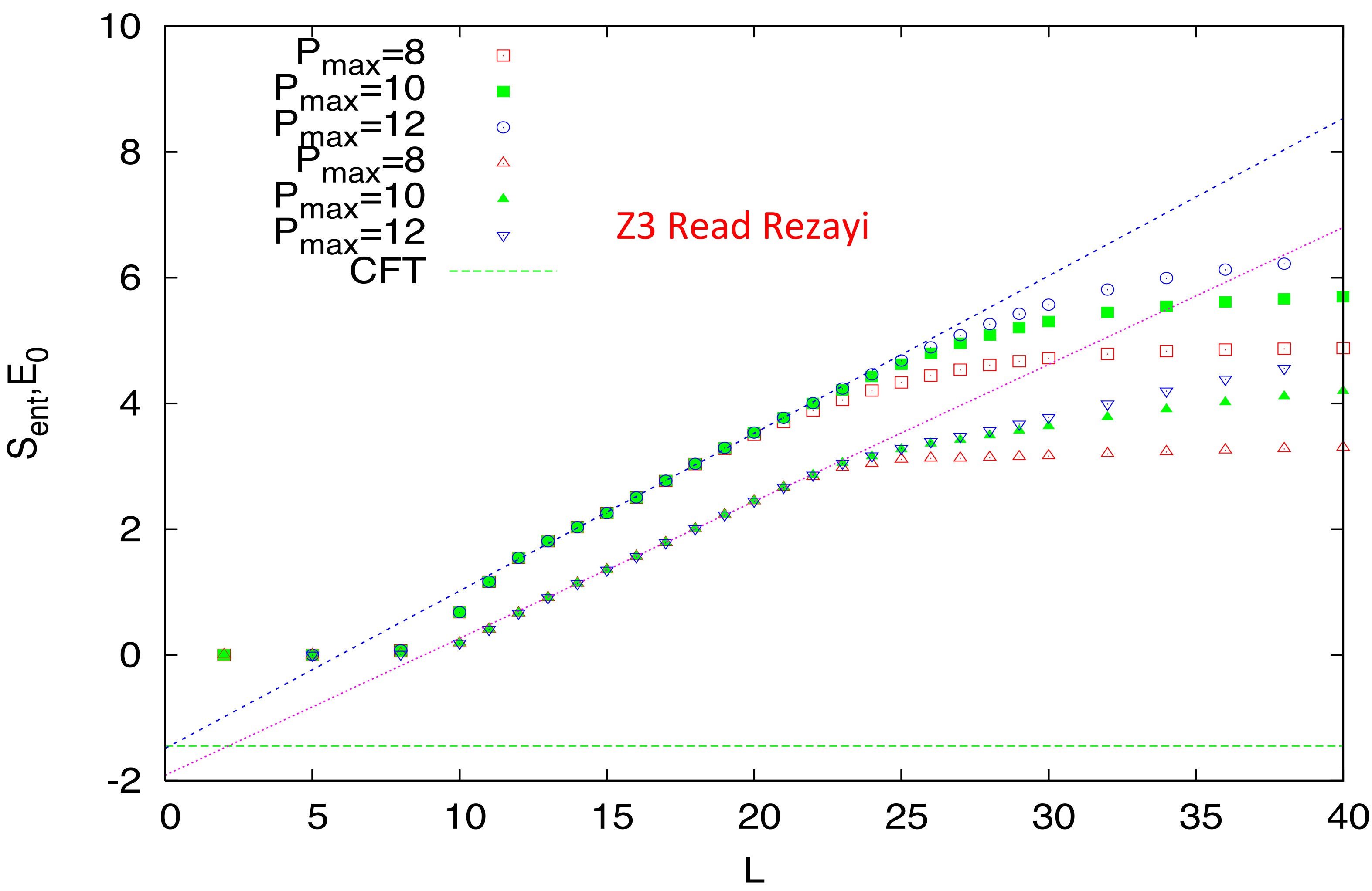
1.03972 CFT prediction

Entanglement Entropy – Moore-Read

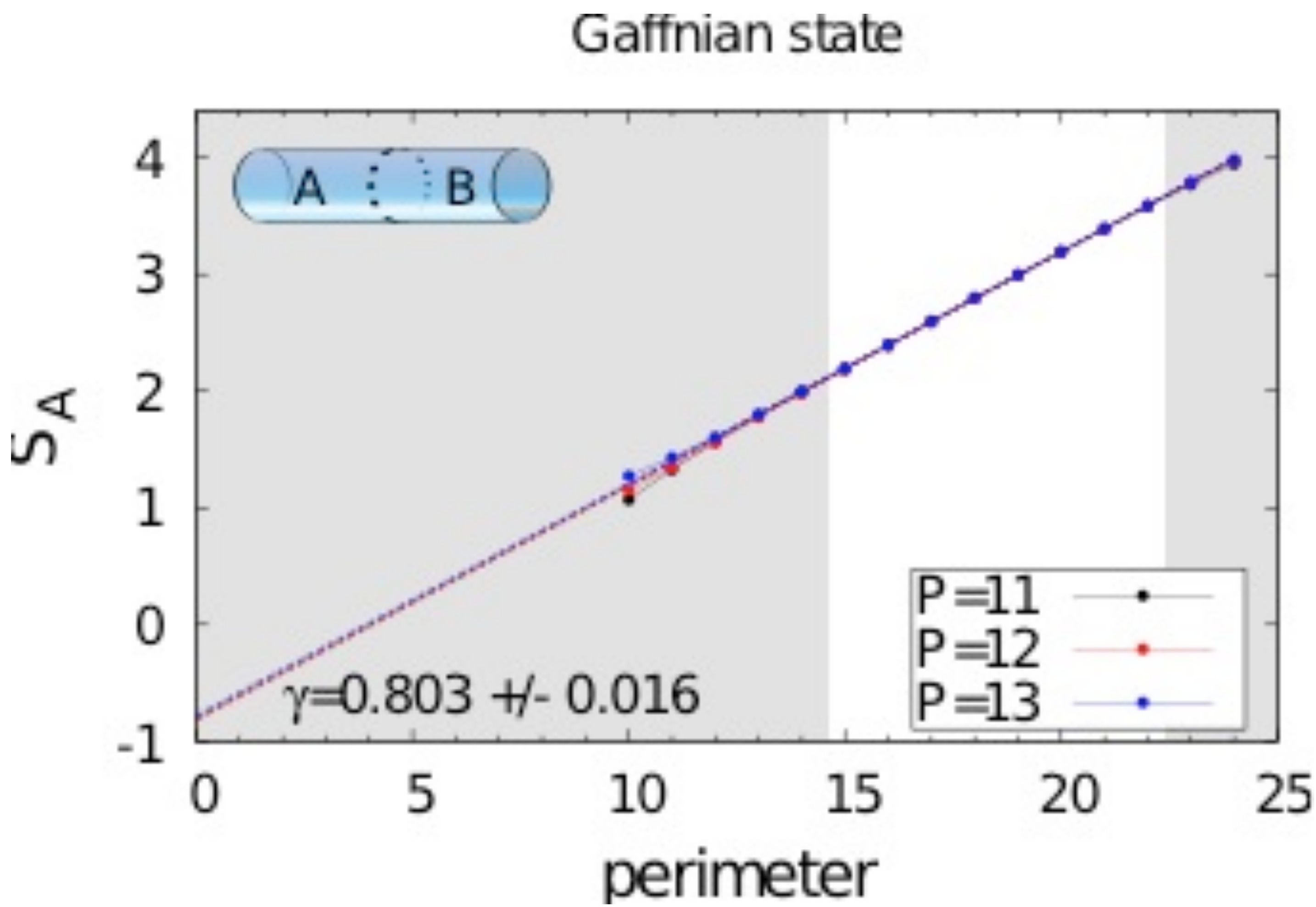


0.6931 CFT Prediction

