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Entanglement Entropy and Temperature

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References

C. P. H. and M. Spillane, "Tracing Through Scalar Entanglement," Phys. Rev. D **87**, 025012 (2013) [arXiv:1209.6368 [hep-th]].

C. P. H. and T. Nishioka, "Entanglement Entropy of a Massive Fermion on a Torus," arXiv:1301.0336 [hep-th].

Entanglement Entropy as a Unifying idea

- relativistic field theory: monotonicity of certain kinds of entanglement entropy (EE) under RG flow [Huerta and Casini 2006, 2012].
- 2d CFTs: rapid numerical method for computing the central charge c, fitting to $S = \frac{c}{3} \log \frac{\ell}{\epsilon}$ result [Korepin 2004, Calabrese and Cardy 2004].
- condensed matter: nonlocal order parameter for exotic phase transitions [Osborne and Nielsen 2002, Vidal et al. 2003].
- string theory and general relativity: black hole entropy [Bombelli et al. 1986, Srednicki 1993] and more recently a proposal for computing EE holographically [Ryu and Takayanagi 2006].

A Definition of Entanglement Entropy

- Partition the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. A and B are usually spatial regions.
- Form the reduced density matrix $\rho_A = \mathrm{tr}_B\, \rho_A$ by tracing over the degrees of freedom in B.
- The EE is then
$$S \equiv -\,\mathrm{tr}\, \rho_A \log \rho_A \ .$$

Motivation for this work:

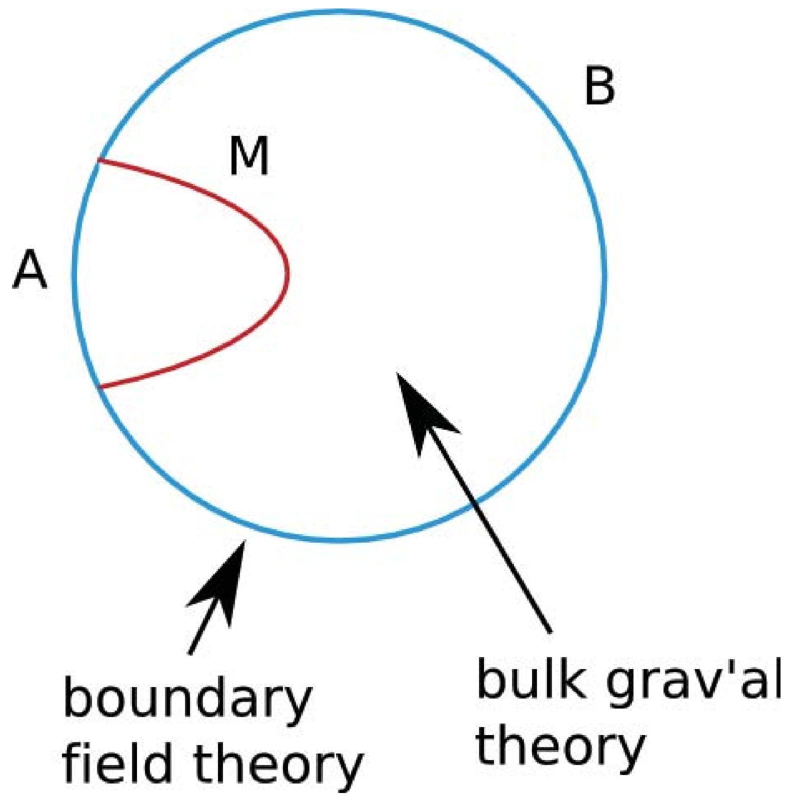
1. Try to understand where the holographic Ryu-Takayanagi (RT) formula comes from by looking at where it breaks down.
2. Investigate the effect of temperature and the presence or absence of a mass gap on EE.

The Ryu-Takayanagi Formula

- Start with a time independent space-time with a holographic dual equilibrium field theory description.
- Consider a spatial slice of the metric.
- Let M be a minimal surface such that $\partial A = \partial B = \partial M$.
- Motivated in part by black hole entropy, RT conjectured

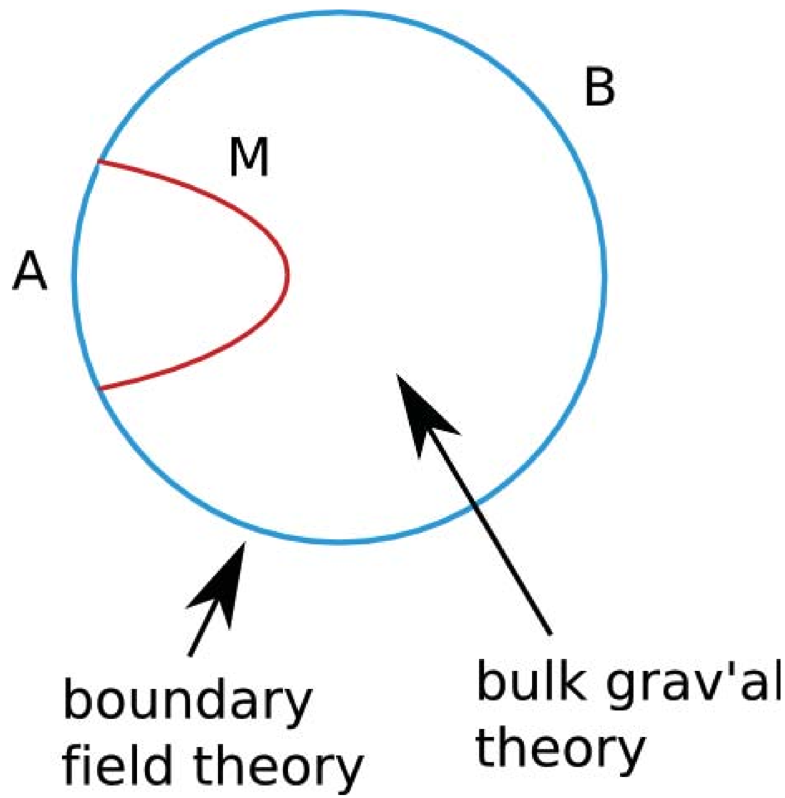
$$S = \frac{\text{Area}(M)}{4G_N} \ .$$

- RT valid when classical gravity is valid. (Fursaev '06, Headrick '10, Faulkner '13, Hartman '13, Lewkowycz-Maldacena '13)



