# Geometry and Spectral Variation

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**The problem** $\quad M \in \mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ $\quad$ $n \times n$ complex matrix

$$M \to M' \qquad [m_{ij}] \to [m_{ij}']$$

induces change in the eigenvalues

$$\lambda_k \to \lambda_k' \quad : \quad \sigma(M) \to \sigma(M')$$

¿ Can we quantify this process ?
"perturbation"

Are there useful inequalities

of the form

$$d(\sigma(M), \sigma(M')) \leq \text{Const.} \, \|M - M'\| \quad ?$$

<u>Some</u> metric on sets ?

<u>Some</u> norm on $\mathbb{M}_n$ ?

Perturbation of polynomial roots

Since $\lambda \in \sigma(M) \iff p_M(\lambda) = 0$ where $p_M$ is the characteristic poly'l $p_M(z) = \det(zI_n - M)$ there is interplay between results on spectral perturbation and perturbation of polynomial roots. Evidently

$$M \simeq M' \iff \text{coeffs of } p_M \simeq \text{coeffs of } p_{M'}$$

$$\iff \text{zeros } Z(p_M) \simeq Z(p_{M'})$$

eg classic result of Ostrowski

$$h(Z(p), Z(q)) \leq \Theta(p, q)$$

where $h$ is "Hausdorff distance"

and $\Theta$ is a function of the coeffs of the monic degree $n$ poly'l $p, q$ that is small when the coeffs are close.

Note that $Z(p)$, $Z(q)$ are simply sets of zeros ie multiplicity of roots is ignored. Recall that

$$h(S,T) = \max\left(\max_{s\in S} d(s,T),\ \max_{t\in T} d(t,S)\right)$$

is defined for any non-empty compact sets $S, T \subseteq \mathbb{C}$, where

$d(s,T) = \min_{t\in T} |s-t|$. Thus $h(S,T) = \varepsilon$

means $S \subseteq T_\varepsilon$ = $T$ fattened up by $\varepsilon$

$= T + \{z : |z| \le \varepsilon\}$

and $T \subseteq S_\varepsilon$.

<u>Weakness</u> $p(z) = z(z-1)^{n-1}$, $q(z) = z^{n-1}(z-1)$

are quite different polys but

$Z(p) = Z(q) = \{0,1\}$ so $h(Z(p), Z(q)) = 0$.

<u>Taking multiplicity into account</u>: consider

lists $\{z_1, \dots z_n\}$, $\{w_1, \dots w_n\}$ from $\mathbb{C}$ and

the "matching distance" or "spectral distance"

in our applications:

$$sd(\{z_1, \dots, z_n\}, \{w_1, \dots, w_n\}) =$$

$$\min_{\pi} \left( \max_{k} |z_k - w_{\pi(k)}| \right)$$

where $\pi$ runs over all permutations of the index set $\{1, 2, \dots, n\}$.

( list order has no effect on sd )

eg $sd(\{1,0,0\}, \{1,1,0\}) = 1$.

<u>Two matrix norms</u> $M \in \mathbb{M}_n$

operator norm $\|M\| = \max \{ \|Mu\| : u \in \mathbb{C}^n, \|u\| = 1 \}$

Euclidean length of $u$

= Lipschitz constant of the mapping

$M : \mathbb{C}^n \longrightarrow \mathbb{C}^n$

Thus if $U, W$ are unitary (linear isometries on $\mathbb{C}^n$)

$\|UMW\| = \|M\|$ ("unitary invariance").

2-norm or Frobenius norm

$$\|M\|_2 = \left(\sum_{i,j} |m_{ij}|^2\right)^{1/2} \equiv \text{Euclidean norm on } \mathbb{C}^{n^2}$$

Check that $\|\cdot\|_2$ is also unitarily invariant and that $\|M\| \le \|M\|_2 \le \sqrt{n}\,\|M\|$

<u>Two landmark results for normal matrices</u>

Recall $M \in \mathbb{M}_n$ <u>normal</u> means

$M = UDU^*$ where $U$ unitary, $D$ diagonal

($\Leftrightarrow$ $M^*M = MM^*$) ; we write $M \in \mathbb{N}_n$

So if $D = \operatorname{diag}(\lambda_1, \dots \lambda_n)$

$\sigma(D)$ as a <u>list</u> is $\{\lambda_1, \dots, \lambda_n\}$.

1. <u>Weyl (c.1912)</u> : If $A, B \in \mathbb{N}_n$ are <u>Hermitian</u>

then $\operatorname{sd}(\sigma(A), \sigma(B)) \le \|A - B\|$

One way to prove this (not the best ?)

$A, B$ Hermitian $\Longleftrightarrow$ normal with <u>real</u> eigenvalues

say $\sigma(A) = \{a_1 \le a_2 \le \cdots \le a_n\}$

$\sigma(B) = \{b_1 \le b_2 \le \cdots \le b_n\}$

We show $|a_k - b_k| \le \|A - B\|$ so that

$$\mathrm{sd}(\sigma(A), \sigma(B)) \le \max_k |a_k - b_k| \le \|A - B\|$$

We may assume $a_k \ge b_k$

Let $u_1, \ldots u_n$ be o.n. eigenvectors for $A$

with $Au_k = a_k u_k$, and

$w_1, \ldots w_n$ o.n. with $Bw_k = b_k w_k$.

Take unit vector

$$u \in \underbrace{\mathrm{span}\{u_k, \ldots u_n\}}_{\dim\ n-k+1} \cap \underbrace{\mathrm{span}\{w_1, \ldots, w_k\}}_{\dim\ k} \qquad \left(\text{sum } \underset{n}{\underset{\vee}{n+1}}\right)$$

$$u = \sum_{j=k}^{n} z_j u_j \qquad 1 = \|u\|^2 = \sum_k^n |z_k|^2$$

$$(Au, u) = \sum_{j=k}^{n} a_j |z_j|^2 \ge a_k \sum_k^n |z_k|^2 = a_k .$$

2

Similarly $(Bu,u) \leq b_k$ so that

$$a_k - b_k \leq ((A-B)u,u) \leq \|A-B\| . \qquad \text{QED}$$

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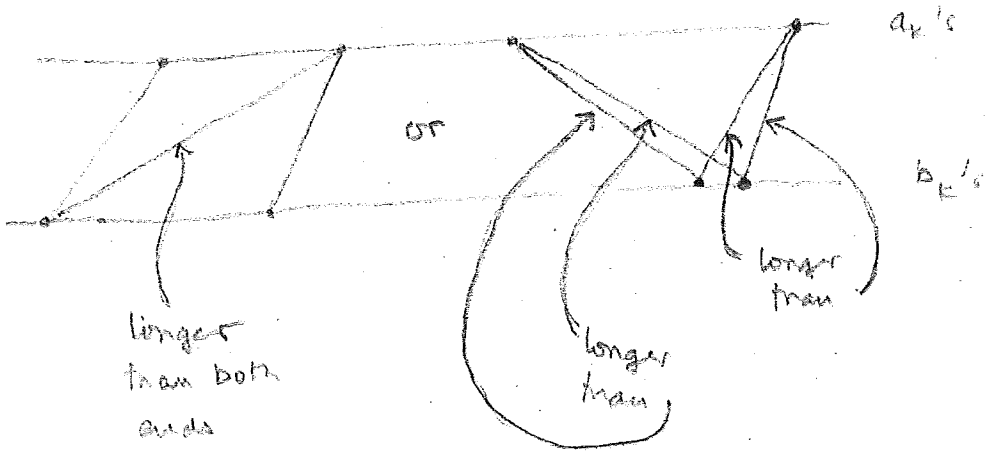
Note that in fact $sd(\sigma(A), \sigma(B)) = \max_k |a_k - b_k|$,

ie $a_k \leftrightarrow b_k$ is the optimal matching:

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2. <u>Hoffman – Wielandt</u> (c. 1953) For <u>all</u> $A, B \in \mathbb{N}_n$

$$sd_2(\sigma(A), \sigma(B)) \leq \|A - B\|_2$$

appropriate spectral distance for $\|\cdot\|_2$ ie

$$\min_{\pi} \|D_A - P(\pi) D_B P^*(\pi)\|_2 \qquad \text{where}$$

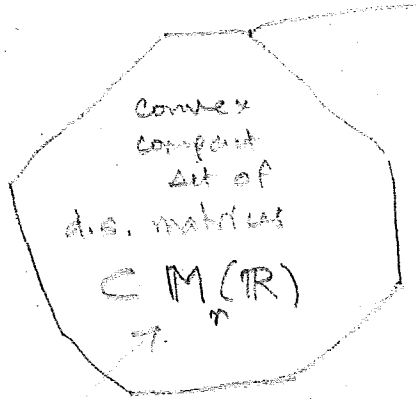
$P(\pi)$ is the permutation matrix corresponding to $\pi$ and $D_A$, $D_B$ are diagonal forms for $A$, $B$

<u>Proof</u> $\|A-B\|_2^2 = \|UD_AU^* - WD_BW^*\|_2^2$

$$= \|D_A \underbrace{U^*W}_{\text{unitary } V} - \underbrace{U^*W}_{V} D_B\|_2^2 = \|[(\alpha_i - \beta_j)v_{ij}]\|_2^2$$

$$= \sum_{i,j} |\alpha_i - \beta_j|^2 |v_{ij}|^2 \quad , \text{ where } [|v_{ij}|^2] \text{ is}$$

doubly stochastic (nonneg entries, each row & column sum $= 1$)




(or lower dim'l space)

each extreme point is a permutation matrix $P(\pi)$

thus (Birkhoff ...)

$$[|v_{ij}|^2] = \sum_\pi t_\pi P(\pi)$$

(convex combination: $t_\pi \geq 0$, $\sum t_\pi = 1$)

Thus $\|A-B\|_2^2 = \sum_\pi t_\pi \left( \sum_{i,j} |a_i - b_j|^2 P_{ij}(\pi) \right)$

$$= \sum_\pi t_\pi \left( \sum_i |a_i - b_{\pi(i)}|^2 \right).$$

For some $\pi$ we must have $\sum_i |a_i - b_{\pi(i)}|^2 \le \|A-B\|_2^2$

$$= \|D_A - P(\pi) D_B P^*(\pi)\|_2^2 \qquad \text{QED}$$

Mirsky (c. 1960) – Wielandt (c. 1953 ?) conjecture

$$\forall A, B \in \mathbb{N}_n \qquad sd(A, B) \le \|A - B\| \qquad (1)$$

ie $sd(\sigma(A), \sigma(B))$

True in many interesting cases – but not all (c. 1992)

Normal path inequality (Bhatia c. 1982, ...)

$$\forall A, B \in \mathbb{N}_n \qquad sd(A, B) \le |\gamma|, \qquad (2)$$

where $|\gamma|$ is the arclength (with respect to $\|\cdot\|$) any normal path $\gamma$ from $A$ to $B$.

Some consequences

Now look at Weyl's thm : $A, B$ Hermitian

$\Rightarrow$ the affine path $[A,B] = \{(1-t)A + tB : t \in [0,1]\}$

is all Hermitian, therefore normal path, so

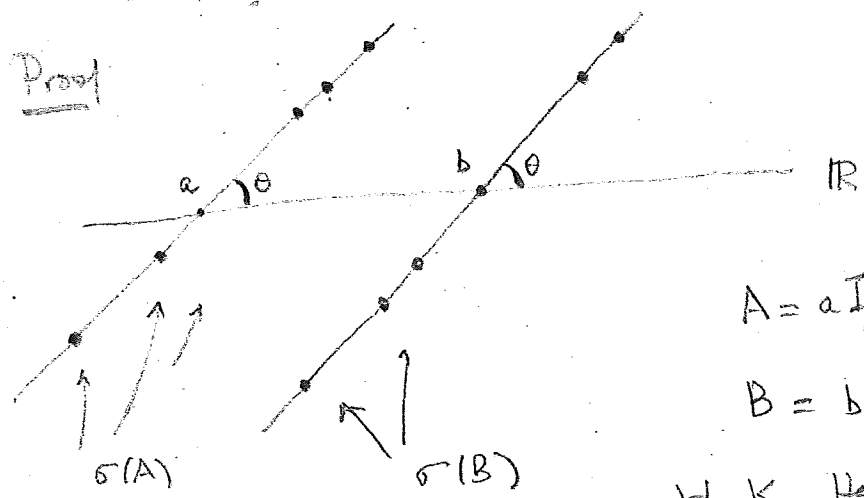
$$sd(A,B) \leq |[A,B]| = \|A-B\| .$$

More generally

Propn 7.1 If $A, B \in \mathbb{M}_n$ and $\sigma(A), \sigma(B)$ lie on parallel lines then

$$sd(A,B) \leq \|A-B\| .$$

Proof




$A = aI_n + e^{i\theta}H$

$B = bI_n + e^{i\theta}K$

$H, K$ Hermitian

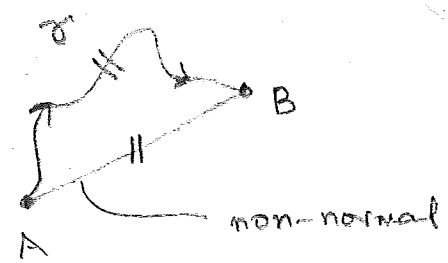
$$(1-t)A + tB = ((1-t)a+tb)I_n + e^{i\theta}(\underbrace{(1-t)H + tK}_{\text{Hermitian}})$$

So again $[A,B]$ is a normal path so that

$$sd(A,B) \leq |[A,B]| = \|A-B\|. \quad \text{QED}$$

<u>Prop'n 7.2</u> If $A, B \in \mathbb{N}_n$ and $\sigma(A)$, $\sigma(B)$ lie on <u>concentric circles</u>, then $sd(A,B) \leq \|A-B\|$ <u>because</u> there is a normal path $\gamma$ from $A$ to $B$ with $|\gamma| = \|A-B\|$.

<u>Note</u> In most cases $[A,B]$ is not normal, in fact one can check that if <u>any</u> $C \in \mathbb{R}A + \mathbb{R}B$, $C \neq A, B$ is normal then all of $\mathbb{R}A + \mathbb{R}B \subseteq \mathbb{N}_n$. Thus the existence of such $\gamma$ depends on the <u>non Euclidean</u> nature of the $\|\cdot\|$-geometry




$C \notin \mathbb{R}A, \mathbb{R}B$

<u>Proof</u> We look at a specific but representative case where $A, B$ are $5\times5$ permutation matrices (unitary)

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Can check that $P+Q$ is not normal to the path $[P, Q]$ is not available and in fact $\gamma$ must lie entirely outside $\mathbb{R}P + \mathbb{R}Q$ (except for the end points)

