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Partition functions and stability criteria of topological insulators

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Partition functions and stability criteria of topological insulators

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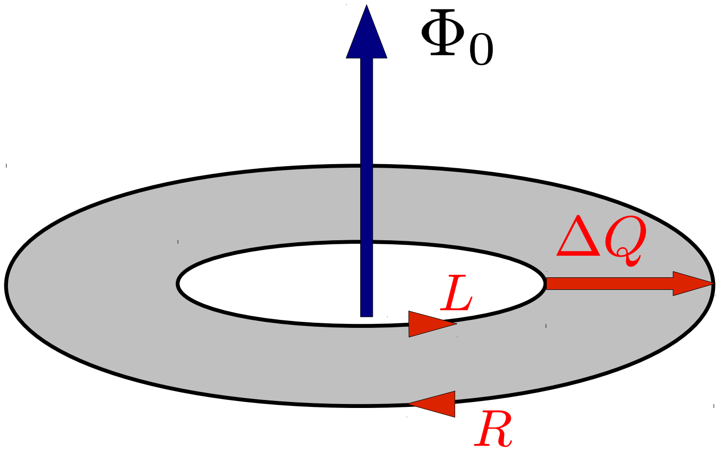
Outline

- Chiral and non-chiral Topological Phases of Matter
- Stability of edge states of Topological Insulators: $\mathbb{Z}_2$ anomaly
- Partition functions and modular invariance
- Electromagnetic and gravitational (discrete) responses
- Stability of TI with interacting & non-Abelian edges

Chiral Topological States

Quantum Hall effect



- chiral edge states
- no Time-Reversal symmetry (TR)
- Laughlin's argument, e.g. $\nu = \frac{1}{3}$

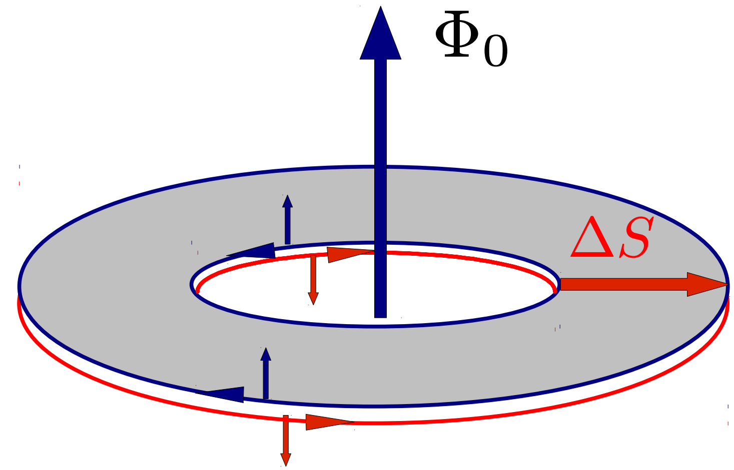
$$\Phi \to \Phi + \Phi_0, \quad H\left[\Phi + \Phi_0\right] = H\left[\Phi\right]$$

$$Q_R \to Q_R + \Delta Q = \nu, \qquad \Delta Q = \int dtdx\ \partial_t J^0_R = \nu \int F = \nu\, n$$ chiral anomaly

- $\Phi \to \Phi + n\Phi_0$ spectral flow between charge sectors $\{0\} \to \{\frac{1}{3}\} \to \{\frac{2}{3}\} \to \{0\}$
- edge chiral anomaly = response of topological bulk to e.m. background
- chiral edge states cannot be gapped ⟷ topological phase is stable
- anomalous response extended to other systems and anomalies in any dimension

(Ryu, J. Moore, Ludwig '10)