Ryu-Takayanagi for mixed states

- Note that if M is unique, then $S_A = S_B$.
- Analog field theory result: For $\rho = |\psi\rangle\langle\psi|$ constructed from a pure state, $S_A = S_B$. (Result follows from Schmidt decomposition of the Hilbert space.)
- Black holes are dual to mixed states. There are now two extremal surfaces M_A and M_B.
- One finds that $S_A(M_A) \neq S_B(M_B)$.



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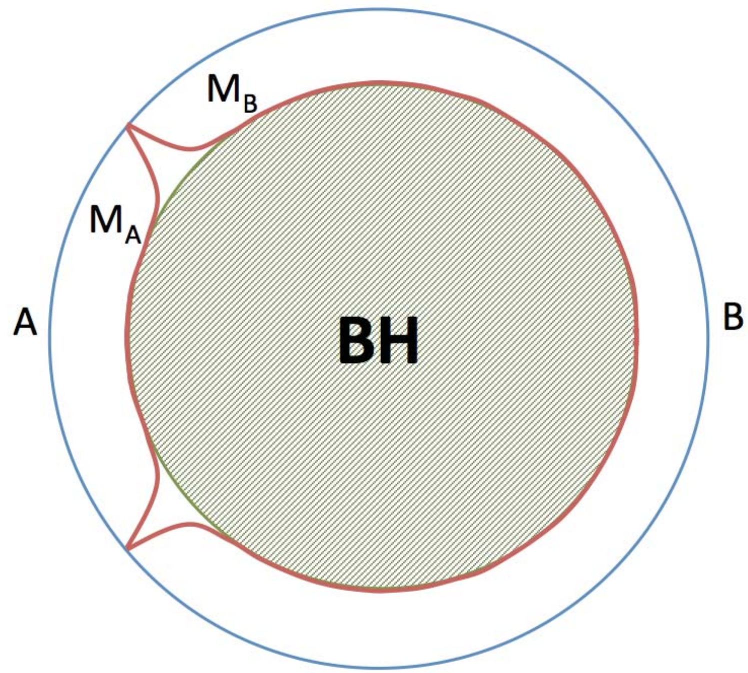


Comparison with Field Theory

Consider $\mathcal{N} = 4$ SU(N) super Yang-Mills in the large coupling, $N \to \infty$ limit on $S^3 \times S^1$. Dual classical gravity description.

Let R be the radius of S^3 and $1/T$ be the radius of S^1.

There exists a first order phase transition at $T_c \sim 1/R$.
[Witten 1998]

- $T < T_c$: Dual geometry is $AdS_5 \times S^5$.
- $T = T_c$: Geometry undergoes a Hawking-Page phase transition.
- $T > T_c$: Geometry contains a black hole.

On the one hand, $RT \implies S_A - S_B = 0$ when $T < T_c$.

On the other, field theory $\implies$ no phase transition at any finite N

$$\implies S_A - S_B = O(1/N) \text{ when } T < T_c.$$

Can We Compute Entanglement Entropy for Gauge Theories?

- In a lattice Hamiltonian formulation, observables are Wilson loops. The partition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is not straight forward. [Buividovich and Polikarpov 2008, Donnelly 2011]
- One can compute Rényi entropies S_n, $n = 2, 3, 4, \ldots$, using the replica trick. Equivalent to a computation of the partition function on a multi-sheeted cover of the space-time. Lattice computations at $T = 0$ exist [Buividovich and Polikarpov 2008, Nakagawa et al. 2011], but how does one take the limit $S = \lim_{n\to 1} S_n$?
- For CFTs, one can map S to a sphere partition function [Casini, Huerta, Myers 2011], but ours are gapped theories.

Outline for the rest of the talk

We will put gauge theories aside and focus on 1+1 dimensional free fermions and bosons on a torus. We find

$$S_A(T) - S_A(0) \sim e^{-m_{\mathrm{gap}}/T}$$

$$S_A(T) - S_B(T) \sim e^{-m_{\mathrm{gap}}/T}$$

in the limit $m_{\mathrm{gap}} \gg T$.

- 1+1d fermion:
  - by bosonization and the replica trick
  - using the lattice
- 1+1d scalar:
  - using the lattice
- 1+1d fermions without a gap

Replica Trick

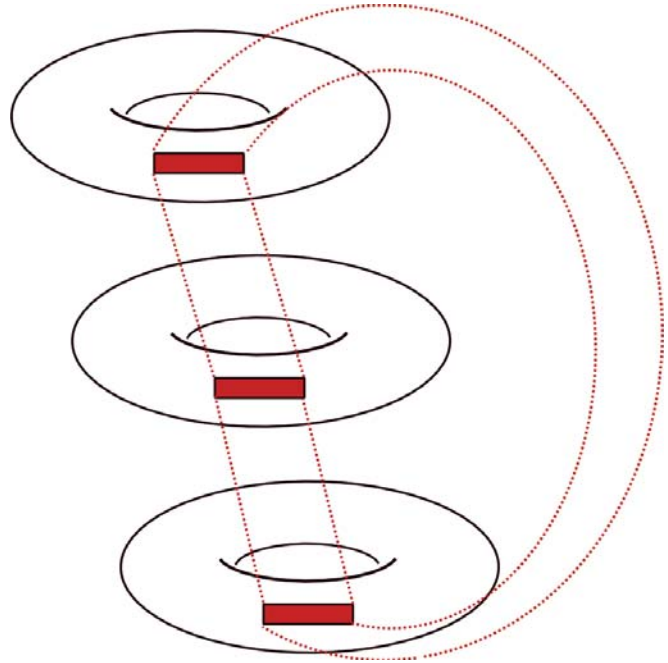
Can compute Rényi entropies from a multi-sheeted cover of the original spacetime.