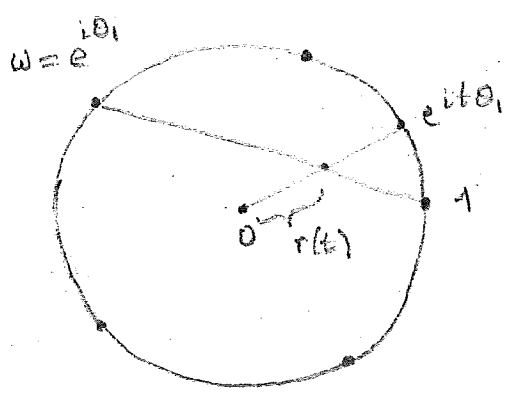
$P^*Q$ corresponds to a 5-cycle so eigenvalues are the 5th roots of unity

$$\|P - Q\| = \|I - P^*Q\| = \|I - e^{iH}\| \quad (\simeq 1.9021)$$

$$= |1 - \omega|$$

$\theta_1 = \max |\theta_k|$

$\theta_k$ eigenvalues of $H$




Let $\gamma : [0,1] \to \mathbb{N}_5$ be defined by

$$\gamma(t) = r(t)\, P e^{itH}$$

$$\gamma(0) = 1 P I = P, \quad \gamma(1) = 1 P P^* Q = Q$$

Magic :

$$|\gamma| = \int_0^1 \|\gamma'(t)\|\,dt = \int_0^1 \|r'(t)\,Pe^{itH} + r(t)\,Pe^{itH}\,iH\|\,dt$$

$$= \int_0^1 \underbrace{\|r'(t)\,I_n + r(t)\,iH\|}_{\substack{\text{modulus of largest of} \\ \text{eigenvalues } r'(t) + r(t)\,i\theta_k}}\,dt$$

$$= \int_0^1 |r'(t) + r(t)\,i\theta_1|\,dt$$

$$= \int_0^1 \underbrace{|r'(t)\,e^{it\theta_1} + r(t)\,i\theta_1 \cdot e^{it\theta_1}|}_{(r(t)e^{it\theta_1})'}\,dt$$

$$= \text{length of } [1, \omega] = |1 - \omega| = \|P - Q\|.$$

QED

(general case

$A = cI + r_0 U$, $B = cI + r_1 W$

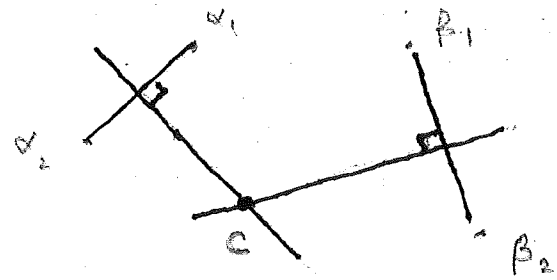
$U, W$ unitary etc.)

(Lecture 11 ...)



<u>Corollary 7.2</u> $\forall A, B \in \mathbb{N}_2$ there is a normal path from $A$ to $B$ of length $\|A-B\|$

<u>Proof</u>




right bisectors

QED

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To understand the normal path inequality note first that (1) for the <u>Hausdorff</u> distance is easy: $A, B \in \mathbb{N}_n \Rightarrow$

$$h(\sigma(A), \sigma(B)) \leq \|A-B\|$$

<u>because</u> of

<u>Prop'n 6.1</u> If $A \in \mathbb{N}_n$ and $M \in \mathbb{M}_n$ (arbitrary matrix) then for each eigenvalue $\mu$ of $M$ there is some eigenvalue $\alpha_k$ of $A$ such that

$$|\mu - \alpha_k| \leq \|A - M\|$$

<u>Proof</u> Let $u_k$ be o.n. with $Au_k = \alpha_k u_k$.

Let $u$ be a unit vector such that $Mu = \mu u$.

Then $\|A - M\|^2 \geq \|Au - Mu\|^2$.

$$= \left\| A \sum_k (u, u_k) u_k - \mu \sum_k (u, u_k) u_k \right\|^2$$

$$= \sum_k |\alpha_k - \mu|^2 |(u, u_k)|^2 .$$ Since

$\sum_k |(u, u_k)|^2 = 1$ some $|\alpha_k - \mu| \leq \|A - M\|$. QED

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Now consider a normal path $\gamma : [0;1] \longrightarrow \mathbb{N}_n$ from $A$ to $B$. Let $\gamma_t = \gamma|_{[0,t]}$

and suppose we have $sd(A, \gamma(t)) \leq |\gamma_t|$.

Then we can extend the inequality further by considering $t' > t$ such that

$sd(\gamma(t), \gamma(t')) < \frac{1}{2}$ (min distance between <u>distinct</u> eigenvalues of $\gamma(t)$)

But then each eigenvalue of $\gamma(t')$ must be matched with the closest eigenvalue of $\gamma(t)$ and Prop 6.1 $\Rightarrow$ $sd(\gamma(t), \gamma(t')) \leq \|\gamma(t) - \gamma(t')\|$

so that

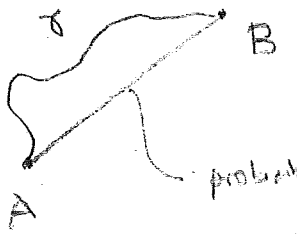
$$sd(A, \gamma(t')) \leq sd(A, \gamma(t)) + sd(\gamma(t), \gamma(t'))$$

$$\leq |\gamma_t| + \|\gamma(t) - \gamma(t')\| \leq |\gamma_{t'}| . \qquad \text{QED}$$

<u>Curvatures</u>

To exploit the normal path inequality we'd like to know the "curvature" $\kappa(S)$ of subsets $S \subseteq \mathbb{N}_n$ :

$$\kappa(S) = \max_{A,B \in S} \left\{ \frac{|\gamma|}{\|A-B\|} : \gamma \text{ is the shortest path in } S \text{ from } A \text{ to } B \right\}$$




probably $(A, B)$ (i.e. $[A,B] \setminus \{A,B\}$) <u>not</u> in $S$

We've seen that

$$\kappa(\mathbb{C}\mathbb{U}_n) = 1 \qquad (\mathbb{U}_n = \text{unitary group in } \mathbb{M}_n)$$

and that

$$\kappa(\mathbb{N}_2) = 1$$

i.e. these classes of normals are "metrically flat"

Not much is known about $\kappa(\mathbb{N}_n)$ for $n > 2$, but Choi (c. 1985) noted that these curvatures are certainly $> 1$:

Consider $A_n = \frac{1}{2} J_n + \frac{1}{2} J_n^*$, $B_n = \frac{1}{2} J_n - \frac{1}{2} J_n^*$

where $J_n$ is the $n \times n$ Jordan nilpotent

eg

$$A_3 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}$$

Note that $\|A_n - B_n\| = \|J_n^*\| = 1$

but that no normal path $\gamma$ from $A_n$ to $B_n$

can have length 1 because the "midpoint" $C$

of such a path would satisfy $\|A_n - C\| = \|C - B_n\| = \frac{1}{2}$

so that $c_{k+1,k} = 0$ $\left( |c_{k+1,k} \pm \frac{1}{2}| \leq \frac{1}{2} \right)$

But then $(A_n - C)_{k+1,k} = \frac{1}{2} = \|A_n - C\|$

requires too many zeros in $C$

eg $n = 3$

$$A_3 - C = \begin{bmatrix} 0 & 0 & -c_{13} \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

ie $C = \begin{bmatrix} 0 & \frac{1}{2} & c_{13} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ which cannot be normal because

$C$ normal $\Rightarrow$ $\|\text{row}_k\| = \|\text{column}_k\|$

Note:

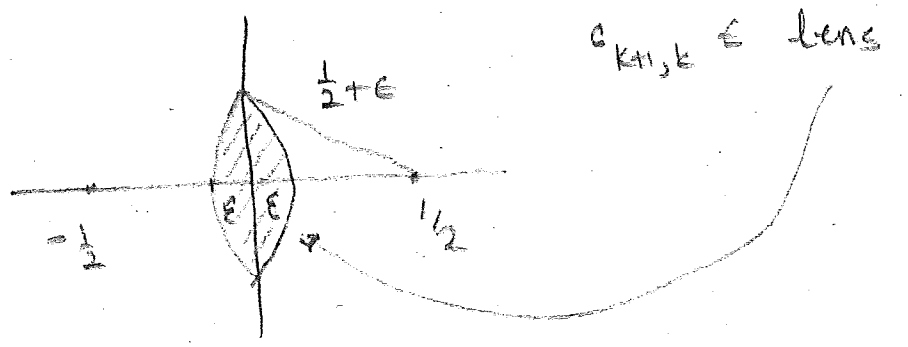
$A_n$ is Hermitian with eigenvalues $\cos\frac{\pi k}{n+1}$

