

# Algebraic Geometry of Matrices I

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# objectives

- give a taste of algebraic geometry
- with minimum prerequisites
- provide pointers for a more serious study
- not intended to be a formal introduction
- tailored specially for this audience:
  - assumes familiarity with linear algebra, matrix analysis
  - maybe even some operator theory, differential geometry
  - but less comfortable with (abstract) algebra

promise: *we shall see lots of matrices and linear algebra*

# Overview

# why algebraic geometry

- possibly the most potent tool in modern mathematics
- applications to other areas of mathematics
  - number theory: Fermat's last theorem
  - partial differential equations: soliton solutions of KdV
  - many more . . . , but not so surprising
- applications to other areas *outside* of mathematics
  - biology: phylogenetic invariants
  - chemistry: chemical reaction networks
  - physics: mirror symmetry
  - statistics: Markov bases
  - optimization: sum-of-squares polynomial optimization
  - computer science: geometric complexity theory
  - communication: Goppa code
  - cryptography: elliptic curve cryptosystem
  - control theory: pole placement
  - machine learning: learning Gaussian mixtures
- why should folks in linear algebra/matrix theory care?

# solves long standing conjectures

Horn:  $A, B \in \mathbb{C}^{n \times n}$  Hermitian,  $I, J, K \subsetneq \{1, \dots, n\}$ ,

$$\sum_{k \in K} \lambda_k(A + B) \leq \sum_{i \in I} \lambda_i(A) + \sum_{j \in J} \lambda_j(B)$$

holds iff Schubert cycle  $s_K$  is component of  $s_I \cdot s_J$   
[Klyachko, 1998], [Knutson-Tao, 1999]

Strassen: no approximate algorithm for  $2 \times 2$  matrix product  
in fewer than 7 multiplications [Landsberg, 2006]

- involve Schubert **varieties** and secant **varieties** respectively
- for now, **variety** = affine variety = zero loci of polynomials

$$\{(x_1, \dots, x_n) \in \mathbb{C}^n : F_j(x_1, \dots, x_n) = 0 \text{ for all } j \in J\}$$

$F_j \in \mathbb{C}[x_1, \dots, x_n]$ ,  $J$  arbitrary index set

# view familiar objects in new light

linear affine variety: solutions to linear equation

$$\{\mathbf{x} \in \mathbb{C}^n : \mathbf{Ax} = \mathbf{b}\}$$

where  $\mathbf{A} \in \mathbb{C}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{C}^m$

determinantal variety: rank- $r$  matrices

$$\{X \in \mathbb{C}^{m \times n} : \text{rank}(X) \leq r\}$$

Segre variety: rank-1 matrices

$$\{X \in \mathbb{C}^{m \times n} : X = \mathbf{uv}^T\}$$

Veronese variety: rank-1 symmetric matrices

$$\{X \in \mathbb{C}^{n \times n} : X = \mathbf{vv}^T\}$$

Grassmann variety:  $n$ -dimensional subspaces in  $\mathbb{C}^m$

$$\{X \in \mathbb{C}^{m \times n} : \text{rank}(X) = n\} / \text{GL}_n(\mathbb{C})$$

# gain new insights

secant variety: rank- $r$  matrices

lines through  $r$  points on  $\{X \in \mathbb{P}^{m \times n} : \text{rank}(X) = 1\}$

dual variety: singular matrices

$$\{X \in \mathbb{P}^{n \times n} : \text{rank}(X) = 1\}^\vee = \{X \in \mathbb{P}^{n \times n} : \det(X) = 0\}$$

Fano variety: vector spaces of matrices of low rank

set of  $k$ -planes in  $\{X \in \mathbb{P}^{m \times n} : \text{rank}(X) \leq r\}$

projective  $n$ -space:  $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$  with equivalence relation  $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$  for  $\lambda \in \mathbb{C}^\times$