- The Rényi entropies are given by

$$S_n = \frac{1}{1-n} \log \mathrm{tr}(\rho_A)^n \ .$$

- The entanglement entropy can be computed from the limit

$$S = \lim_{n \to 1} S_n$$



Graphic representation of $Z[3]$ where $\mathrm{tr}(\rho_A)^n = Z[n]/Z[1]^n$. Red rectangle formed by cutting along region A.

Replica Trick for Fermions

- Start with n-sheeted covering of the torus with branch points u and v bounding interval A.
- Replace single Ψ on n-sheeted cover with n decoupled fermions $\tilde{\Psi}^k$, $k = -\frac{n-1}{2}, \ldots, \frac{n-1}{2}$ living on a single torus, multivalued around u and v with phases $e^{\pm i \frac{2\pi k}{n}}$.
- Introduce external gauge field and make replacement

$$\tilde{\Psi}^k = e^{i \int_{x_0}^{x} dx'^{\mu} \mathcal{A}_\mu^k(x')} \Psi^k(x) \ ,$$

where $\mathcal{A}_\mu^k$ is almost pure gauge

$$\epsilon^{\mu\nu} \partial_\nu \mathcal{A}_\mu^k = \frac{2\pi k}{n} (\delta(x-u) - \delta(x-v)) \ .$$

[Casini, Fosco, Huerta 2005, Azeyanagi et al. 2007, Ogawa et al. 2011]

Replica Trick and Bosonization for Fermions

- With this rewriting, the partition function $Z[n]$ takes the form

$$Z[n] = \prod_k Z_k = \prod_k \langle e^{i \int \mathcal{A}^k_\mu j^\mu_k d^2x} \rangle ,$$

  where $j^\mu_k = \bar{\Psi}^k \gamma^\mu \Psi^k$.

- Massive fermion bosonizes to sine-Gordon model

$$\mathcal{L} = \frac{1}{8\pi} \partial_\mu \phi \partial^\mu \phi + \frac{m}{\pi \epsilon L} \cos \phi ,$$

  with the current $j^\mu_k \to \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi$.

- When $m = 0$, the correlation functions $Z_k = \langle V_k(u) V_{-k}(v) \rangle$ where $V_k(x) = e^{-i \frac{k}{n} \phi(x)}$ are known.

(Suppressing a discussion of spin structure.)

Multi-interval Rényi entropies for fermions

Multi-interval generalization of Ogawa et al. 2011:

$$
\begin{aligned}
S_n^{(\nu)} &= S_{n,0} + S_{n,1}^{(\nu)} \\
S_{n,0} &= -\frac{n+1}{6n} \log \left| \frac{\prod_{a<b} \vartheta_1(u_a - u_b|\tau)\, \vartheta_1(v_a - v_b|\tau)}{\prod_{a,b} \vartheta_1(u_a - v_b|\tau)} \cdot \right. \\
&\quad \cdot \left. (\epsilon\, \partial_z \vartheta_1(0|\tau))^p \right| , \\
S_{n,1}^{(\nu)} &= \frac{2}{1-n} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \left| \frac{\vartheta_\nu(\frac{k}{n}\frac{\ell_t}{L} + \frac{i\beta\mu}{2\pi}|i\beta)}{\vartheta_\nu(\frac{i\beta\mu}{2\pi}|i\beta)} \right| .
\end{aligned}
$$

where $\ell_t = L\sum_a (v_a - u_a)$ and $\nu = 1, 2, 3,$ or 4 denotes the spin structure.

An Analytic Result

In the massless case, we find analytic evidence of our scaling hypothesis.

- Consider fermion anti-periodic in both time (thermal) and space.
- Such a fermion has a mass gap $m_{\rm gap} = \pi/L$ where L is the size of the spatial circle.
- We find

$$
\begin{aligned}
S_A(T) - S_A(0) &= 4(1 - \pi r \cot(\pi r))e^{-m_{\rm gap}/T} + O(e^{-2m_{\rm gap}/T}) \ , \\
S_B(T) - S_A(T) &= 4\pi \cot(\pi r) e^{-m_{\rm gap}/T} + O(e^{-2m_{\rm gap}/T}) \ ,
\end{aligned}
$$

where $r = \ell/L$ and ℓ is the length of A.

Update!

- Giombi, Maloney, and Yin '08 demonstrated that the one loop contribution from a bulk field to the AdS_3 partition function with a genus n Riemann surface as boundary is given by a Selberg zeta function.
- Using the replica trick, this Selberg zeta function can be used to evaluate corrections to the Ryu-Takayanagi formula.
- Barrella et al. '13 evaluated this zeta function in a low temperature expansion for gravitons. The contribution to the entanglement entropy takes the form

$$\delta S_E \sim \left(1 - \frac{\pi \ell}{L} \cot \frac{\pi \ell}{L}\right) e^{-4\pi/LT} \ .$$

- The gravitons in the bulk are dual to the stress tensor operator in the boundary CFT which has scaling dimension $h = 2$. For fields of conformal dimension h, $e^{-4\pi LT}$ is trivially replaced by $e^{-2\pi h/LT}$.