$k = 1, 2, \dots, n$ and

$B_n$ is skew-Hermitian with eigenvalues $i\cos\frac{\pi k}{n+1}$

Argument above can be "quantified" & consider a normal path $\gamma$ from $A_n$ to $B_n$ such that $|\gamma| < 1 + 2\epsilon$ : midpoint $C$ would satisfy $\|A_n - C\| < \frac{1}{2} + \epsilon$, $\|C - B_n\| < \frac{1}{2} + \epsilon$

$\Rightarrow$




etc yields a (weak, surely) estimate

$$\kappa(\mathbb{N}_n) > 1.06 \quad \text{if } n \geq 3$$

For $n = 3$ we have also an upper bound (again, surely weak)

<u>Prop'n 10.1</u> $\kappa(\mathbb{N}_3) \leq 3$

<u>Proof</u> Consider $A, B \in \mathbb{N}_3$. We may take $A = \mathrm{diag}(a)$, $B = U\,\mathrm{diag}(b)\,U^*$ with $U$ unitary, $a, b \in \mathbb{C}^3$. Since Hausdorff distance $h(\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}) \leq \|A - B\|$ we have $c_k \in \{a_1, a_2, a_3\}$ such that $|a_k - b_k| \leq \|A - B\|$ (not necessarily a matching eg could have $c_1 = c_2 = a_1$)

Let $C = U\,\mathrm{diag}(c)\,U^*$ so the path $\gamma_1 = [B, C]$ is normal of length $\leq \|A - B\|$.

The eigenvalues of $A$ and $C$ lie on the circle through $a_1, a_2, a_3$ <u>so</u>

there is normal path $\gamma_2$ from $C$ to $A$ with

$$|\gamma_2| \leq \|C - A\| \leq \|C - B\| + \|B - A\|$$

$$\leq 2\|A - B\| .$$

Thus $\quad |\gamma_1 + \gamma_2| \leq 3\|A - B\|$ $\qquad$ QED

<u>Returning to spectral variation</u>

Since the shortest normal path from $A_n$ to $B_n$ has length at least $1.06$ we cannot use the normal path inequality to conclude that $sd(A_n, B_n) \leq \|A_n - B_n\| = 1$

However, knowing the eigenvalues

$$\left\{ \cos \frac{k\pi}{n+1} \right\} \quad \text{and} \quad \left\{ i \cos \frac{k\pi}{n+1} \right\}$$

we can check directly that

$n$ even $\Rightarrow sd(A_n, B_n) < 1$ $\qquad$ and

$n$ odd $\Rightarrow sd(A_n, B_n) = 1$

More generally:

<u>Theorem</u> (Sunder c. 1982) $A, B \in \mathbb{M}_n$ with $A$ Hermitian, $B$ skew-Hermitian $\Rightarrow$

$$\mathrm{sd}(A, B) \leq \|A - B\|.$$

Proof procedes somewhat like our first proof of Weyl's inequality; look at a specific matching: if

$$|a_1| \leq |a_2| \leq \cdots \leq |a_n|, \quad |b_1| \leq |b_2| \leq \cdots \leq |b_n|$$

match $a_k$ with $b_{n+1-k}$ and consider a unit vector

$$u \in \mathrm{span}\{u_{n+1-k}, \ldots, u_n\} \cap \mathrm{span}\{w_k, \ldots, w_n\}$$

where $\{u_j\}_1^n$, $\{w_j\}_1^n$ are orthonormal

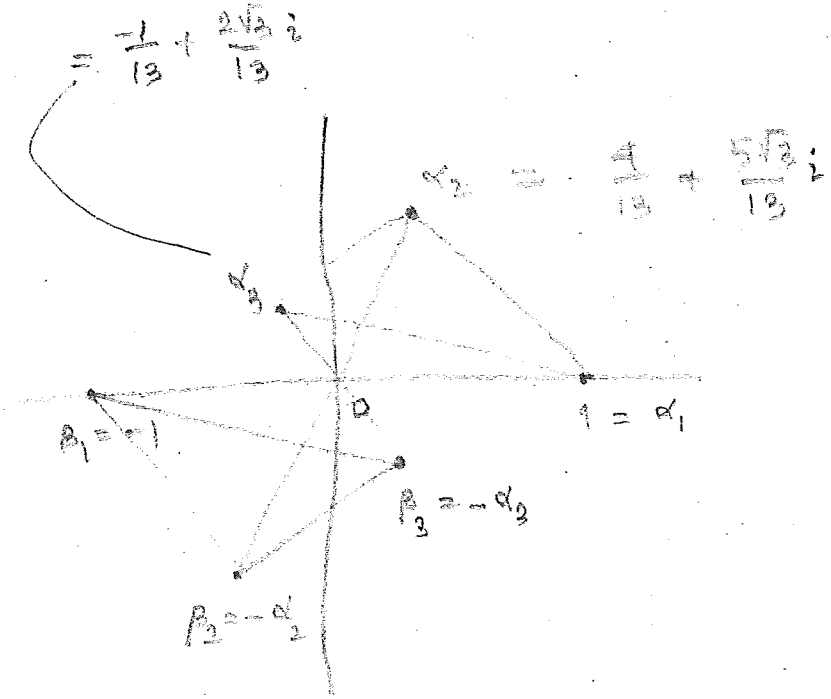
and $Au_j = a_j u_j$, $Bw_j = b_j w_j$

real pure imaginary

. . .

Nevertheless, the Mirsky - Wielandt conjecture does fail — even for $n = 3$ : (c. 1992) carefully optimized computer searches found normal $A, B \in \mathbb{N}_3$ with $sd(A,B) > \|A - B\|$ and Gert Krause described a specific example that could be analysed precisely

Krause's spectral geometry :




(p 163 Bhatia : Matrix Analysis)

$A = \mathrm{diag}(\alpha)$

$B = U \,\mathrm{diag}(\beta)\, U^*$

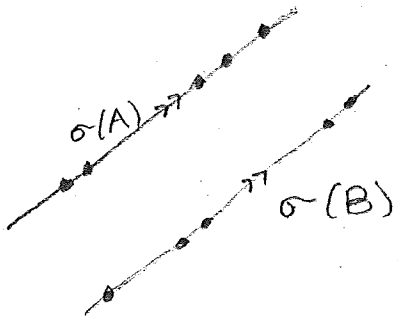
for a certain unitary ...

Can compute $\dfrac{sd(A,B)}{\|A - B\|} = \sqrt{\dfrac{28}{27}} \simeq 1.018$

On the other hand, we know a few <u>favorable spectral geometries</u> :

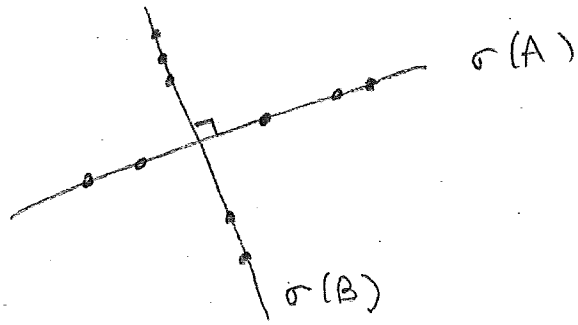
$$sd(A, B) \leq \|A - B\| \quad \text{provided}$$

$A, B \in M_n$ and




spectra lie on <u>parallel lines</u>

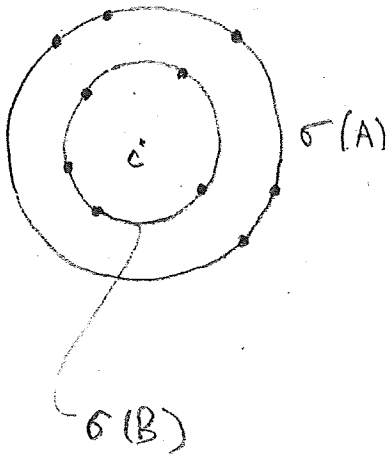
OR




spectra lie on <u>perpendicular lines</u>

OR

(Sunder ...)




spectra lie on <u>concentric circles</u>

In fact "random" spectral geometries seem to be favorable: perform following experiment

1) choose random $a, b \in \mathbb{C}^3$

2) for <u>many</u> ($10^7$, say) $U \in \mathbb{U}_3$

compute
$$\frac{sd(\mathrm{diag}(a), U\,\mathrm{diag}(b)\,U^*)}{\|\mathrm{diag}(a) - U\,\mathrm{diag}(b)\,U^*\|}$$

It "never" exceeds 1.