Entanglement Entropy – Read-Rezayi

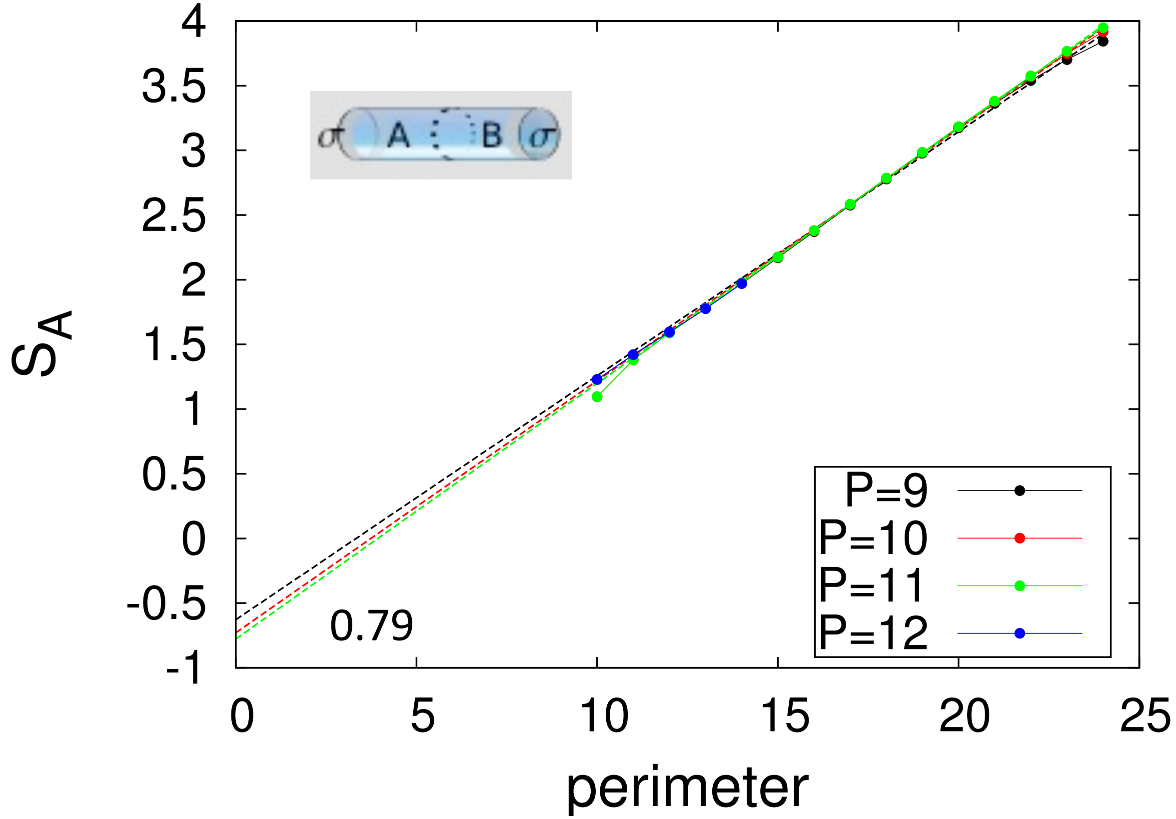


Entanglement Entropy NonUnitary State Gaffnian



This corresponds to a total quantum dimensions = 5 of the theory. This