Further Thoughts

- D'Hoker and Phong '86 showed that the determinant of the Laplacian operator on a genus n Riemann surface is a Selberg zeta function $Z_{s-\lfloor s \rfloor}(s)$ where s is the spin of the field.
- We may conjecture that this scaling form of the entanglement entropy is valid for a general CFT gapped by finite volume where h is the smallest conformal operator dimension in the theory.
- There is likely a simple functional identity that relates the Selberg theta function in the fermionic case to the product of theta functions we found through bosonization.

moving on to truly massive theories...

Fermions on the Lattice

[Srednicki 1993, Peschel and Eisler 2009]

- Discretize the Hamiltonian

$$H = \sum_{j=1}^{N} \left[-\frac{i}{2} (\Psi_j^\dagger \sigma^3 \Psi_{j+1} - \Psi_{j+1}^\dagger \sigma^3 \Psi_j) + m\epsilon\, \Psi_j^\dagger \sigma^1 \Psi_j \right] \; .$$

- Spectrum $\omega_a^2 = m^2 + \frac{\sin^2(2\pi a/N)}{\epsilon^2}$, $a = 1, \ldots, N$, exhibits the usual fermion doubling problem. Divide final answer by two.
- Compute the two-point function $C = \epsilon \langle \Psi \Psi^\dagger \rangle$. Restrict two-point function C to the region A.
- Entanglement entropy computed numerically from

$$S = -\operatorname{tr}[(1-C)\log(1-C) + C \log C] \; .$$

Form of the fermion two point function

We can compute entanglement entropy numerically for arbitrary spin structure (ν_1, ν_2) and arbitrary chemical potential μ,

$$\langle \Psi_j \Psi_k^\dagger \rangle =$$

$$\frac{1}{2L} \sum_{a=1-\nu_1}^{N-\nu_1} e^{i\theta_a(j-k)} \Bigg[\left(1 + \frac{\sinh(\beta\mu)}{\cosh(\beta\mu) + (-1)^{2\nu_2+1} \cosh(\beta\omega(\theta_a))} \right)$$

$$+ \left(\frac{\sin\theta_a}{\omega(\theta_a)\epsilon} \sigma_3 + \frac{m}{\omega(\theta_a)} \sigma_1 \right) \frac{\sinh(\beta\omega(\theta_a))}{(-1)^{2\nu_2+1} \cosh(\beta\mu) + \cosh(\beta\omega(\theta_a))} \Bigg] ,$$

where $\theta_a = 2\pi a/N$ and $\beta = 1/T$.

Lattice Results for Fermions

Data confirming $e^{-m/T}$ scaling for thermal fermions with periodic spatial boundary conditions.

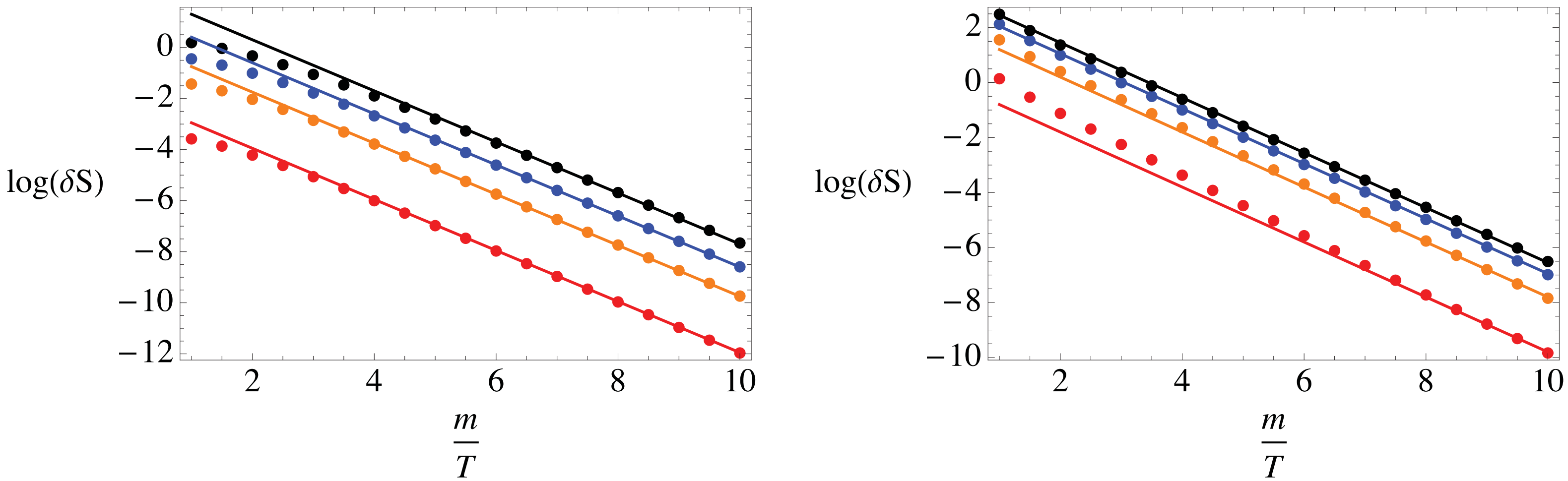


Figure: The entanglement entropy difference $\delta S = S(T) - S(0)$: [Left] $mL = 1/10$; [Right] $mL = 10$. The points are computed from a lattice, and the lines are fits with slope -1. From bottom to top, $\ell/L = 1/10$, $3/10$, $1/2$, $7/10$.

Note that $S_A(T) \neq S_B(T)$.