Non-chiral topological states



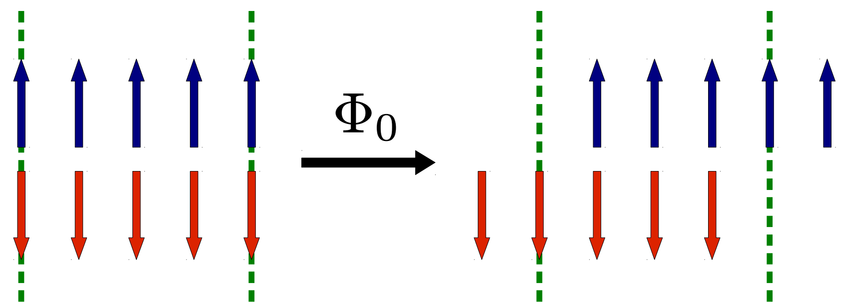
Quantum Spin Hall Effect

- take two $\nu = 1$ Hall states of spins ↑↓
- system is Time Reversal invariant:

$$\mathcal{T}: \ \psi_{k\uparrow} \to \psi_{-k\downarrow}, \qquad \psi_{k\downarrow} \to -\psi_{-k\uparrow}$$

- non-chiral CFT with $U(1)_Q \times U(1)_S$ symmetry
- adding flux pumps spin → $U(1)_S$ anomaly

(X-L Qi, S-C Zhang '08)



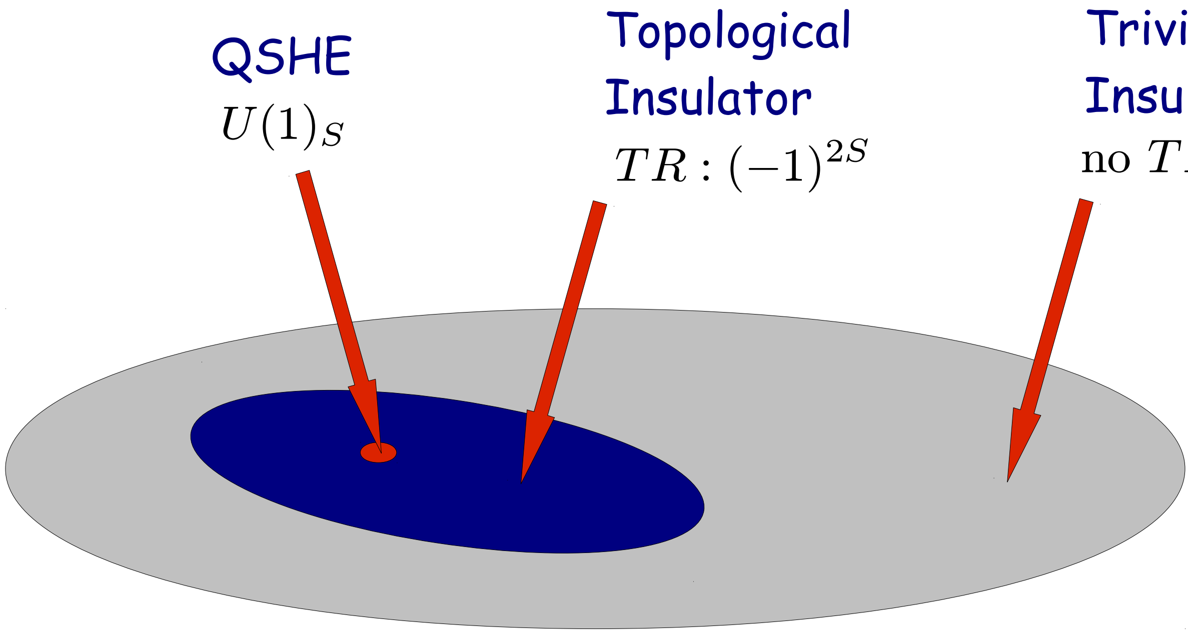
$$\Delta Q = \Delta Q^{\uparrow} + \Delta Q^{\downarrow} = 1 - 1 = 0$$

$$\Delta S = \tfrac{1}{2} - \left(-\tfrac{1}{2}\right) = 1$$

$$\Delta S = \Delta Q^{\uparrow} = \nu^{\uparrow}$$

- in Top. Insulators $U(1)_S$ is explicity broken by Spin-Orbit Coupling etc.
- TR symmetry is kept
- no anomalous currents $\quad \sigma_H = \sigma_{sH} = \kappa_H = 0$

Symmetry Protected Topological Phases



- QSHE edge theory is used to describe Topological Insulator with Time-Reversal symmetry only $U(1)_S \rightarrow (-1)^{2S}$