would be the quantum dimension of the theory without the quasihole sector.

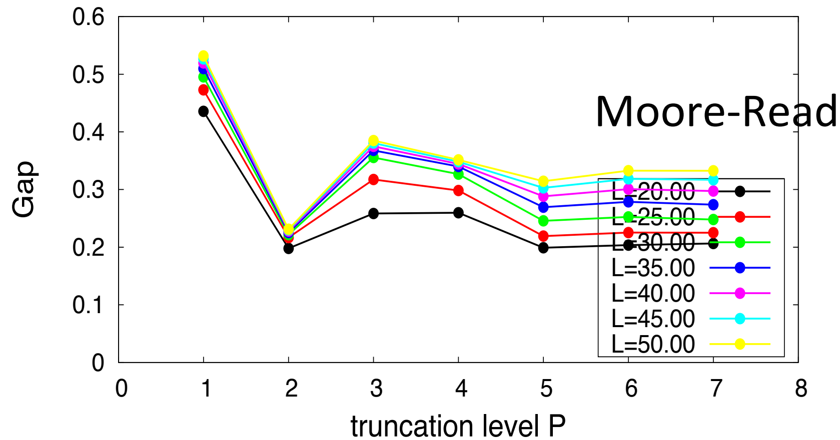
Entanglement Entropy NonUnitary State Gaffnian Plus Quasiholes