# encouraging observation

last two slides: if you know linear algebra/matrix theory, you  
have seen many examples in algebraic geometry

next three slides: more such examples

moral: you have already encountered quite a bit of  
algebraic geometry

# zero loci of matrices

twisted cubic:  $2 \times 3$  rank-1 Hankel matrices

$$\left\{ [x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 : \text{rank} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} = 1 \right\}$$

rational normal curve:  $2 \times d$  rank-1 Hankel matrices

$$\left\{ [x_0 : x_1 : \cdots : x_d] \in \mathbb{P}^d : \text{rank} \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_{d-1} & x_d \end{pmatrix} = 1 \right\}$$

rational normal scroll:  $(d - k + 1) \times (k + 1)$  rank-1 Hankel matrices

$$\left\{ [x_0 : x_1 : \cdots : x_d] \in \mathbb{P}^d : \text{rank} \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_k \\ x_1 & x_2 & \cdots & \cdots & x_{k+1} \\ x_2 & \cdots & \cdots & \cdots & x_{k+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & x_{d-1} \\ x_{d-k} & \cdots & \cdots & x_{d-1} & x_d \end{pmatrix} = 1 \right\}$$

discriminant hypersurface of singular quadrics in  $\mathbb{P}^n$ :

$$\left\{ [x_{00} : x_{01} : \cdots : x_{nn}] \in \mathbb{P}^{n(n+3)/2} : \det \begin{pmatrix} x_{00} & x_{01} & \cdots & x_{0n} \\ x_{01} & x_{11} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{0n} & x_{1n} & \cdots & x_{nn} \end{pmatrix} = 0 \right\}$$

# algebraic groups

- elliptic curve:  $y^2 = (x - a)(x - b)(x - c)$ ,

$$E = \left\{ (x, y) \in \mathbb{C}^2 : \det \begin{pmatrix} x-a & 0 & y \\ 0 & 1 & \frac{1}{2}(b+c)+x \\ y & \frac{1}{2}(b+c)-x & -\frac{1}{4}(b-c)^2 \end{pmatrix} = 0 \right\}$$

- $E$  is abelian variety, i.e., variety that is abelian group
- generalization: algebraic groups
- multiplication/inversion defined locally by rational functions
- two most important classes:

projective: abelian varieties

affine: linear algebraic groups

- examples:

general linear group:  $\mathrm{GL}_n(\mathbb{F}) = \{X \in \mathbb{F}^{n \times n} : \det(X) \neq 0\}$

special linear group:  $\mathrm{SL}_n(\mathbb{F}) = \{X \in \mathbb{F}^{n \times n} : \det(X) = 1\}$

projective linear group:  $\mathrm{PGL}_n(\mathbb{F}) = \mathrm{GL}_n(\mathbb{F}) / \{\lambda I : \lambda \in \mathbb{F}^\times\}$

# linear algebraic groups

$\text{char}(\mathbb{F}) \neq 2$

orthogonal group:  $q$  symmetric nondegenerate bilinear

$$\mathbf{O}_n(\mathbb{F}, q) = \{X \in \mathbf{GL}_n(\mathbb{F}) : q(X\mathbf{v}, X\mathbf{w}) = q(\mathbf{v}, \mathbf{w})\}$$

special orthogonal group:  $q$  symmetric nondegenerate bilinear

$$\mathbf{SO}_n(\mathbb{F}, q) = \{X \in \mathbf{SL}_n(\mathbb{F}) : q(X\mathbf{v}, X\mathbf{w}) = q(\mathbf{v}, \mathbf{w})\}$$

symplectic group:  $q$  skew-symmetric nondegenerate bilinear

$$\mathbf{Sp}_{2n}(\mathbb{F}, q) = \{X \in \mathbf{SL}_{2n}(\mathbb{F}) : q(X\mathbf{v}, X\mathbf{w}) = q(\mathbf{v}, \mathbf{w})\}$$

special case:  $q(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \mathbf{w}$ , get  $\mathbf{O}_n(\mathbb{F})$ ,  $\mathbf{SO}_n(\mathbb{F})$ ,  $\mathbf{Sp}_{2n}(\mathbb{F})$ ,

$$\mathbf{PO}_n(\mathbb{F}) = \mathbf{O}_n(\mathbb{F})/\{\pm I\}, \quad \mathbf{PSO}_{2n}(\mathbb{F}) = \mathbf{SO}_{2n}(\mathbb{F})/\{\pm I\}$$