  Main issue: stability of TI ⟷ stability of non-chiral edge states

- e.g. TR symmetry forbids mass term in CFT with odd number of free fermions

$$\mathcal{T} : H_{\text{int.}} = m \int \psi_\uparrow^\dagger \psi_\downarrow + h.c. \ \rightarrow \ -H_{\text{int.}}$$

$\mathbb{Z}_2$ classification (free fermions)

Topological band theory & flux argument

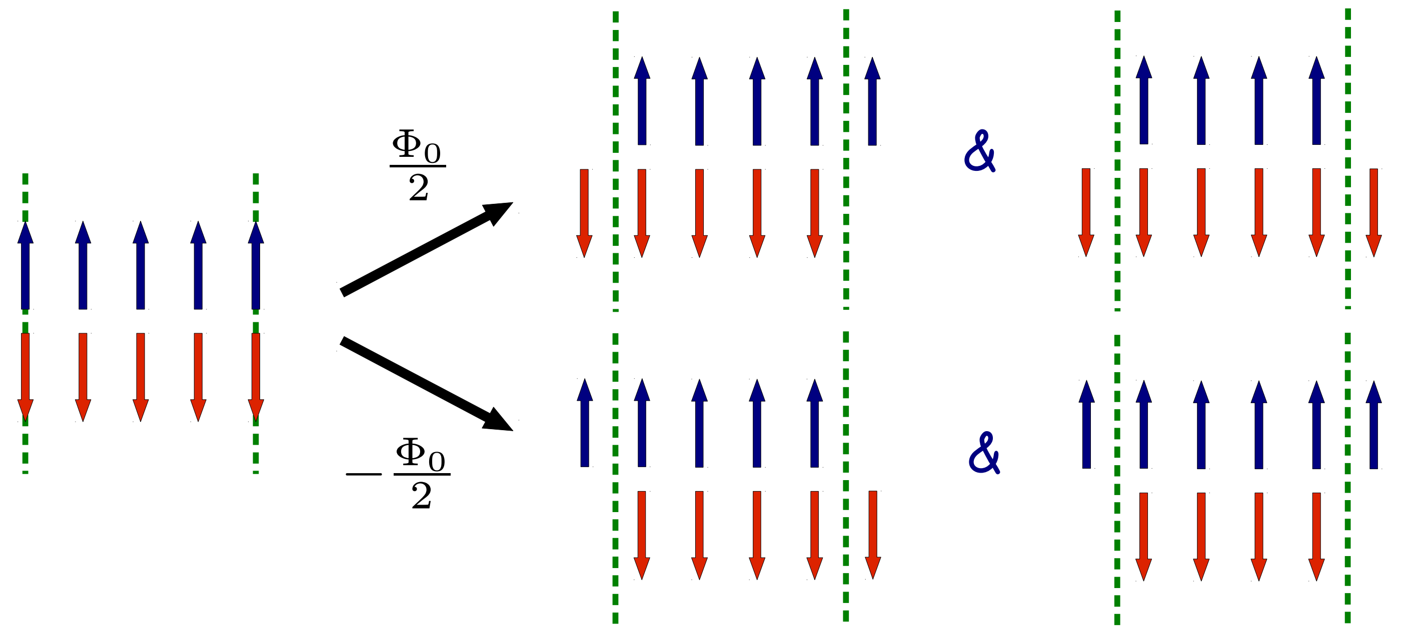
(Fu, Kane, Mele '05-06; Levin, Stern '10-13)

- TR symmetry: $\mathcal{T} H[\Phi] \mathcal{T}^{-1} = H[-\Phi]$ ⟷ $H[\Phi + \Phi_0] = H[\Phi]$
- TR invariant points: $\Phi = 0, \frac{\Phi_0}{2}, \Phi_0, \frac{3\Phi_0}{2}, \ldots$
- we can define TR-invariant polarization $(-1)^{P_\theta}$ (bulk quantity) that:

- is topological, conserved by TR invariance, changes sign for $\Delta\Phi = \frac{\Phi_0}{2}$