Of course, if we force $a, b$ to be close to the Krause geometry we <u>will</u> find ratios $> 1$.

<u>Mystery</u>: What <u>are</u> the favorable spectral geometries — even for $n = 3$.

Consider the best constants $c(n)$ that can replace 1 in the Mirsky–Wielandt conjecture:

$$c(n) = \max\left\{ \frac{sd(A,B)}{\|A-B\|} : A, B \in \mathbb{N}_n \right\}.$$

We have $c(2) = 1$ (Corollary 7.3 or elementary argument) and $c(3) > 1.018$ but know little about exact values. However:

<u>Theorem 11.1</u> (Bhatia, Davis, McIntosh, Kosaki)

$\exists$ constant $c \simeq 2.9$ such that

$$c(n) \leq c \quad \text{for } \underline{\text{all } n}.$$

(see chapter <u>VII</u> of Bhatia, Matrix Analysis)

For $n < 9$, we get better estimates using the Hoffman–Wielandt Theorem:

$$A, B \in \mathbb{N}_n \Rightarrow sd(A,B) \le sd_2(A,B) \le \|A-B\|_2 \le \sqrt{n}\,\|A-B\|$$

$$\min_{\pi} \max_k |\alpha_k - \beta_{\pi(k)}|$$

Hoffman–Wielandt

$$\min_{\pi} \left( \sum_k |\alpha_k - \beta_{\pi(k)}|^2 \right)^{1/2}$$

Hence $c(n) \le \sqrt{n}$ $\quad (< 2.9$ if $n < 9)$

---

$c(n) \le K(n)$ but we seem to know very little about $K(n)$

---

What can we say about particular matrices?

Let $sp(A,B) = \min \left\{ \dfrac{|\gamma|}{\|A-B\|} : \gamma \text{ is a normal path from } A \text{ to } B \right\}$.

"shortest (normal) path"

We have seen that

$$1.06 < sp(A_n, B_n) \quad \text{and} \quad sp(A_3, B_3) \le 3$$

Can we do better?

The pairs $A_n, B_n$ are unusual because there are nontrivial normals in $\operatorname{span}\{A_n, B_n\} = \operatorname{span}\{J_n, J_n^*\}$

Thus it is interesting to look at

$$\underset{\text{"span"}}{\text{spsn}}(A_n, B_n) = \min\left\{ \frac{|\gamma|}{\|A_n - B_n\|} : \begin{array}{l}\gamma \text{ is a normal path} \\ \text{in span}\{A_n, B_n\} \\ \text{from } A_n \text{ to } B_n\end{array} \right\}$$

Of course $\quad sp(A_n, B_n) \leq spsn(A_n, B_n)$

(could this be $=$ ?)

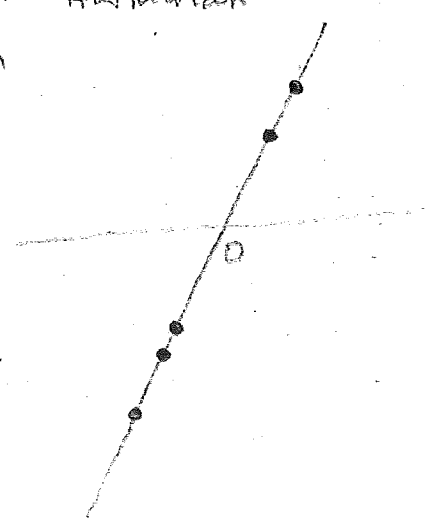
Can check that span$\{S, S^*\}$ is either entirely normal ($\Leftrightarrow$ $S$ is normal) or

$A$ normal in span$\{S, S^*\}$ $\Leftrightarrow$ $A = \alpha S + \beta S^*$ with $|\alpha| = |\beta|$

In this case $A$ is normal for a simple reason:

$$A = \alpha S + e^{i\theta}\alpha S^* = \alpha e^{+i\theta/2}\underbrace{\left(e^{-i\theta/2} S + e^{+i\theta/2} S^*\right)}_{\text{Hermitian}}$$

In particular, $\sigma(A)$ lies on a line through $0$



<u>Prop'n 10.2</u> $\operatorname{span}(A_3, B_3) = \sqrt{2}\sin\left(\frac{\pi}{2\sqrt{2}}\right) (\simeq 1.2672)$

$= \min_{\gamma} \{ |\gamma| : \gamma$ normal path in $\operatorname{span}\{J_3, J_3^*, I_3\}$ from $A_3$ to $B_3$

<u>Proof</u> A normal path in $\operatorname{span}\{J_3, J_3^*, I_3\}$

$(= \operatorname{span}\{A_3, B_3, I_3\})$ has the form

$$\gamma(t) = a(t) J_3 + b(t) J_3^* + c(t) I_3$$

where $a, b, c : [0, T] \longrightarrow \mathbb{C}$

are continuous (let's assume differentiable ...)

and $|a(t)| = |b(t)|$. It will be convenient

to parametrize on $[0, \pi]$.

If $\gamma(0) = A_3$ and $\gamma(\pi) = B_3$ we have

$a(0) = a(\pi) = \frac{1}{2}$, $b(0) = \frac{1}{2}$ & $b(\pi) = -\frac{1}{2}$, and

$c(0) = c(\pi) = 0$.

Since

$$|\gamma| = \int_0^{\pi} \| a'(t) J_3 + b'(t) J_3^* + c'(t) I_3 \| \, dt$$

we make $|\gamma|$ shorter by taking $c(t) \equiv 0$ ;

$$\| \alpha J_3 + \beta J_3^* + z I_3 \| \geq \| \alpha J_3 + \beta J_3^* \|$$

by the $\Delta$ inequality because

$$\left\| \begin{bmatrix} -z & \alpha & 0 \\ \beta & -z & \alpha \\ 0 & \beta & -z \end{bmatrix} \right\| = \| - \mathrm{diag}(1,-1,1) \begin{bmatrix} z & \alpha & 0 \\ \beta & z & \alpha \\ 0 & \beta & z \end{bmatrix} \mathrm{diag}(1,-1,1) \|$$

Thus we need to minimize

$$|\gamma| = \int_0^{\pi} \| a'(t) J_3 + b'(t) J_3^* \| \, dt = \int_0^{\pi} \left( |a'(t)|^2 + |b'(t)|^2 \right)^{1/2} dt$$

(Can check that $X = \alpha J_3 + \beta J_3^*$ has norm $\sqrt{|\alpha|^2 + |\beta|^2}$ by computing eigenvalues of $X^* X$ ...)