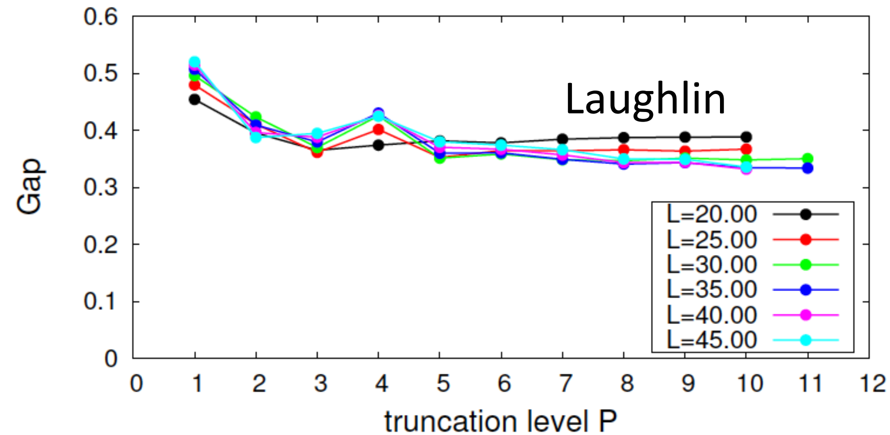


Similar Value to the ent entropy without the nonabelian nonunitary quasihole sector.

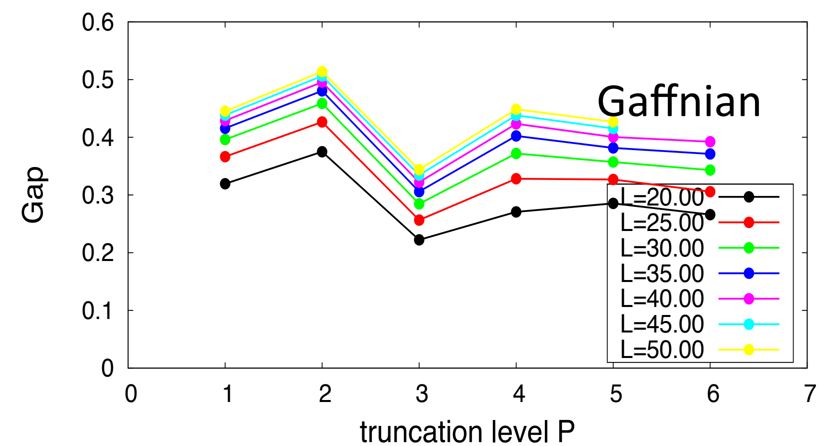
Correlations in Moore-Read and Laughlin

The E MPS matrix is gapped for More Read and Laughlin, reaches convergence fast, for small truncation level.





The electron sector of the Gaffnian matrix also exhibits exponentially decaying correlations. “Gaplessness” of the Gaffnian not in the electron sector.