- is equal to edge spin parity $(-1)^{P_\theta} = (-1)^{2S} = (-1)^{N_\uparrow + N_\downarrow}$

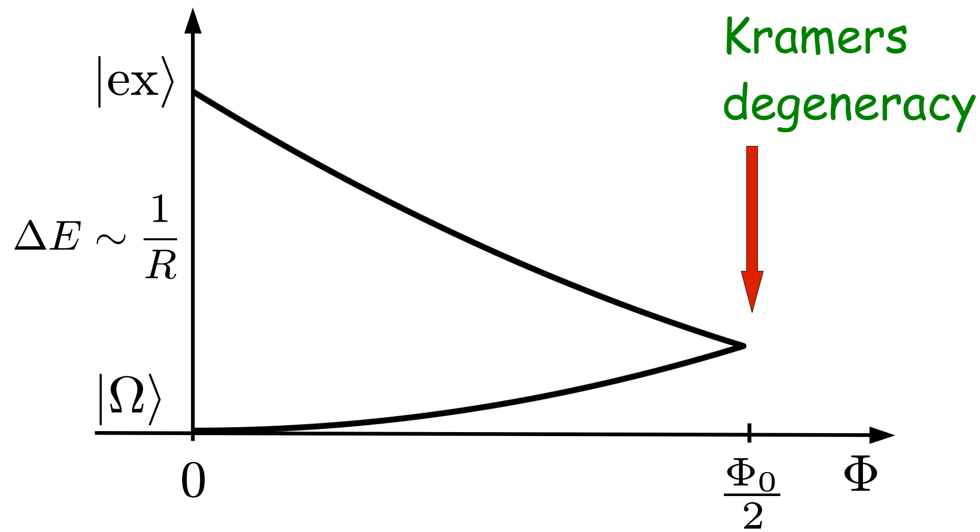
- $(-1)^{2S} = -1$ signals pair of edge states degenerate by Kramers theorem



$$\Phi = 0 : \quad (-1)^{2S} = 1$$

$$\Phi = \frac{\phi_0}{2} : \quad (-1)^{2S} = -1$$

$|\text{ex}\rangle$ gapless edge state



Conclusions

- topological phase is protected by TR symmetry if $\exists$ edge Kramers pair (N_f odd)
- spin parity is anomalous, discrete remnant of spin anomaly $U(1)_S \to (-1)^{2S}$
- Fu-Kane flux argument is Laughlin's argument for discrete $\mathbb{Z}_2$ anomaly $(-1)^{2S}$

Questions

- Stability of TR-symmetric Topological Insulators with interacting fermions?
- Stability of Top. Superconductors, i.e. of neutral (Majorana) edge fermions?
- Can we use modular non-invariance, i.e. discrete gravitational anomaly of the partition function as another probe?

(Ryu, S-C Zhang '12)

Answers in this talk

- partition functions of edge states for QHE, QSHE and TI are completely understood (AC, Zemba '97; AC, Georgiev, Todorov '01; AC, Viola '11)
- we can study flux insertions and discuss stability
- we can study modular transformations and compare e.m. & grav. responses

→ TI: $\mathbb{Z}_2$ classification extends to interacting & non-Abelian edges

$(-1)^{2\Delta S} = +1$ unstable, Z modular invariant

$\phantom{(-1)^{2\Delta S}} = -1$ stable, Z not modular invariant

$$2\Delta S = \frac{\sigma_{sH}}{e^*} = \frac{\nu^\uparrow}{e^*}$$

spin-Hall conductivity = chiral conduct.
minimal fractional charge

(Levin, Stern)

→ Top. Superconductors: richer structure not fully understood yet

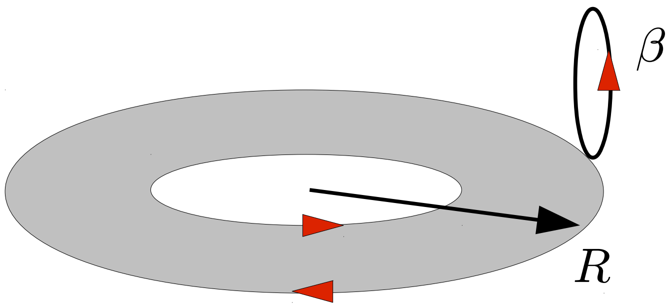
QHE partition function

Consider states of outer edge for $\nu = \frac{1}{p},\ p\ \mathrm{odd}$ described by $c = 1$ Luttinger CFT

$E \sim P \sim v\frac{L_0}{R}$ edge energy & momentum

$\tau = v\frac{i\beta}{2\pi R} + t,$ modular parameter

$\zeta = \frac{\beta}{2\pi}(iV_o + \mu)$ electric & chemical pot.