Moreover $|\gamma|$ gets shorter if we replace $a(t)$ by $|a(t)|$ :

$a(t) \longrightarrow |a(t)|$ respects the requirements $|a(t)| = |b(t)|$ and $a(0) = a(\pi) = 1/2$, and $||a|'(t)| \leq |a'(t)|$. Thus we consider

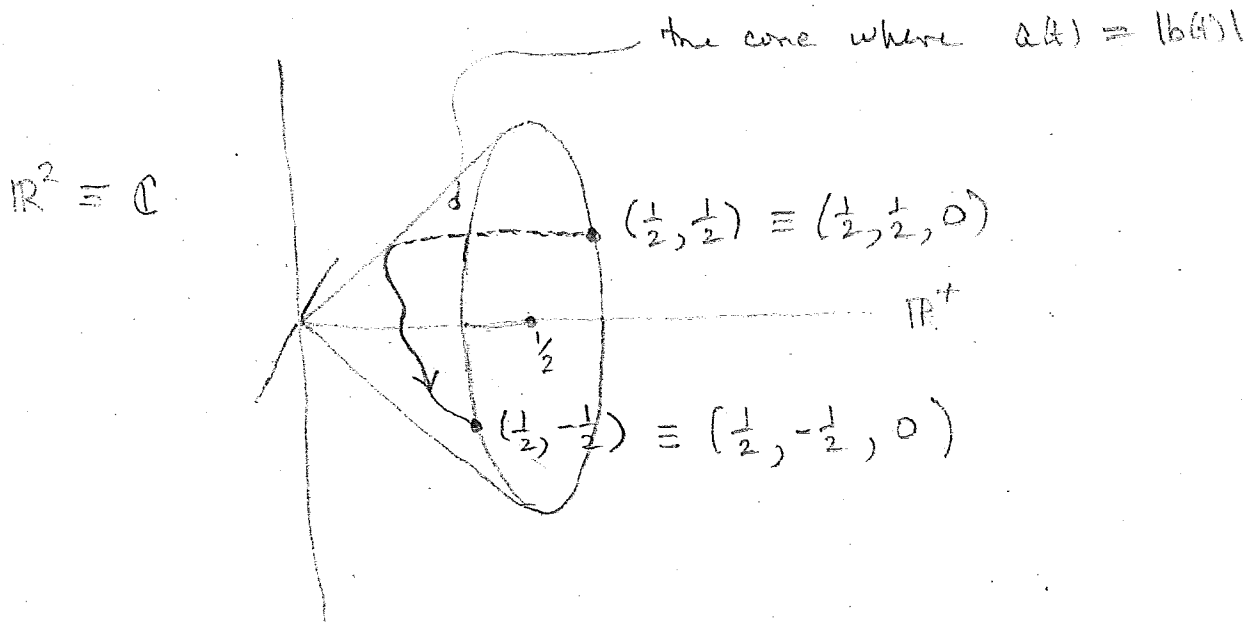
$$(a(t), b(t)) \in \mathbb{R}^+ \times \mathbb{C} \equiv \mathbb{R}^+ \times \mathbb{R}^2$$

a path in $\mathbb{R}^3$ and

$$|\sigma| = \int_0^\pi \left(|a'(t)|^2 + |b'(t)|^2\right)^{1/2} dt =$$

length of this path $\sigma$ in 3-space.

Picture the restrictions:



So $\sigma$ lies on the cone $\{r, r\cos t, r\sin t : r \geq 0\}$

and we need to minimize

$$|\sigma| = \int_0^{\pi} \left( |r'(t)|^2 + |(r(t)\cos t)'|^2 + |(r(t)\sin(t))'|^2 \right)^{\frac{1}{2}} dt$$

$$= \int_0^{\pi} \left( 2(r'(t))^2 + r^2(t) \right)^{\frac{1}{2}} dt$$

(over $r : [0,\pi] \to \mathbb{R}^+$ with $r(0) = r(\pi) = \frac{1}{2}$)

except for the 2 this looks like arclength of the polar curve $(r(t), t)$. Letting $s(t) = r(\sqrt{2}t)$ and $\theta = t/\sqrt{2}$ we find that

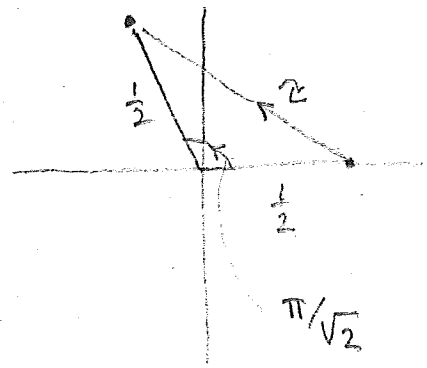
$$|\gamma| = \sqrt{2} \int_0^{\pi/\sqrt{2}} \left( (s'(\theta))^2 + s^2(\theta) \right)^{\frac{1}{2}} d\theta$$

$= \sqrt{2}\,|\tau|$ where $\tau$ is the plane curve

$\tau(\theta) = (s(\theta), \theta)$ from $(\frac{1}{2}, 0)$ to $(\frac{1}{2}, \pi/\sqrt{2})$

polar coords

and we minimize $|\gamma|$ by taking $\tau$ to be the straight line between these points =




$$\min |\gamma| = \sqrt{2}\,|z|$$
$$= \sqrt{2}\,\sin\frac{\pi}{2\sqrt{2}}\,.$$

QED

---

We have a credible (?) upper bound

$$sp(A_3, B_3) \le spsn(A_3, B_3) \simeq 1.2672$$

but what can we say about

$sp(A_n, B_n)$, $spsn(A_n, B_n)$ for larger $n$ ?

The "dream" of using the normal path inequality to improve the constant in

$c(n) \le 2.9$ via $c(n) \le K(n)$

seem remote since we can't even rule out $K(n) \uparrow \infty$ as $n \uparrow$.

It would be "comforting" to find bounds on $sp(A_n, B_n) \leq spsn(A_n, B_n)$ ...

Note that $\alpha J_3 + \beta J_3^*$ is a $(3\times 3)$ submatrix of $\alpha J_n + \beta J_n^*$ so that

$$\sqrt{|\alpha|^2+|\beta|^2} = \|\alpha J_3 + \beta J_3^*\| \leq \|\alpha J_n + \beta J_n^*\|.$$

On the other hand

$$\|\alpha J_n + \beta J_n^*\| \leq q(n)\sqrt{|\alpha|^2+|\beta|^2}$$

where $q(n) = \sup\{\|J_n + t J_n^*\| / \sqrt{1+t^2} : t \in [0,1]\}$.

$q(4) \simeq 1.1547$

$q(5) \simeq 1.2247$

$q(6) \simeq 1.2750$

and it appears that $q(n) \uparrow \sqrt{2}$ ...

Thus, mimicking the proof of Prop 10.2 we obtain

<u>Prop 12.1</u> $\sqrt{2}\sin\frac{\pi}{2\sqrt{2}}$ $(\cong 1.2672) \leq \operatorname{spsn}(A_n, B_n)$,

$\operatorname{spsn}(A_4, B_4) \leq (1.2672)(1.154) \cong 1.4632$, and

$\operatorname{spsn}(A_m, B_n) \leq 2\sin\frac{\pi}{2\sqrt{2}} \cong 1.7920$.