Bosons on the Lattice

[Srednicki 1993, Peschel and Eisler 2009]

- Discretize the Hamiltonian

$$H = \frac{1}{2\epsilon} \sum_{j=1}^{N} \left[\pi_j^2 + (\phi_{j+1} - \phi_j)^2 + m^2 \epsilon^2 \phi_j^2\right] \ .$$

- No doubling problem in the spectrum, $\omega_a^2 = m^2 + \frac{4}{\epsilon^2} \sin^2 \frac{\pi a}{N}$, $a = 1, \ldots, N$.
- Compute the two point functions $\langle \phi_j \phi_k \rangle$ and $\langle \pi_j \pi_k \rangle$.
- Restrict the range of the two point functions to the region A and define $C^2 = \langle \phi\phi \rangle \cdot \langle \pi\pi \rangle$.
- The EE may be computed from

$$S = \mathrm{tr}[(C + \frac{1}{2}) \log(C + \frac{1}{2}) - (C - \frac{1}{2}) \log(C - \frac{1}{2})] \ .$$

Form of the scalar two point functions

$$
\begin{aligned}
\langle \phi_j \phi_k \rangle &= \frac{1}{2N} \sum_{a=0}^{N-1} \frac{1}{\epsilon \omega_a} \coth\left(\frac{\omega_a}{2T}\right) \cos\left(\frac{2\pi(j-k)a}{N}\right) \ , \\
\langle \pi_j \pi_k \rangle &= \frac{1}{2N} \sum_{a=0}^{N-1} \epsilon \omega_a \coth\left(\frac{\omega_a}{2T}\right) \cos\left(\frac{2\pi(j-k)a}{N}\right) \ ,
\end{aligned}
$$

The zero mode $a = 0$ terms in the sum determine much of the physics in the $mL \ll 1$ limit. Recall

$$
\omega_a^2 = m^2 + \frac{4}{\epsilon^2} \sin^2 \frac{\pi a}{N} \ .
$$

Analytic results from C^2

We were able to estimate the largest pair of of eigenvalues of C^2 in the limit $m, T \ll 1/L$.

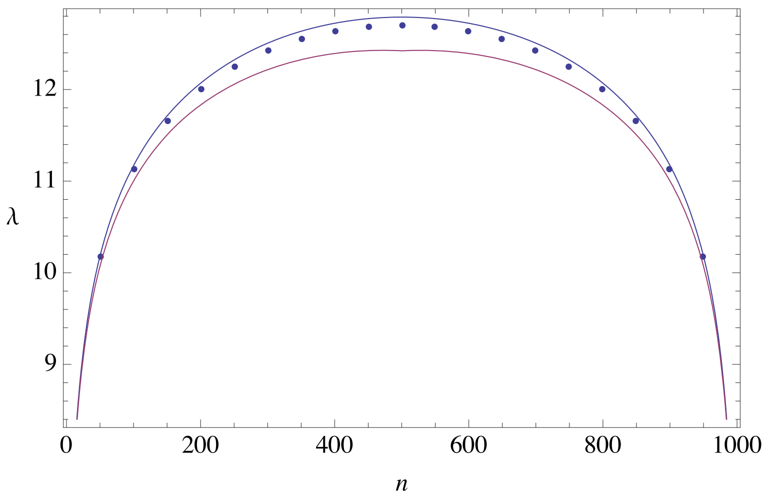
- For a free scalar field $\rho_A \sim e^{-H_A}$ where $H_A = \sum_k \epsilon_k b_k^\dagger b_k$ [Peschel and Eisler 2009].
- If λ_k are eigenvalues of C^2, we find $\lambda_k = \frac{1}{4}\coth^2 \frac{\epsilon_k}{2}$.
- We conclude that $\lambda_k \geq 1/4$.
- The trace gives an upper bound on the largest eigenvalue:

$$\frac{\ell}{4\epsilon} + \lambda_{\max} \leq \text{tr}\, C^2 \ ,$$

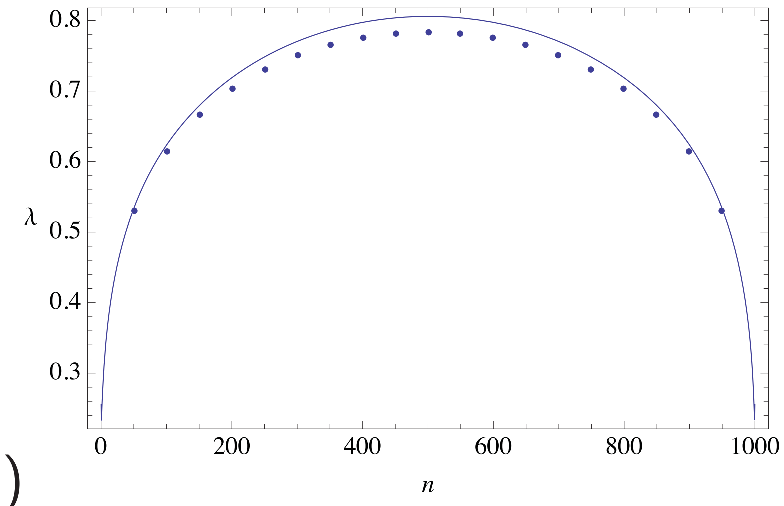
where ℓ is the length of A.

Analytics (continued)

We can do better by dividing C^2 into parity even and odd pieces.



a)



b)

Figure: The largest (a) and second largest (b) eigenvalue of C^2 plotted against the interval length for $mL = 1/10$, $T = 0$, and $N = 1000$. The points are numerically computed. The curves above the points are the analytic upper bounds computed from the traces. The solid curve below the points on the left is the lower bound computed from a variational principle.