Partition function for one charge sector $Q = \frac{\lambda}{p} + n, \quad n \in \mathbb{Z}$

sum of characters for representations of $\widehat{U(1)}$ current algebra

$$K_\lambda(\tau,\zeta;p) = \mathrm{Tr}_{\mathcal{H}^{(\lambda)}}\left[\exp\left(i2\pi\tau L_0 + i2\pi\zeta Q\right)\right]$$

$$= \frac{1}{\eta(\tau)}\sum_n \exp\left(i2\pi\left(\tau\frac{(np+\lambda)^2}{2p} + \zeta\frac{np+\lambda}{p}\right)\right)$$

$\frac{\text{theta function}}{\text{Dedekind function}}$

$K_{\lambda+p} = K_\lambda$

Modular trasformations

$$K_\lambda(\tau,\zeta;p) = \frac{1}{\eta(\tau)} \sum_n \exp\left(i2\pi \left(\tau \frac{(np+\lambda)^2}{2p} + \zeta \frac{np+\lambda}{p} \right) \right), \qquad Q = \frac{\lambda}{p} + \mathbb{Z}$$

discrete coordinate changes respecting double periodicity

$S : \tau \to -1/\tau$ exchanges periods

$T : \tau \to \tau + 1$ adds a twist, i.e. a finite translation

also consider e.m. background changes:

$U : \zeta \to \zeta + 1$ adds the weight $e^{i2\pi Q}$

$V : \zeta \to \zeta + \tau$ adds a flux quantum $\Phi \to \Phi + \Phi_0,$

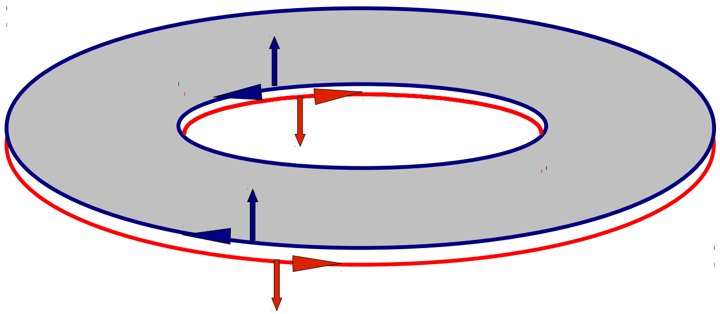
$S: \quad K_\lambda\left(\frac{-1}{\tau}, \frac{-\zeta}{\tau}\right) = \sum_{\mu=1}^{p} S_{\lambda\mu} K_\mu(\tau,\zeta)$ unitary S matrix, completeness

$T^2: \quad K_\lambda(\tau+2,\zeta) = \exp\left(i4\pi h_\lambda\right) K_\lambda(\tau,z)$ odd-integer electron statistics

$U: \quad K_\lambda(\tau,\zeta+1) = \exp\left(i2\pi\lambda/p\right) K_\lambda(\tau,z)$ integer electron charge

$V: \quad K_\lambda(\tau,\zeta+\tau) \sim K_{\lambda+1}(\tau,z)$ Φ_0 flux insertion: $Q \to Q + \nu$

Partition function of Topological Insulators



Partition function for a single edge:

- combine two chiralities $K_{\lambda}^{\uparrow}\,\overline{K}_{\mu}^{\downarrow}$

- fractional charges should match locally

$$Z^{NS}(\tau,\zeta) = \sum_{\lambda=1}^{p} K_{\lambda}^{\uparrow}\,\overline{K}_{-\lambda}^{\downarrow}, \quad S, T^2, U, V \ \text{invariant}$$