---

However, we get better information for the larger $m$ by noting that

$$\sqrt{(1-2/n)}\,(|\alpha|+|\beta|) \leq \|\alpha J_n + \beta J_n^*\| \leq |\alpha|+|\beta| .$$

$\uparrow_n$ $|\alpha|+|\beta|$ via general properties of Toeplitz matrices

vs

specific inequality following from

$$((\alpha J_n + \beta J_n^*)u)_k = \alpha u_{k+1} + \beta u_{k-1} \quad k = 2, 3, \ldots, n-1$$

control $\arg(u_{k+1}/u_{k-1})$ to maximize $|\alpha u_{k+1} + \beta u_{k-1}|$

<u>Prop'n 12.2:</u> For $n \geq 2$ we have

$$(1 - 2/n)^{\frac{1}{2}} (1 + \cos 1) \leq \operatorname{spsn}(A_n, B_n) \leq (1 + \cos 1) \simeq 1.5403$$

<u>Proof:</u> Again we can replace matrix paths by paths in the Euclidean plane. Consider a normal path

$$\gamma(t) = a(t) J_n + b(t) J_n^* \qquad t \in [0,1]$$

from $A_n$ to $B_n$ : $|a(t)| = |b(t)|$ and

$a(0) = a(1) = \frac{1}{2}$, $b(0) = \frac{1}{2}$ & $b(1) = -\frac{1}{2}$

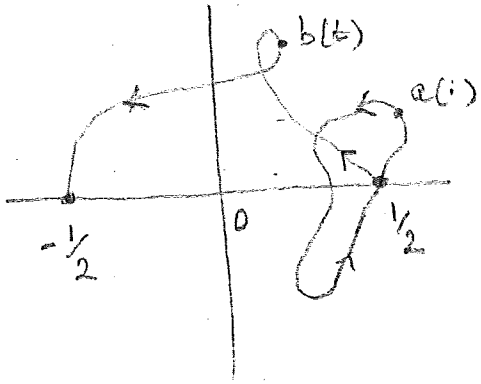
Combine

$$|\gamma| = \int_0^1 \| a'(t) J_n + b'(t) J_n^* \| \, dt$$

with inequalities above to obtain

$$\sqrt{1 - 2/n} \underbrace{\int_0^1 (|a'(t)| + |b'(t)|) \, dt}_{} \leq |\gamma| \leq \int_0^1 (|a'(t)| + |b'(t)|) \, dt$$

But is the sum of the lengths of the curves $a(\cdot)$ and $b(\cdot)$ in $\mathbb{C} \equiv \mathbb{R}^2$



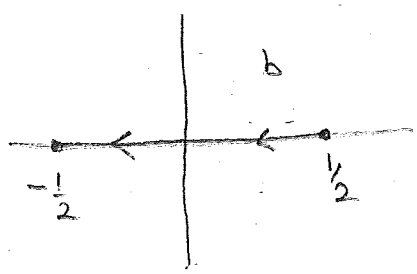

Minimize

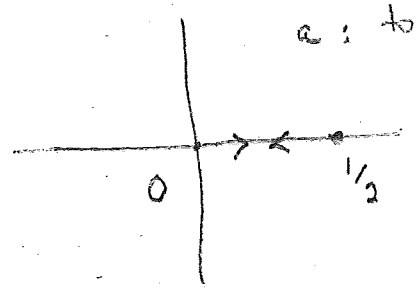
$|a| + |b|$ subject to the restrictions

$(\,|a(t)| \equiv |b(t)|\,)$

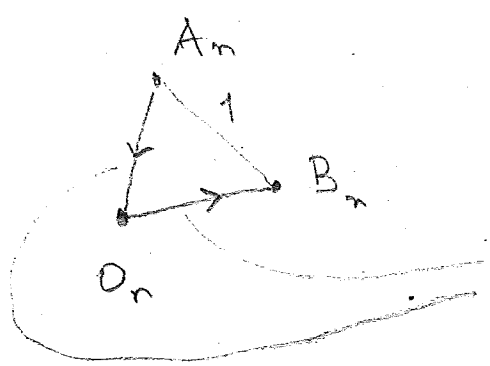
Consider $r = \min_t |b(t)| \quad \left( = \min_t |a(t)| \right)$

$r = 0$ is a bad choice : the best we could do is



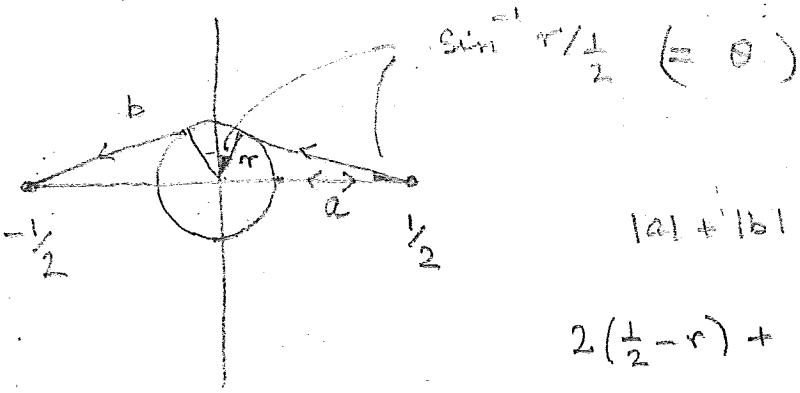




$|\gamma| = 1 + 1$ and $\gamma$ passes thru $O_n$




$\cos \frac{\pi}{n+1} \uparrow_n 1$

$r > 0$ :




$$|a| + |b| =$$

$$2\left(\tfrac{1}{2} - r\right) + 2\sqrt{\left(\tfrac{1}{2}\right)^2 - r^2}$$

$$+ \; 2r\sin^{-1}(2r) \quad = f(r)$$

$0 = f'(r)$ finds the minimum : $2r = \sin 1$

$$f(r) = 1 + \cos 1 \qquad \text{QED}$$

---

compute or

(easier) reparametrize using $\theta$

---

Experimental evidence that

$$sp(A_3, B_3) = \text{span}\,(A_3, B_3) = \sqrt{2}\,\sin\frac{\pi}{2\sqrt{2}}$$

$$\simeq 1.2672$$

?

- We have noted that

$$sp(A_3, B_3) < 1 + 2\epsilon \implies$$

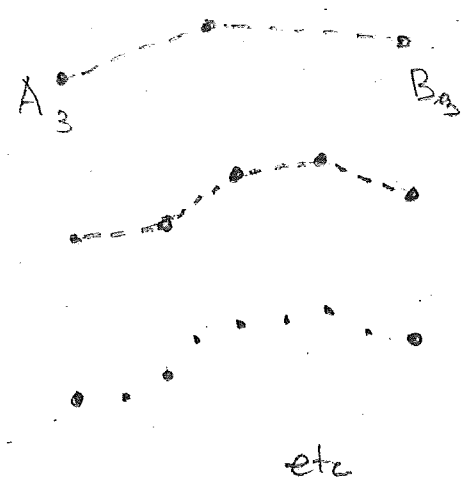
$\exists$ normal $C \in \mathbb{N}_3$ such that

$\|A_3 - C\|, \|C - B_3\| < \frac{1}{2} + \epsilon$

so why not a computer experiment to estimate

$$\min_{C \in \mathbb{N}_3} \left( \max \{ \|A_3 - C\|, \|C - B_3\| \} \right) \quad ?$$

- Given an effective algorithm for finding good "normal mid-point" $C$ we could iterate to construct good "approximate normal paths"



Return to problem of identifying

$$sp(A_3, B_3) \underset{?}{=} (\text{certainly} \leq)\ spsn(A_3, B_3)$$

$$= \sqrt{2} \sin \frac{\pi}{2\sqrt{2}} \simeq 1.2672$$

Note that the path found in Prop'n 10.2 can be parametrized as $\gamma(\theta) = e^{i\theta} H(\theta)$ where $\theta \in [0, \pi/2]$ and each $H(\theta)$ is Hermitian; we may consider this construction in general (ie moving <u>beyond</u> span $(A_3, B_3)$):

$$\gamma : [0, \pi/2] \longrightarrow \mathbb{N}_3 \quad \gamma(0) = A_3, \ \gamma(\pi/2) = B_3$$

with $\gamma(\theta) = e^{i\theta} H(\theta)$, $H(\theta)$ Hermitian

and $H(0) = A_3$, $H(\pi/2) = \frac{1}{i} B_3$.