# comes in different flavors

complex algebraic geometry: varieties over  $\mathbb{C}$

real algebraic geometry: semialgebraic sets & varieties over  $\mathbb{R}$ ,  
e.g. hyperbolic cone,  $A \succ 0$ ,  $\mathbf{b} \in \mathbb{R}^n$ ,

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T \mathbf{A} \mathbf{x} \leq (\mathbf{b}^T \mathbf{x})^2, \mathbf{b}^T \mathbf{x} \geq 0\}$$

convex algebraic geometry: convex sets with algebraic  
structure, e.g. spectrahedron,  $A_0, \dots, A_n \in \mathbb{S}^{m \times m}$ ,

$$\{A_0 + x_1 A_1 + \dots + x_n A_n \succeq \mathbf{0} : \mathbf{x} \in \mathbb{R}^n\}$$

tropical algebraic geometry: varieties over  $(\mathbb{R} \cup \{\infty\}, \min, +)$ ,  
e.g. tropical linear space, tropical polytope, tropical  
eigenspace, tropical Grassmannian

many others: diophantine geometry (over  $\mathbb{Q}, \mathbb{Q}_p, \mathbb{F}_q, \mathbb{F}_q((t)), \mathbb{Z}$ ,  
etc), noncommutative algebraic geometry, etc

# going beyond matrices

provides groundwork to go beyond linear algebra and matrices

linear to multilinear:  $f : V_1 \times \cdots \times V_d \rightarrow W$ ,

$$f(\mathbf{v}_1, \dots, \alpha \mathbf{u}_k + \beta \mathbf{w}_k, \dots, \mathbf{v}_d) = \alpha f(\mathbf{v}_1, \dots, \mathbf{u}_k, \dots, \mathbf{v}_d) \\ + \beta f(\mathbf{v}_1, \dots, \mathbf{w}_k, \dots, \mathbf{v}_d)$$

matrices to hypermatrices:

$$(\mathbf{a}_{ij}) \in \mathbb{C}^{m \times n}, \quad (\mathbf{a}_{ijk}) \in \mathbb{C}^{l \times m \times n}, \quad (\mathbf{a}_{ijkl}) \in \mathbb{C}^{l \times m \times n \times p}, \dots$$

linear/quadratic to polynomial:  $\mathbf{a}^T \mathbf{x}$ ,  $\mathbf{A}\mathbf{x}$ ,  $\mathbf{x}^T \mathbf{A}\mathbf{x}$ ,  $\mathbf{x}^T \mathbf{A}\mathbf{y}$ , to

$$f(\mathbf{x}) = a + \sum_{i=1}^n b_i x_i + \sum_{i,j=1}^n c_{ij} x_i x_j + \sum_{i,j,k=1}^n d_{ijk} x_i x_j x_k + \sum_{i,j,k,l=1}^n e_{ijkl} x_i x_j x_k x_l + \cdots$$

vector spaces to vector bundles: family of vector spaces (later)