In fermionic systems there are always four sectors of the spectrum:

$$NS,\ \widetilde{NS},\ R,\ \widetilde{R},\ \text{risp.}\ (AA),\ (AP),\ (PA),\ (PP)$$

Ramond sector describes half-flux insertions: $\frac{\Phi_0}{2}, \frac{3\Phi_0}{2}, \ldots$

$$V^{\frac{1}{2}} : Z^{NS}(\tau,\zeta) \to Z^{NS}\left(\tau, \zeta + \frac{\tau}{2}\right) \sim Z^{R}(\zeta,\tau)$$

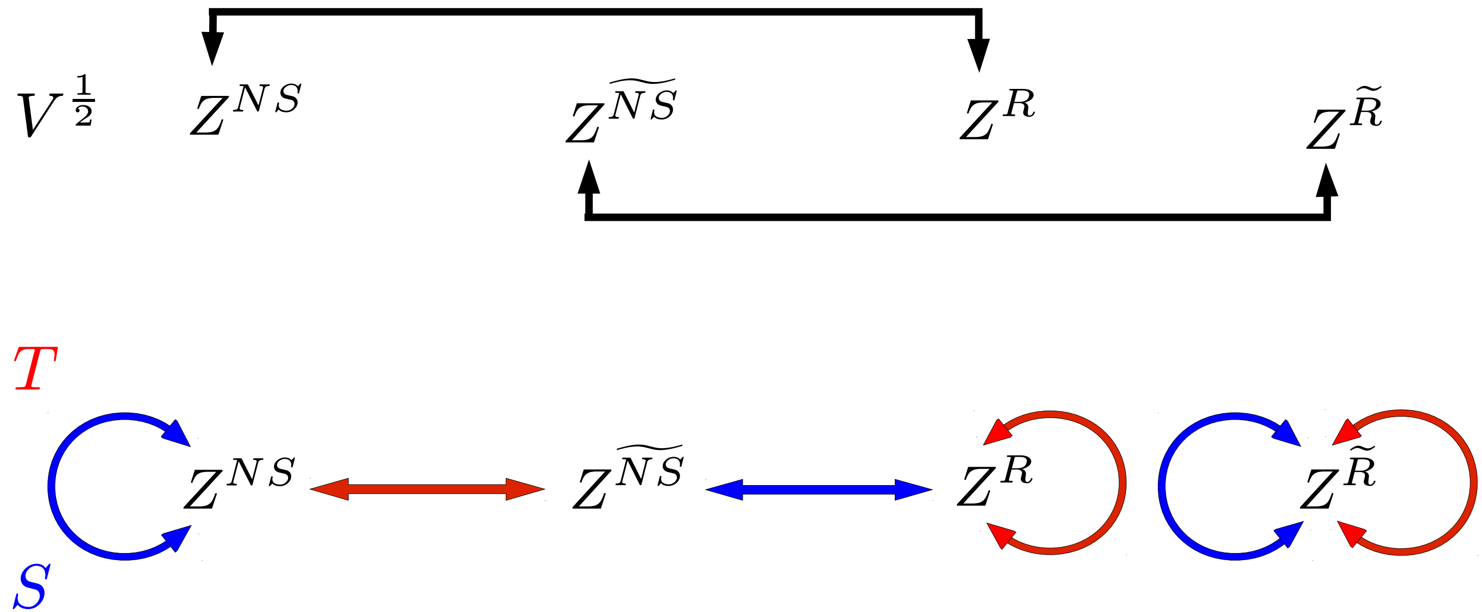
Each sector has $\lambda = 1, \ldots, p$ vacua for the would-be fractional charges, e.g.

$$Z^{R} = \sum_{\lambda=1}^{p} \widehat{K}_{\lambda}^{\uparrow}\,\overline{\widehat{K}}_{-\lambda}^{\downarrow}, \qquad \widehat{K}_{\lambda}(\tau,\zeta) \sim K_{\lambda}\left(\tau, \zeta + \frac{\tau}{2}\right), \qquad V : Z^{NS} \to Z^{NS}, \quad Z^{R} \to Z^{R}$$

E.m. & gravitational responses

e.m.

grav.



