# Matrix Geometries and Applications
# Exercises

1. Show that $\mathbb{P}_n$, the set of $n \times n$ positive definite matrices, is an open subset of $\mathbb{H}_n$, the space of $n \times n$ Hermitian matrices. (Find as many proofs as you can.)

2. Let $A$ be a positive definite matrix. Show that there exists a Hermitian matrix $H$ such that $A = e^H$. Let $U$ be a unitary matrix. Show that there exists a Hermitian matrix $H$ such that $U = e^{\iota H}$.

3. If $t > 0$ and $0 < r < 1$, then

$$t^r = \frac{\sin r\pi}{\pi} \int_0^\infty \frac{t}{\lambda + t} \lambda^{r-1} d\lambda \tag{1}$$

4. (i) Let $A, B$ be $n \times n$ matrices, and let $m$ be any natural number. Calculate $(A+B)^m$.

   (ii) Let $f : \mathbb{M}_n \to \mathbb{M}_n$ be any differentiable map on the space of matrices. Its derivative at $A$ is a linear map $Df(A)$ on $\mathbb{M}_n$. Its action is given as

$$Df(A)(X) = \frac{d}{dt}|_{t=0} f(A + tX). \tag{2}$$

   Let $f(A) = A^m$. Use Part(i) to calculate $Df(A)$.

   (iii) Suppose $A$ is Hermitian. Choose a basis in which $A$ is diagonal, $A=$ diag$(\lambda_1, ..., \lambda_n)$. Use Part(ii) to show that when $f(A) = A^m$,

$$Df(A)(X) = \left[ \frac{f(\lambda_\iota) - f(\lambda_j)}{\lambda_\iota - \lambda_j} x_{\iota j} \right]. \tag{3}$$

   ( Here $[y_{\iota j}]$ stands for the matrix with entries $y_{\iota j}$).

   (iv) Show that the formula(3) is valid when$f$ is a polynomial. Standard approximation arguments then show it is valid when $f$ is any $C^1$ function.

   (v) We denote by $A \circ B$ the entrywise product $[a_{ij} b_{ij}]$ of two matrices $A$ and $B$. If $A$ is as in Part (iii) we write

$$f^{[1]}(A) = \left[ \frac{f(\lambda_\iota) - f(\lambda_j)}{\lambda_\iota - \lambda_j} \right]. \tag{4}$$

   Then we have

$$Df(A)(X) = f^{[1]}(A) \circ X.$$

   This is called the Daleckii-Krein formula.

5. Let$K$ be a closed convex set in a Hilbert space $\mathcal{H}$. Given any point $x$ of $\mathcal{H}$ there exists a unique point $x_0$ in $K$ which is closest to $x$. Show that for every point $y$ in $K$

$$||x-y||^2 \geq ||x-x_0||^2 + ||x_0-y||^2. \tag{5}$$

If $K$ is a linear subspace of $\mathcal{H}$, then this is an equality.

6. Let $x_1, ..., x_m$ be $m$ vectors in $\mathcal{H}$, and let $\bar{x} = \frac{x_1+...+x_m}{m}$. Then for every $z$ in $\mathcal{H}$

$$||z-\bar{x}||^2 = \sum_{j=1}^{m} \frac{1}{m}\left[||z-x_j||^2 - ||\bar{x}-x_j||^2\right]. \tag{6}$$

7. (i) If $A, B$ are $n \times n$ matrices, then $AB$ and $BA$ have the same eigenvalues.
   (ii) if $A, B$ are positive definite, then all eigenvalues of $AB$ are positive.

8. Let $A, B$ be positive definite. Show that all eigenvalues of $(A-B)(A^{-1}-B^{-1})$ are nonpositive.

9. Let $A, B$ be positive definite. Show that

$$\left(\frac{A+B}{2}\right)^2 \leq \frac{A^2+B^2}{2} \tag{7}$$

and

$$\left(\frac{A+B}{2}\right)^{-1} \leq \frac{A^{-1}+B^{-1}}{2}. \tag{8}$$

10. Let $A \geq B \geq 0$ and $X \geq 0$. Are the following statements true

$$X^{1/2}A\, X^{1/2} \geq X^{1/2}B\, X^{1/2} \tag{9}$$

and

$$A^{1/2}X\, A^{1/2} \geq B^{1/2}X\, B^{1/2} \tag{10}$$