# advertisement

L.-H. Lim, *Lectures on Tensors and Hypermatrices*

basic notions: tensor, multilinear functions, hypermatrices, tensor fields, covariance & contravariance, symmetric hypermatrices & homogeneous polynomials, skew-symmetric hypermatrices & exterior forms, hypermatrices with partial skew-symmetry/symmetry & Schur functors, Dirac & Einstein notations

ranks & decompositions: tensor rank, multilinear rank & multilinear nullity, rank-retaining decompositions, border rank, generic & typical rank, maximal rank, nonexistence of canonical forms, symmetric rank, nonnegative rank, Waring rank, Segre, Veronese, & Segre-Veronese varieties, secant varieties

eigenvalues & singular values: symmetric eigenvalues & eigenvectors, eigenvalues & eigenvectors, singular values & singular vectors, nonnegative hypermatrices, Perron-Frobenius theorem, positive semidefinite & Gram hypermatrices

norms, hyperdeterminants, & other loose ends spectral norm, nuclear norm, Holder  $p$ -norms, geometric hyperdeterminant, combinatorial hyperdeterminant, tensor products of other objects: modules, Hilbert space, Banach space, matrices, operators, representations, operator spaces, computational complexity

# advertisement

biology: phylogenetic invariants

chemistry: fluorescence spectroscopy, matrix product state DMRG

computer science: computational complexity, quantum information theory

optimization: self-concordance, higher-order optimality conditions,  
polynomial optimization

applied physics: elasticity, piezoelectricity, X-ray crystallography

theoretical physics: quantum mechanics (state space of multiple quantum  
systems), statistical mechanics (Yang-Baxter equations),  
particle physics (quark states), relativity (Einstein equation)

signal processing: antenna array processing, blind source separation,  
CDMA communication

statistics: multivariate moments and cumulants, sparse recovery and  
matrix completion

venue: Room B3-01, Instituto para a Investigação  
Interdisciplinar da Universidade de Lisboa

dates: July 23–24, 2013

# Affine Varieties

# for further information

## easy

- S. Abhyankar, *Algebraic Geometry for Scientists and Engineers*, 1990
- B. Hassett, *Introduction to Algebraic Geometry*, 2007
- K. Hulek, *Elementary Algebraic Geometry*, 2003
- M. Reid, *Undergraduate Algebraic Geometry*, 1989
- K. Smith et al., *An Invitation to Algebraic Geometry*, 2004 (our main text)

## medium

- J. Harris, *Algebraic Geometry: A First Course*, 1992
- I. Shafarevich, *Basic Algebraic Geometry*, Vols. I & II, 2nd Ed., 1994

## standard

- P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, 1978
- R. Hartshorne, *Algebraic Geometry*, 1979

## recent

- D. Arapura, *Algebraic Geometry over the Complex Numbers*, 2012
- S. Bosch, *Algebraic Geometry and Commutative Algebra*, 2013
- T. Garrity et al., *Algebraic Geometry: A Problem Solving Approach*, 2013
- A. Holme, *A Royal Road to Algebraic Geometry*, 2012

# basic and not-so-basic objects

affine varieties: subsets of  $\mathbb{C}^n$  cut out by polynomials

projective varieties: subsets of  $\mathbb{P}^n$  cut out by homogeneous polynomials

quasi-projective varieties: open subsets of projective varieties

algebraic varieties: affine varieties glued together

affine schemes: affine varieties with 'non-closed points' added

schemes: affine schemes glued together

furthermore: schemes  $\subseteq$  algebraic spaces  $\subseteq$  Deligne–Mumford stacks  $\subseteq$  algebraic stacks  $\subseteq$  stacks

but to a first-order approximation,

*algebraic geometry is the study of algebraic varieties*

just like differential geometry is, to a first-order approximation, the study of differential manifolds

# what is an algebraic variety

manifold: objects locally resembling Euclidean spaces

algebraic variety: objects locally resembling affine varieties

or:

manifold: open subsets glued together

algebraic variety: affine varieties glued together

differences:

## 1 machinery for gluing things

manifold: usually charts/atlasses/transition maps

algebraic varieties: usually **sheaves**

## 2 dimension

manifold: glue together subsets of same dimension

algebraic varieties: can have different dimensions

**sheaf**: neatest tool for gluing things — works for Riemann surfaces, manifolds, algebraic varieties, schemes, etc

# what is an affine variety

- zero loci of polynomials, i.e., common zeros of a collection of complex polynomials in  $n$  variables  $\{F_j\}_{j \in J}$ ,

$$\{(x_1, \dots, x_n) \in \mathbb{C}^n : F_j(x_1, \dots, x_n) = 0 \text{ for all } j \in J\}$$

- $J$  arbitrary index set, can be uncountable
- notation:  $\mathbb{V}(\{F_j\}_{j \in J})$  or  $\mathbb{V}(F_1, \dots, F_n)$  if finite
- caution: actually these are just Zariski closed subsets of  $\mathbb{C}^n$ , the **actual definition** of affine variety will come later
- may define manifolds as subsets of  $\mathbb{R}^n$  but unwise; want affine varieties to be independent of embedding in  $\mathbb{C}^n$  too
- simplest examples

empty set:  $\emptyset = \mathbb{V}(1)$

singleton:  $\{(a_1, \dots, a_n)\} = \mathbb{V}(x_1 - a_1, \dots, x_n - a_n)$

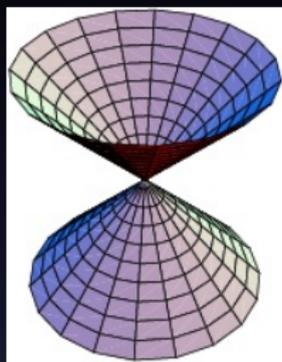
hyperplane:  $\mathbb{V}(a_0 + a_1x_1 + \dots + a_nx_n)$

hypersurface:  $\mathbb{V}(F)$

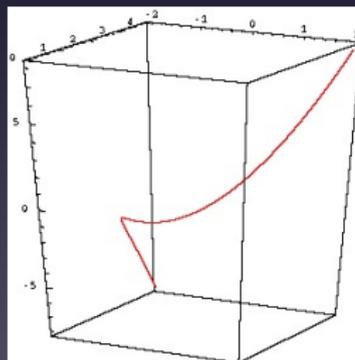
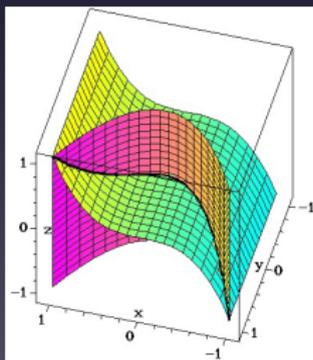
whole space:  $\mathbb{C}^n = \mathbb{V}(0)$

# more affine varieties

quadratic cone:  $\mathbb{V}(x^2 + y^2 - z^2) = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 = z^2\}$