Results for traces of C^2

$$\begin{aligned}
\mathrm{tr}\, C_e^2 \;=\;& \frac{1}{2\pi mL}\coth\left(\frac{m}{2T}\right)\left[\gamma+\ln\left(\frac{4N\sin(\pi r)}{\pi}\right)\right]+\frac{r^2}{4}\mathrm{csch}^2\left(\frac{m}{2T}\right)\\
&+\frac{1}{4}\Bigg[s+\frac{11}{12}-\frac{1}{\pi^2}+\frac{1}{2\pi^2}\Bigg(-2+\gamma+4\ln\frac{2N}{\pi}+\\
&\quad -3\ln\frac{4N\sin(\pi r)}{\pi}\Bigg)\left(\gamma+\ln\frac{4N\sin(\pi r)}{\pi}\right)\Bigg]\\
&-\frac{3mL}{32\pi^3}\coth\left(\frac{m}{2T}\right)\left[\mathrm{Li}_3(e^{2\pi i r})+\mathrm{Li}_3(e^{-2\pi i r})-2\zeta(3)\right]\\
&+O((mL)^2, e^{-2\pi/TL}, \log N/N)\ ,
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}\, C_o^2 \;=\;& \frac{1}{4}\left[s+\frac{1}{12}-\frac{3}{2\pi^2}+\frac{1}{2\pi^2}\left(\gamma-1+\ln\frac{4N\sin(\pi r)}{\pi}\right)^2\right]\\
&+O((mL)^2, e^{-2\pi/TL}, \log N/N)\ ,
\end{aligned}$$

A closer look at the traces

The odd parity piece has no T dependence at this order because it does not contain the zero mode.

The even parity piece confirms our scaling hypothesis:

$$\mathrm{tr}\, C_e^2 = \frac{1}{2\pi mL} \coth\left(\frac{m}{2T}\right) \left[\gamma + \ln\left(\frac{4N\sin(\pi r)}{\pi}\right)\right] + \frac{r^2}{4} \mathrm{csch}^2\left(\frac{m}{2T}\right) + \dots$$

The leading term in $1/mL$ suggests

$$S_A(T) - S_A(0) \sim e^{-m/T}$$

and the subleading term suggests

$$S_A(T) - S_B(T) \sim e^{-m/T}$$

in the limit $T \ll m$.

Numeric Results for Bosons I

Numeric confirmation of $S_A(T) - S_A(0) \sim e^{-m/T}$ scaling for bosons.

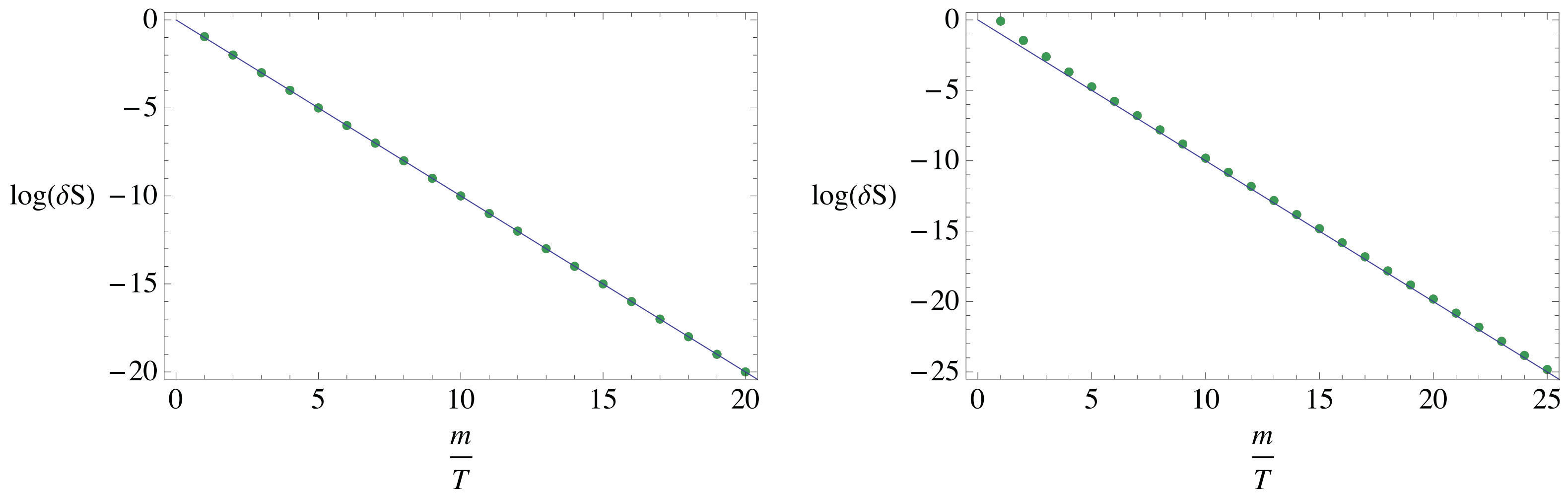


Figure: A log plot of the entanglement entropy $\delta S = S(T) - S(0)$ versus m/T with an interval size $\ell/L = 3/10$. The points are numerically computed. The line $\log(\delta S) = -m/T$ is a guide to the eye: [Left] $mL = 5 \times 10^{-3}$; [Right] $mL = 5$.

Numeric Results for Bosons II

Numeric confirmation of $S_B(T) - S_A(T) \sim e^{-m/T}$ scaling for bosons.