Modular invariant partition function of single TI edge can always be found

$$Z_{\text{Ising}} = Z^{NS} + Z^{\widetilde{NS}} + Z^{R} + Z^{\widetilde{R}}, \qquad S, T, U, V^{\frac{1}{2}} \quad \text{invariant}$$

contrary to that of chiral QHE edge K_λ e.g. $S : K_\lambda \to \sum S_{\lambda\mu} K_\mu$

But: is Z_{Ising} consistent with TR symmetry?

Stability and modular (non)invariance

- p fluxes are needed to create one electron excit. in the same charge sector

$$V^p : K_\lambda^\uparrow \rightarrow K_{\lambda+p}^\uparrow = K_\lambda^\uparrow, \qquad \Delta Q^\uparrow = \frac{p}{p} = 1 \qquad\qquad \nu = \frac{1}{p}$$

- Flux argument: add $\frac{p}{2}$ fluxes and check odd spin parity $(-1)^{2S}$ for having degenerate Kramers pair

$$V^{\frac{p}{2}} : \ K_0 \rightarrow K_{p/2} \sim K_0^R, \qquad |\Omega\rangle_{NS} \rightarrow |\Omega\rangle_R \, , \qquad \Delta Q^\uparrow = \Delta S = \tfrac{1}{2}$$

$$(-1)^{2S} |\Omega\rangle_{NS} = |\Omega\rangle_{NS} \rightarrow (-1)^{2S} |\Omega\rangle_R = -|\Omega\rangle_R$$

stable TI

- Spin parity of Ramond ground state is different from that of Neveu-Schwarz gs

→ $\mathbb{Z}_2$ spin-parity anomaly recovered

→ $Z_{\text{Ising}} = Z^{NS} + Z^{\widetilde{NS}} + Z^{R} + Z^{\widetilde{R}}$ not consistent with TR symmetry

$(-1)^{2S} = 1 \qquad 1 \qquad -1 \qquad -1$ (also violates spin-statistics)

TR symmetry + anomaly → no modular invariance → stable insulator

modular invariance +TR symmetry → no anomaly → trivial insulator

General stability analysis

- QHE edge theory involves neutral excitations (possibly non-Abelian)
- neutral part of electron is Abelian: “simple current” modular invariant
- fractional charge sectors $K_\lambda(\tau,\zeta)$ are replaced by $\Theta^\alpha_\lambda(\tau,\zeta)$ (A.C., Viola '11)

  - two integers (k,p)

$$\Theta^\alpha_\lambda(\tau,\zeta) = \sum_{a=1}^{k} K_{\lambda+ap}(\tau, k\zeta; kp)\ \chi^\alpha_{\lambda+ap}(\tau,0)$$

= {g.s.} + {1 el.} + {2 el.} +......

charged neutral

- minimal charge: $Q = \frac{k\lambda}{kp}, \quad \lambda = 1, \quad e^* = \frac{1}{p}$ # charged sectors = p
- Hall current: $V : \zeta \to \zeta + \tau, \quad \lambda \to \lambda + k, \quad \Delta Q = \nu^\uparrow = \frac{k}{p}$
- construct TI partition function as before $Z^{NS} = \sum_{\lambda\alpha} \Theta^\alpha_\lambda \overline{\Theta}^\alpha_{-\lambda}$
- Stability: add fluxes to create an excitation in the same charge sector

$$V^{\frac{p}{2}} : \quad \Delta S = \Delta Q^\uparrow = \frac{p}{2}\,\nu^\uparrow = \frac{k}{2}$$

Kramers pairs if k odd ⟶ stable TI

Levin-Stern index	$2\Delta S = \frac{\nu^\uparrow}{e^*},$	$(-1)^{2\Delta S} = (-1)^k$	fully general

- earlier analysis of modular invariance vs. stability can be extended:

  k even = unstable, TR-inv. modular invariant $\quad Z_{\text{Ising}} = Z^{NS} + Z^{\widetilde{NS}} + Z^{R} + Z^{\widetilde{R}}$

  k odd = stable, modular vector $\quad Z = \left(Z^{NS}, Z^{\widetilde{NS}}, Z^{R}, Z^{\widetilde{R}}\right)$