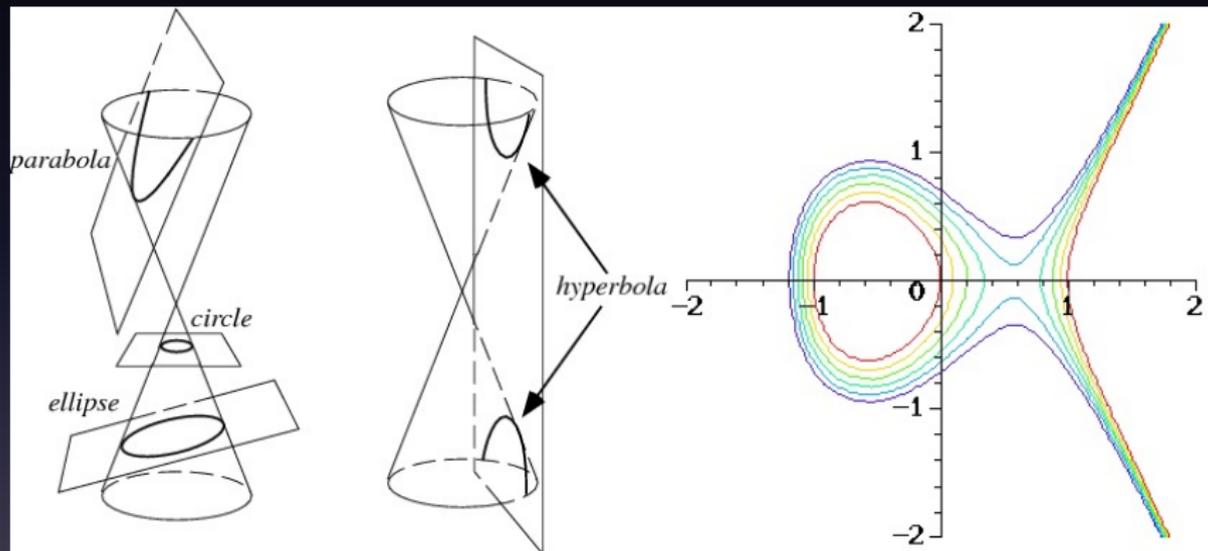


twisted cubic:  $\mathbb{V}(x^2 - y, x^3 - z) = \{(t, t^2, t^3) \in \mathbb{C}^3 : t \in \mathbb{C}\}$



# more affine varieties

conic sections:  $\mathbb{V}(x^2 + y^2 - z^2, ax + by + cz)$



elliptic curve:  $\mathbb{V}(y^2 - x^3 + x - a)$  for  $a = 0, 0.1, 0.2, 0.3, 0.4, 0.5$

# earlier examples revisited

linear affine variety: solutions to linear equation

$$\{\mathbf{x} \in \mathbb{C}^n : \mathbf{Ax} = \mathbf{b}\} = \mathbb{V}(\{a_{i1}x_1 + \cdots + a_{in}x_n - b_i\}_{i=1,\dots,m})$$

determinantal variety: rank- $r$  matrices

$$\{X \in \mathbb{C}^{m \times n} : \text{rank}(X) \leq r\} = \mathbb{V}(\{\text{all } (r+1) \times (r+1) \text{ minors}\})$$

special linear group: determinant-1 matrices

$$\text{SL}_n(\mathbb{C}) = \{X \in \mathbb{C}^{n \times n} : \det(X) = 1\} = \mathbb{V}(\det - 1)$$

- in algebraic geometry, we identify  $\mathbb{C}^{m \times n} \cong \mathbb{C}^{mn}$
- why did we say  $\text{GL}_n(\mathbb{C}) = \{X \in \mathbb{C}^{n \times n} : \det(X) \neq 0\}$  is an affine variety?

# non-examples

assume Euclidean/norm topology, following *not* affine varieties:

open ball:  $B_\varepsilon(\mathbf{x}) = \{\mathbf{x} \in \mathbb{C}^n : \|\mathbf{x}\| < \varepsilon\}$

closed ball:  $B_\varepsilon[\mathbf{x}] = \{\mathbf{x} \in \mathbb{C}^n : \|\mathbf{x}\| \leq \varepsilon\}$

unitary group:  $U_n(\mathbb{C}) = \{X \in \mathbb{C}^{n \times n} : X^* X = I\}$

general linear group:  $GL_n(\mathbb{C}) = \{X \in \mathbb{C}^{n \times n} : \det(X) \neq 0\}$

punctured line/plane:  $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ ,  $\mathbb{C}^2 \setminus \{(0, 0)\}$

set with interior points:  $S \supseteq B_\varepsilon(\mathbf{x})$  for some  $\varepsilon > 0$

graphs of transcendental functions:  $\{(x, y) \in \mathbb{C}^2 : y = e^x\}$

- $GL_n(\mathbb{C})$  and  $\mathbb{C}^\times$  affine varieties via actual definition
- complex conjugation is not an algebraic operation
- inner product  $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i \bar{y}_i$  not polynomial
- every affine variety is closed in Euclidean topology
- converse almost never true
- need another topology: Zariski