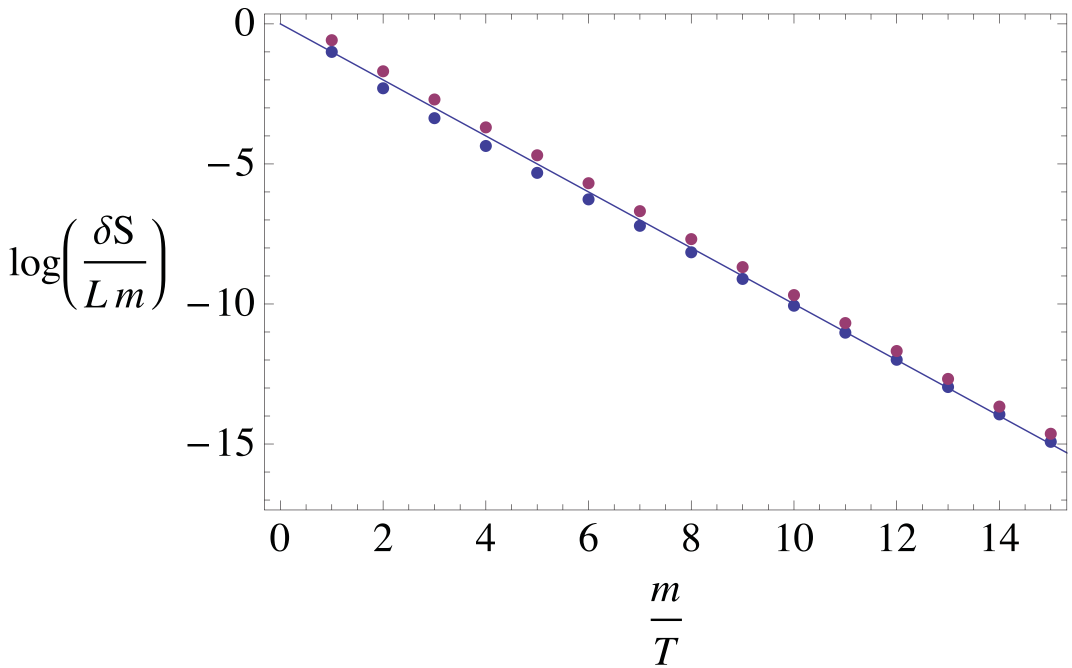


Figure: A log plot of the entanglement entropy difference $\delta S = S_A - S_B$ versus m/T for $mL = 5$ and 5×10^{-3}, and an interval B of size $\ell/L = 1/5$. At fixed m/T, the larger mass points lie below the smaller ones. The line $\log(\delta S/mL) = -m/T$ is a guide to the eye.

What happens when there is no gap?

An observation: The spectrum of the C matrix for the spatially antiperiodic fermion in the limit where $m \to 0$ and then $T \to 0$ is the same as the spectrum of the C matrix for the spatially periodic fermion in the limit where $T \to 0$ first followed by m. From this observation, one obtains analytically for the Rényi entropies of the spatially periodic fermion

$$\left[\lim_{T\to 0}, \lim_{m \to 0} \right] S_n(m, T) = \frac{2}{1-n} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \left| \cos \frac{\pi k r}{n} \right| .$$

In other words, the well known result $\frac{1}{3} \log \left(\frac{L}{\pi \epsilon} \sin \frac{\pi \ell}{L} \right) + c_0$ is not correct in the limit $m \to 0$ first, where one has a mixed state.

We have not been able to take the $n \to 1$ limit of this sum except in the case where $r = 1$ in which case it yields $2 \log 2$, in agreement with the fact that there are four degenerate ground states.

Results I didn't have time to talk about

- Chemical potential dependence of the multi-interval Rényi entropies for massless fermions on the torus.
- Mutual information for massless fermions on the torus.
- Leading small mass correction to the torus fermion Rényi entropies from the sine-Gordon model.

Discussion

Argued that $e^{-m_{\mathrm{gap}}/T}$ for EE is a generic behavior for a gapped system at low temperature. If $1/N$ corrections to the RT formula for $\mathcal{N} = 4$ SYM could be calculated, they should exhibit this scaling.

- For AdS_3, Barrella et al. show that the right type of corrections are produced by the Selberg zeta function which Giombi, Maloney, and Yin argue gives the one loop gravity correction to the partition function.
- Can we compute corrections to the gravity side in AdS_d, $d > 3$? Can we abstract any lessons from the AdS_3 example?
- Can we compute these corrections at weak coupling in $\mathcal{N} = 4$ SYM using field theory?

Further Discussion

Regarding the fermionic and bosonic systems studied here:

- The two-point functions $\langle \Psi\Psi \rangle$, $\langle \pi\pi \rangle$, and $\langle \phi\phi \rangle$ are Toeplitz. Can results such as the Szëgo limit theorem or the Fisher-Hartwig formula be used? For the XY model Korepin et al. 2003 have used such techniques successfully.
- Can the correlation function $\langle e^{\phi(u)} e^{-\phi(v)} \rangle$ be computed in the sine-Gordon model (i.e. for $m > 0$)? On flat space, the answer is yes [Bernard and LeClair 1994].
- Can we compute entanglement entropy for interacting fermions? For more general couplings, sine-Gordon is dual to interacting fermions. Note however that the replica trick is not compatible with the quartic interaction term.

Extra Slides

Mutual Information

$$
\begin{aligned}
I_n(A,B) &= S_n(A)+S_n(B)-S_n(A\cup B)\\
&= \frac{n+1}{6n}\log\left|\frac{\vartheta_1(\frac{\ell_1+\ell_3}{L}|\tau)\vartheta_1(\frac{\ell_2+\ell_3}{L}|\tau)}{\vartheta_1(\frac{\ell_1+\ell_2+\ell_3}{L}|\tau)\vartheta_1(\frac{\ell_3}{L}|\tau)}\right|\\
&\qquad -\frac{2}{1-n}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}}\log\left|\frac{\vartheta_\nu(\frac{k}{n}\frac{\ell_1+\ell_2}{L}|\tau)\vartheta_\nu(0|\tau)}{\vartheta_\nu(\frac{k}{n}\frac{\ell_1}{L}|\tau)\vartheta_\nu(\frac{k}{n}\frac{\ell_2}{L}|\tau)}\right| .
\end{aligned}
$$