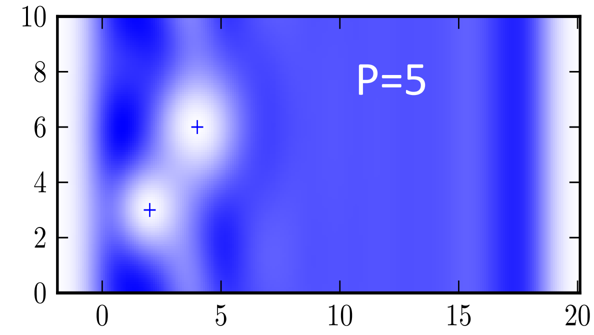
Quasiholes: Laughlin

with Yangle Wu

$$\psi(\eta_1 \dots \eta_{n_{qh}}; z_1 \dots z_{N_e}) =$$

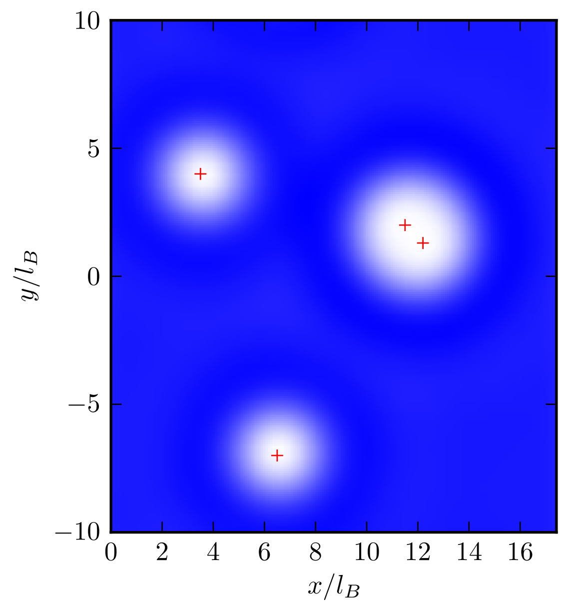
$$= \sum_P \left\langle N_e\sqrt{q} + \frac{n_{qh}}{\sqrt{q}}, 0 \right| V_{\frac{1}{\sqrt{q}}}(\eta_1) \dots V_{\frac{1}{\sqrt{q}}}(\eta_{n_{qh}}) \left| N_e\sqrt{q}, P \right\rangle \left\langle N_e\sqrt{q}, P \right| V_{\sqrt{q}}(z_1) \dots V_{\sqrt{q}}(z_{N_e}) \left| 0, 0 \right\rangle$$

$$= \prod_{\alpha<\beta}^{n_{qh}} (\eta_\alpha - \eta_\beta)^{\frac{1}{q}} \left\langle N_e\sqrt{q} + \frac{n_{qh}}{\sqrt{q}}, 0 \right| : e^{i\frac{1}{\sqrt{q}}\sum_{\alpha=1}^{n_{qh}} \phi(\eta_\alpha)} : \left| N_e\sqrt{q}, P \right\rangle$$

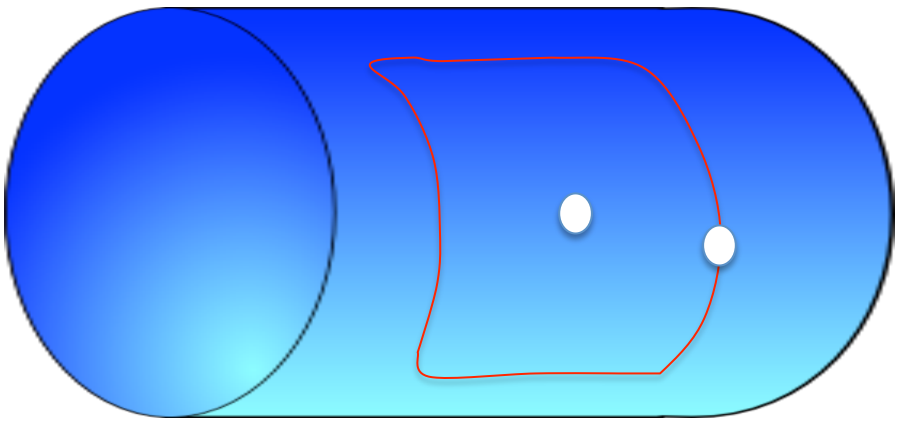


There are a large number of quasi-hole representations depending on where we insert the quashiholes. Edge representation can be made exact, bulk representation cannot, due to branch-cut. Better resolution for braiding - in the bulk, for edge coefficients – on the edge.

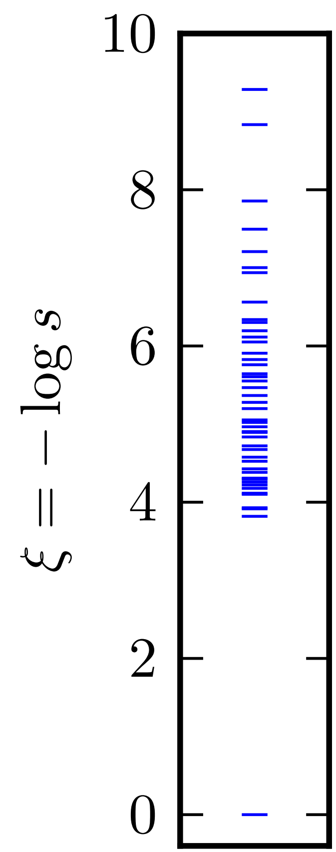
$$\sum_{Q,P',P} [B^{N_\phi} \dots B^{M+1}]_{\frac{q-1}{\sqrt{q}},0;Q+\frac{1}{\sqrt{q}},P'} V_{Q;P',P}(\eta) [B^{m_M} \dots B^{m_0}]_{Q,P;0,0}$$