Examples

- Jain-like TI $\quad \nu^{\uparrow} = \frac{k}{2nk+1}, \quad e^{*} = \frac{1}{2nk+1}, \quad 2\Delta S = \frac{\nu^{\uparrow}}{e^{*}} = k \quad$ stable / unstable

- (331) & Pfaffian TI $\quad \nu^{\uparrow} = \frac{1}{2}, \quad e^{*} = \frac{1}{4}, \quad 2\Delta S = 2 \quad$ unstable

- Abelian TI $\quad K = \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} \quad \nu^{\uparrow} = \frac{3}{7}, \quad e^{*} = \frac{1}{7}, \quad 2\Delta S = 3 \quad$ stable

- Read-Rezayi TI $\quad \nu^{\uparrow} = \frac{k}{kM+2}, \quad e^{*} = \frac{1}{kM+2}, \quad 2\Delta S = k \quad$ stable / unstable

Remarks

- general expression of partition function allows to extend Levin-Stern stability criterium to any TI with interacting fermions

$\mathbb{Z}_2$ classification of TI protected by TR invariance

- neutral states are invariant under flux changes
- neutral states recombined by modular transformations but the structure in four sectors $NS, \widetilde{NS}, R, \widetilde{R}$ is always present
- unprotected edge states do become fully gapped?

- Abelian states: yes, by careful analysis of possible TR-invariant interactions

(Levin, Stern; Neupert et al,; Y-M Lu, Vishwanath)

- non-Abelian states: yes, use projection from "parent Abelian states",

e.g. (331) -> Pfaffian, and [projection, TR-inv.] (A.C., Georgiev, Todorov)

Topological Superconductors

- stability of TS ⟷ gapless non-chiral Majorana edge states
- N_f free Majorana $\sim \frac{N_f}{2}$ charged fermions but neutral, no flux insertion argument
- $\mathbb{Z}_2 \times \mathbb{Z}_2$ chiral spin parity $(-1)^{2S^\uparrow}(-1)^{2S^\downarrow} = (-1)^{N_\uparrow}(-1)^{N_\downarrow}$ no spin flip

$\mathbb{Z}$ classification (free fermions)

- $N_f = 8$ unstable by non-trivial quartic interaction: $\mathbb{Z} \to \mathbb{Z}_8$ (many people) (Ryu, S-C-Zhang; Sule, X. Chen,Ryu)

→ proposal: study partition function and gravitational response

- Standard invariant for any N_f

$$Z_{\text{Ising}} = Z^{NS} + Z^{\widetilde{NS}} + Z^R + Z^{\widetilde{R}}$$

Is it consistent with $\mathbb{Z}_2 \times \mathbb{Z}_2$ parity?

- Yes, for $N_f = 8$ mod. trasf. $ST \sim V^{\frac{1}{2}}$ creates $\Delta S^\uparrow = \Delta S^\downarrow = 1$ in R sector OK
- Ryu-Zhang: test modular invariance of subsector $(-1)^{N_R} = (-1)^{N_L} = 1$

→ GSO projection $N_f = 8$

- general analysis of modular invariance + discrete symm. not understood yet
- no charge matching allows many more modular invariants $Z = \sum \mathcal{N}_{\lambda\mu} K_\lambda^\uparrow \overline{K}_\mu^\downarrow$

Conclusions

- partition functions of QSHE and TI:
  - general structure
  - response to e.m. background & discrete reparameterizations
  - $\mathbb{Z}_2$ spin parity anomaly
  - recover and generalize stability analysis of TI with TR symmetry protection

  → $\mathbb{Z}_2$ classification   index $(-1)^{2\Delta S}$, $2\Delta S = \frac{\nu^{\uparrow}}{e^*} = \frac{k}{p}$

- work in progress:
  - partition functions for TS, form & stability ↔ Ryu-Zhang stability criterium
  - modular invariance vs. fusion rules ↔ Levin stability criterium