Exercises related to the lecture series
Geometry and Spectral Variation

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Notations are as in the Lecture Notes

X1. It is clear from the definition that the operator norm $\|\cdot\|$ is unitarily invariant, ie $\|UMW\| = \|M\|$ for any unitary U, W and that $\|XY\| \leq \|X\|\,\|Y\|$. Show that these properties also hold for the Frobenius norm $\|\cdot\|_2$.

By compactness any two norms on a finite–dimensional vector space are equivalent, ie their ratios are bounded above and below. Show that $\|M\| \leq \|M\|_2 \leq \sqrt{n}\|M\|$ for any $M \in \mathbb{M}_n$ and that these inequalities are best possible.

X2. If $S \in \mathbb{M}_n$ is normal, note that $\mathrm{span}\{S, S^*\}$ consists entirely of normals. If S is not normal, show that the normal matrices in $\mathrm{span}\{S, S^*\}$ are precisely those of the form $\alpha S + \beta S^*$ where $|\alpha| = |\beta|$.

X3. Show, by computing eigenvalues of X^*X, that $X = \alpha J_3 + \beta J_3^*$ has norm $\|X\| = \sqrt{|\alpha|^2 + |\beta|^2}$.

X4. Show that $\sqrt{1 - 2/n}(\|\alpha| + |\beta|) \leq \|\alpha J_n + \beta J_n^*\|$ by choosing unit $u \in \mathbb{C}^n$ so as to make $\|(\alpha J_n + \beta J_n^*)u\|$ large.

X5. Show that if $A, B \in \mathbb{N}_n$ and the real span of $\{A, B\}$, ie $\mathbb{R}A + \mathbb{R}B$, contains a nontrivial normal C (ie C is not a scalar multiple of A or B) then all of $\mathbb{R}A + \mathbb{R}B$ is normal.

X6. Find 2×2 unitary U, W so that $U + W$ is not normal; by X5 the affine path $[U, W]$ is not normal, yet there IS a normal path of the same length ($\|U - W\|$), by Proposition 7.2; what can you say about the short normal path in this simple case?

X7. In place of the fancy proof using Corollary 7.3 and the normal path inequality, give an elementary proof that

$$A, B \in \mathbb{N}_2 \quad \Rightarrow \quad \mathrm{sd}(A, B) \leq \|A - B\|.$$

For example, show that for lists of length 2, sd is the Hausdorff distance, and use Proposition 6.1 ...

X8. If $p_n(\lambda) = \det(\lambda I_n - A_n)$ (the characteristic polynomial of A_n), show that

$$p_n(\lambda) = \lambda p_{n-1}(\lambda) - \frac{1}{4} p_{n-2}(\lambda).$$

Verify by induction that

$$p_n(\lambda) = \frac{\sin(n+1)\theta}{2^n \sin\theta}, \text{ where } \theta = \cos^{-1}(\lambda), |\lambda| \leq 1.$$

Conclude that the eigenvalues of A_n are $\cos(k\pi/(n+1))$ $(k = 1, 2, \ldots, n)$.

X9. Show that $B_n = -iU_n A_n U_n^*$ where U_n is the diagonal unitary $\mathrm{diag}(i, i^2, \ldots, i^n)$. Conclude that the eigenvalues of B_n are $i\cos(k\pi/(n+1))$ $(k = 1, 2, \ldots, n)$.

X10. In the situation of Sunder's Theorem, ie $A, B \in \mathbb{N}_n$ with A Hermitian and B skew–Hermitian, with eigenvalues numbered so that $|a_1| \leq |a_2| \leq \cdots \leq |a_n|$ and $|b_1| \leq |b_2| \leq \cdots \leq |b_n|$, show that a best matching for $\mathrm{sd}(A, B)$ is a_k with b_{n-k+1}.

X11. In the Sunder situation (X10), show that in computing sd_2 the permutations play no role.

X12. In the Sunder situation (X10), the Hoffman–Wielandt inequality is always equality; in fact

$$sd_2(A, B) = \sqrt{\|A\|_2^2 + \|B\|_2^2} = \|A - B\|_2.$$

Applied to A_n, B_n, this yields the identity

$$\sum_1^n \cos^2(k\pi/(n+1)) = (n-1)/2.$$

X13. Show that $\mathrm{sd}(A_n, B_n) = 1$ if n is odd and that $\mathrm{sd}(A_n, B_n) < 1$ if n is even.

X14. In the Sunder situation (X10), consider

$$u \in \mathrm{span}\{u_{n+1-k}, \ldots, u_n\} \cap \mathrm{span}\{w_k, \ldots, w_n\},$$

where the u_k are orthonormal with $Au_k = a_k u_k$ and the w_k are orthonormal with $Bw_k = b_k w_k$. Show that $\|Au\| \geq |a_{n+1-k}|$ and $\|Bu\| \geq |b_k|$, and complete the proof of Sunder's Theorem.

X15. Show that the normal path from A_3 to B_3 found in Proposition 10.2 can be reparametrized in the form $\rho(t) = e^{it}H(t)$ where $t \in [0, \pi/2]$ and $H(t)$ traces out a Hermitian path from A_3 to B_3/i. Find $H(t)$.

X16. Suppose in the setup of X15 we use instead the affine path $[A_3, B_3/i]$, ie $H(t) = \frac{2}{\pi}((\frac{\pi}{2} - t)A_3 + tB_3/i)$. Compare the length of this path with $\sqrt{2}\sin(\pi/(2\sqrt{2}))$.