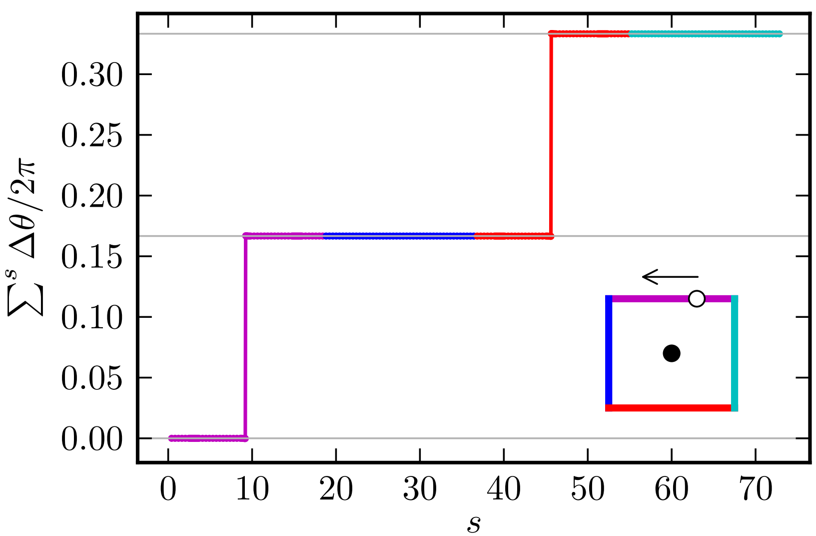


Braiding of Laughlin Quasiholes

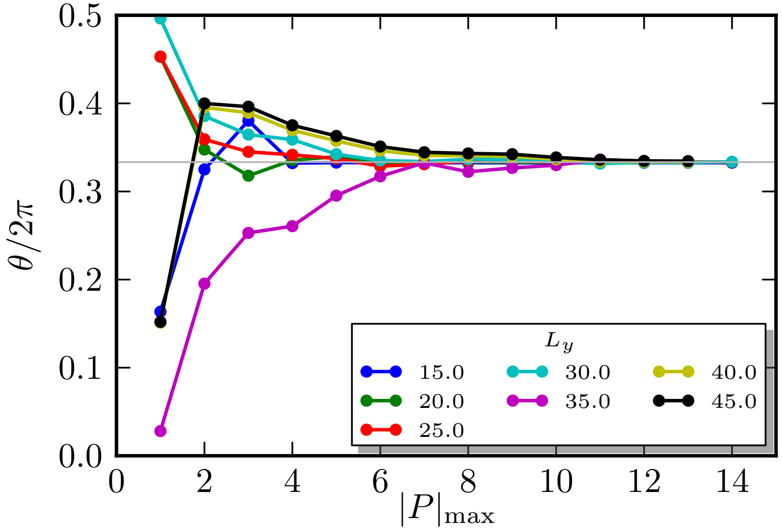


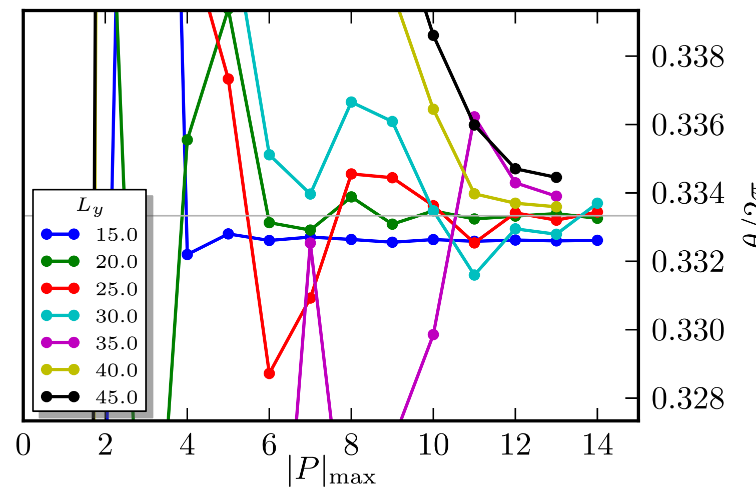
Uniformity of Norm of Laughlin Conformal Block