Mutual Information

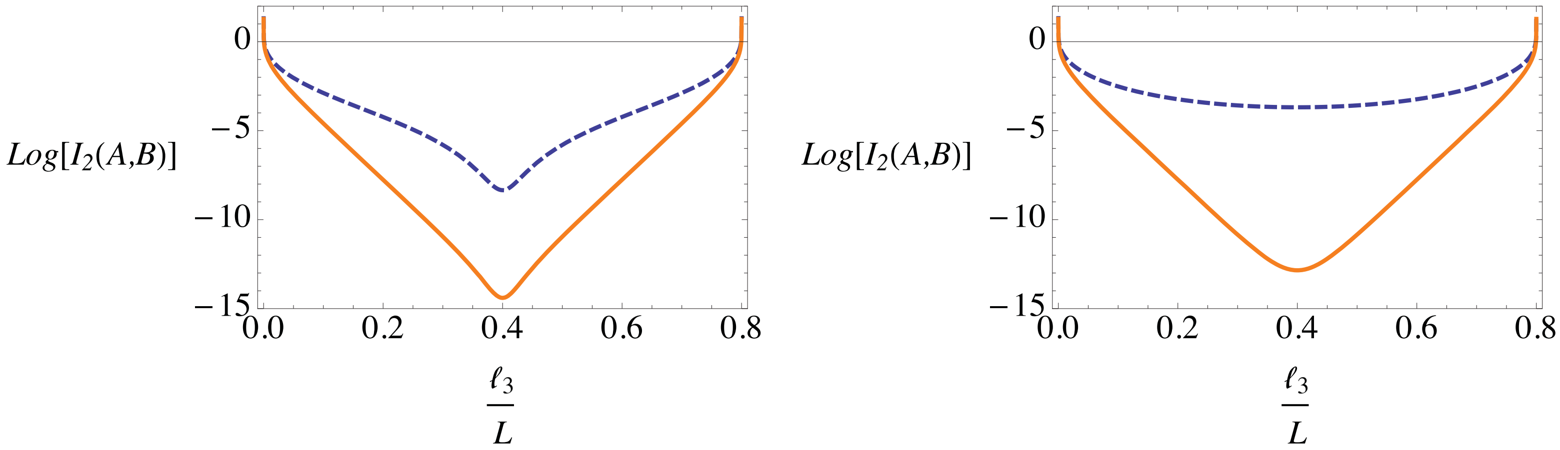


Figure: The mutual Rényi informations of two intervals A and B of width $\ell_1 = \ell_2 = L/10$ with $n = 2$ in the spatially periodic [Left] and antiperiodic [Right] spin sectors. ℓ_3 is the distance between the two intervals. The blue dashed and orange solid curves are for $\beta = 10, 1/5$, respectively.

Mass correction and sine-Gordon

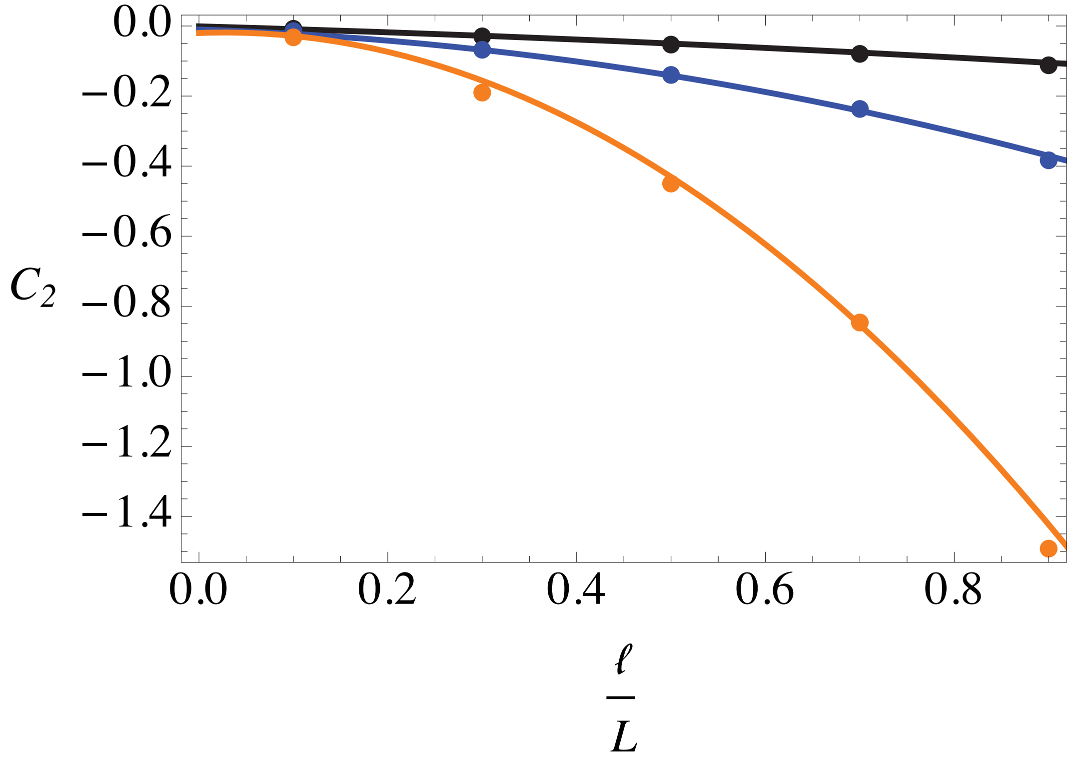


Figure: The ℓ dependence of the $O(m^2)$ correction to the $n = 2$ Rényi entropy for $\nu = 2$. The curves are produced by numerical integration of a result from sine-Gordon. The points are from a lattice computation. From top to bottom, $\beta = 1/2$, 1, and 2.