Thus

$$H(\theta) = \begin{bmatrix} a(\theta) & z(\theta) & q(\theta) \\ \bar{z}(\theta) & b(\theta) & w(\theta) \\ \bar{q}(\theta) & \bar{w}(\theta) & c(\theta) \end{bmatrix}$$

$a, b, c : [0, \pi/2] \to \mathbb{R}$

$z, w, q : [0, \pi/2] \to \mathbb{C}$

(assume "smooth")

end points of $a, b, c, q$ must be $0$

$z(0) = w(0) = 1/2$

$z(\frac{\pi}{2}) = w(\pi/2) = -i/2$

We want to minimize

$$|\gamma| = \int_0^{\pi/2} \| (e^{i\theta} H(\theta))' \| \, d\theta$$

$$\begin{bmatrix} ie^{i\theta}a(\theta) + e^{i\theta}a'(\theta) & ie^{i\theta}z(\theta) + e^{i\theta}z'(\theta) & * \\ ie^{i\theta}\bar{z}(\theta) + e^{i\theta}\bar{z}'(\theta) & * & * \\ * & * & * \end{bmatrix}$$

We make $\| \cdot \|$ smaller by arranging that

diagonal entries and corner entries $\equiv 0$

because $X \in \mathbb{M}_3$ and $U = \operatorname{diag}(1,-1,1)$

$$\Rightarrow \quad -UXU = \begin{bmatrix} -x_{11} & x_{12} & -x_{13} \\ x_{21} & -x_{22} & x_{23} \\ -x_{31} & x_{32} & -x_{33} \end{bmatrix} = Y$$

$$\Rightarrow \quad \|Y\| = \|X\|$$

$$\Rightarrow \quad \Big\| \underbrace{\frac{X+Y}{2}}_{\begin{bmatrix} 0 & x_{12} & 0 \\ x_{21} & 0 & x_{23} \\ 0 & x_{32} & 0 \end{bmatrix}} \Big\| \leq \|X\|$$

This suggest we take $a(\theta), b(\theta), c(\theta), q(\theta) \equiv 0$

and, in fact, this is the only way to force

the relevant entries of $(e^{i\theta} H(\theta))'$ to be

$0$ $\Big($ eg $(e^{i\theta} q(\theta))' \equiv 0 \Rightarrow q(\theta) = Q e^{-i\theta}$

& $q(0) = 0 \Rightarrow Q = 0$ $\Big)$

Thus we want to minimize

$$|\gamma| = \int_0^{\pi/2} \left\| \begin{bmatrix} 0 & (e^{i\theta} z(\theta))' & 0 \\ (e^{i\theta}\bar{z}(\theta))' & 0 & (e^{i\theta} w(\theta))' \\ 0 & (e^{i\theta}\bar{w}(\theta))' & 0 \end{bmatrix} \right\| d\theta$$

Computing eigenvalues of $X^* X$ with

$$X = \begin{bmatrix} 0 & \alpha & 0 \\ \beta & 0 & \sigma \\ 0 & \tau & 0 \end{bmatrix}$$ we find that

$$\|X\| = \max\left\{ \sqrt{|\alpha|^2 + |\tau|^2},\ \sqrt{|\beta|^2 + |\sigma|^2} \right\}.$$

Again, since $||\alpha|'| \leq |\alpha'|$ we should

replace $e^{i\theta} z(\theta)$ with $|e^{i\theta} z(\theta)| = r(\theta) \geq 0$

ie $z(\theta) = r(\theta) e^{-i\theta}$ and similarly

$w(\theta) = s(\theta) e^{-i\theta}$ for some $s(\theta) \geq 0$.

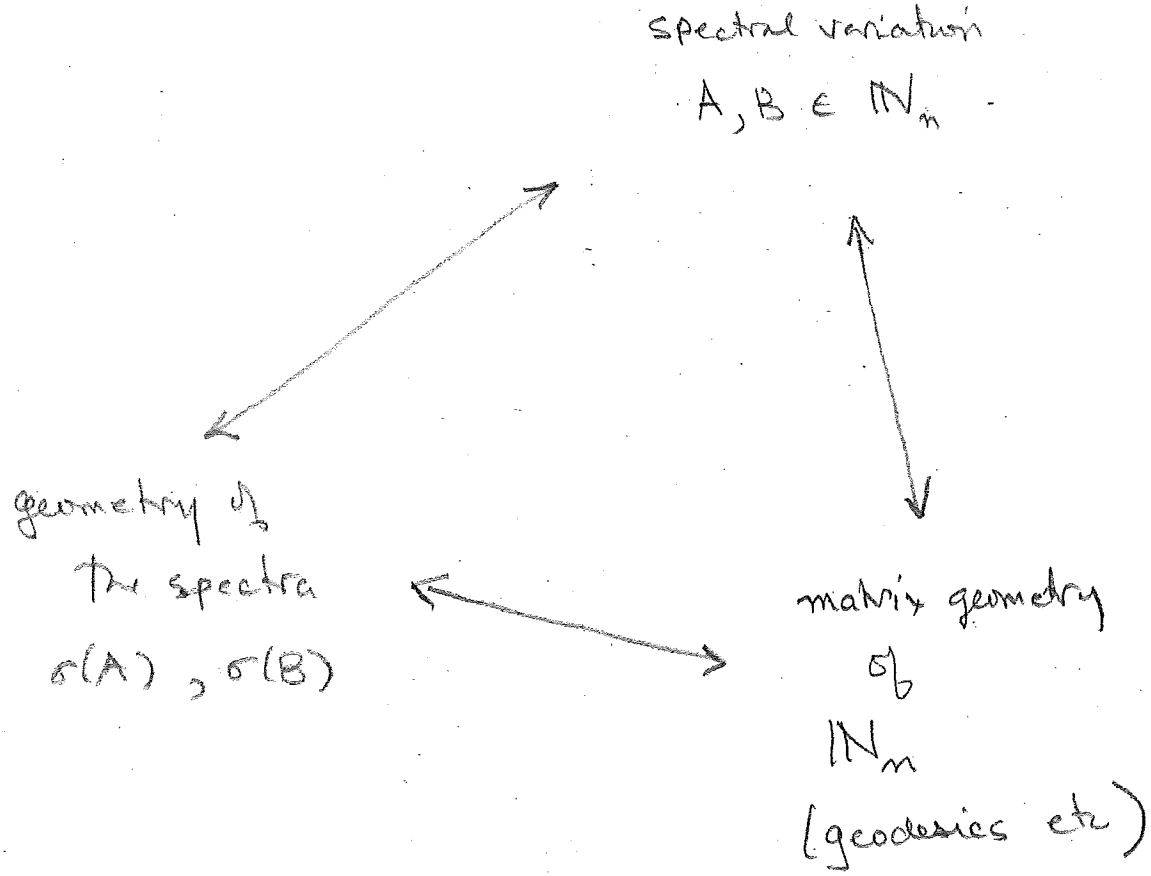
Then $$\gamma'(\theta) = \begin{bmatrix} 0 & r'(\theta) & 0 \\ e^{i2\theta}(r(\theta) + 2i r'(\theta)) & 0 & s'(\theta) \\ 0 & e^{i2\theta}(s(\theta) + 2i s'(\theta)) & 0 \end{bmatrix}$$

and we want to minimize

$$|\gamma| = \int_0^{\pi/2} \max\left\{ \sqrt{|r'(\theta)|^2 + 4|s'(\theta)|^2 + s^2(\theta)}\,,\ \sqrt{|s'(\theta)|^2 + 4|r'(\theta)|^2 + r^2(\theta)} \right\} d\theta$$

Can we analyse in terms of paths in $\mathbb{R}^3$ or ... ?

## Summary


spectral variation
$A, B \in \mathbb{N}_n$
geometry of the spectra
$\sigma(A), \sigma(B)$
matrix geometry of $\mathbb{N}_n$
(geodesics etc)