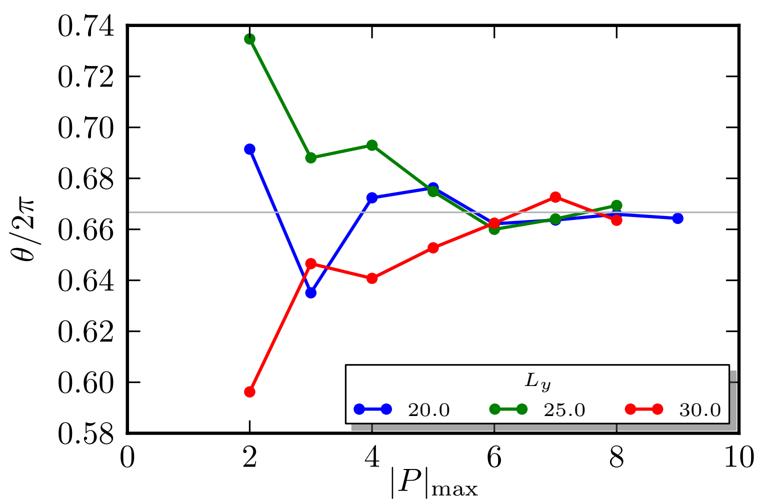


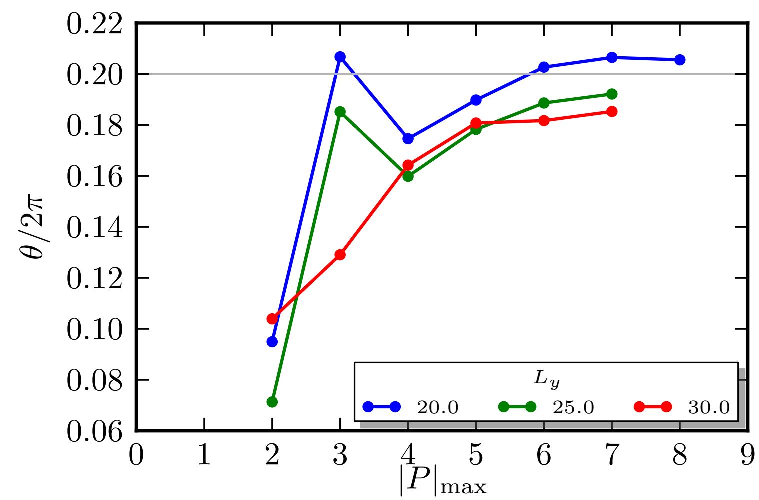
Braiding





Similar numbers 0.3341 quoted in Zalatel and Mong





Quasiholes: (2,2) and (2,3) Jacks. Moore-Read and Gaffnian

For the nonabelian quasiholes, the MPS representation cannot be made exact (except if going to infinity in truncation)

$$V_{qh}(\eta) = \sigma(\eta) e^{i\phi(\eta)/2\sqrt{2}}$$

$$\left\langle N_e\sqrt{q} + \frac{1}{\sqrt{q}}, 0, 0 \right| V_{qh}(\eta_1) V_{qh}(\eta_2) \left| N_e\sqrt{q}, \lambda, \mu, 0 \right\rangle \sim \left\langle 0, 1 \right| \sigma(\eta_1)\sigma(\eta_2) \left| \lambda, 1 \right\rangle (\eta_1 - \eta_2)^{\frac{1}{8}}$$

$$\sum_\lambda \left\langle 0, 1 \right| \sigma(\eta_1)\sigma(\eta_2) \left| \lambda, 1 \right\rangle \left\langle \lambda, 1 \right| \sigma(\eta_3)\sigma(\eta_4) \left| 0, 1 \right\rangle , \qquad \sum_\lambda \left\langle 0, 1 \right| \sigma(\eta_1)\sigma(\eta_2) \left| \lambda, \psi \right\rangle \left\langle \lambda, \psi \right| \sigma(\eta_3)\sigma(\eta_4) \left| 0, 1 \right\rangle$$

Conclusions

MPS description for large series of FQH states is now available.

Zalatel and Mong showed how to do it for Laughlin and Moore-Read.

We obtained the MPS description for a large series of states (most Jack polynomials, other)

This approximation scheme opens new avenues

It becomes possible to test many of the predictions of CFT, sort out what non-unitary theories do

DMRG should NOT be done on the sphere, contrary to many previous calculations.

Braiding, topological quantities, flows, etc