Algebraic Geometry of Matrices II

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today

- Zariski topology
- irreducibility
- maps between varieties
- answer our last question from yesterday
- again, relate to linear algebra/matrix theory

Zariski Topology

basic properties of affine varieties

- recall: affine variety = common zeros of a collection of complex polynomials

$$\mathbb{V}(\{F_j\}_{j\in J}) = \{(x_1,\ldots,x_n) \in \mathbb{C}^n : F_j(x_1,\ldots,x_n) = 0 \text{ for all } j \in J\}$$

- recall: $\varnothing = \mathbb{V}(1)$ and $\mathbb{C}^n = \mathbb{V}(0)$
- intersection of two affine varieties is affine variety

$$\mathbb{V}(\{F_i\}_{i\in I}) \cap \mathbb{V}(\{F_j\}_{j\in J}) = \mathbb{V}(\{F_i\}_{i\in I\cup J})$$

- union of two affine varieties is affine variety

$$\mathbb{V}(\{F_i\}_{i\in I}) \cup \mathbb{V}(\{F_j\}_{j\in J}) = \mathbb{V}(\{F_iF_j\}_{(i,j)\in I\times J})$$

- easiest to see for hypersurfaces

$$\mathbb{V}(F_1) \cup \mathbb{V}(F_2) = \mathbb{V}(F_1F_2)$$

since $F_1(\mathbf{x})F_2(\mathbf{x}) = 0$ iff $F_1(\mathbf{x}) = 0$ or $F_2(\mathbf{x}) = 0$

Zariski topology

- let $\mathcal{V} = \{$all affine varieties in $\mathbb{C}^n\}$, then
  1. $\varnothing \in \mathcal{V}$
  2. $\mathbb{C}^n \in \mathcal{V}$
  3. if $V_1, \dots, V_n \in \mathcal{V}$, then $\bigcup_{i=1}^n V_i \in \mathcal{V}$
  4. if $V_\alpha \in \mathcal{V}$ for all $\alpha \in A$, then $\bigcap_{\alpha \in A} V_\alpha \in \mathcal{V}$
- let $\mathcal{Z} = \{\mathbb{C}^n \backslash V : V \in \mathcal{V}\}$
- then $\mathcal{Z}$ is topology on $\mathbb{C}^n$: Zariski topology
- write $\mathbb{A}^n$ for topological space $(\mathbb{C}^n, \mathcal{Z})$: affine n-space
- Zariski open sets are complements of affine varieties
- Zariski closed sets are affine varieties
- write $\mathcal{E}$ for Euclidean topology, then $\mathcal{Z} \subset \mathcal{E}$, i.e.,
  - Zariski open $\Rightarrow$ Euclidean open
  - Zariski closed $\Rightarrow$ Euclidean closed

Zariski topology is weird

- $\mathcal{Z}$ is much smaller than $\mathcal{E}$: Zariski topology is very coarse
  - basis for $\mathcal{E}$: $B_\varepsilon(\mathbf{x})$ where $\mathbf{x} \in \mathbb{C}^n$, $\varepsilon > 0$
  - basis for $\mathcal{Z}$: $\{\mathbf{x} \in \mathbb{A}^n : f(\mathbf{x}) \neq 0\}$ where $f \in \mathbb{C}[\mathbf{x}]$
- $\varnothing \neq S \in \mathcal{Z}$
  - S is unbounded under $\mathcal{E}$
  - S is dense under both $\mathcal{Z}$ and $\mathcal{E}$
- nonempty Zariski open $\Rightarrow$ generic $\Rightarrow$ almost everywhere $\Rightarrow$ Euclidean dense
- $\mathcal{Z}$ not Hausdorff, e.g. on $\mathbb{A}^1$, $\mathcal{Z} =$ cofinite topology
- Zariski compact $\not\Rightarrow$ Zariski closed, e.g. $\mathbb{A}^n \backslash \{\mathbf{0}\}$ compact
- Zariski topology on $\mathbb{A}^2$ not product topology on $\mathbb{A}^1 \times \mathbb{A}^1$, e.g. $\{(x, x) : x \in \mathbb{A}^1\}$ closed in $\mathbb{A}^2$, not in $\mathbb{A}^1 \times \mathbb{A}^1$

two cool examples

Zariski closed:

common roots: $\{(\mathbf{a},\mathbf{b}) \in \mathbb{A}^4 \times \mathbb{A}^3 : a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and $b_0 + b_1 x + b_2 x^2$ have common roots$\}$

$$\left\{ (\mathbf{a},\mathbf{b}) \in \mathbb{A}^7 : \det\left(\begin{bmatrix} a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_2 & b_1 & b_0 \end{bmatrix} \right) = 0 \right\}$$

repeat roots: $\{\mathbf{a} \in \mathbb{A}^4 : a_0 + a_1 x + a_2 x^2 + a_3 x^3$ repeat roots$\}$

$$\left\{ \mathbf{a} \in \mathbb{A}^4 : \det\left(\begin{bmatrix} a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 \\ 2a_2 & a_1 & 0 & 0 & 0 \\ 0 & 2a_2 & a_1 & 0 & 0 \\ 0 & 0 & 2a_2 & a_1 & 0 \end{bmatrix} \right) = 0 \right\}$$

generally: first determinant is resultant, $\operatorname{res}(f,g)$, defined likewise for f and g of aribtrary degrees; second determinant is discriminant, $\operatorname{disc}(f) := \operatorname{res}(f, f')$

more examples

Zariski closed:

nilpotent matrices: $\{A \in \mathbb{A}^{n\times n} : A^k = 0\}$ for any fixed $k \in \mathbb{N}$

eigenvectors: $\{(A, \mathbf{x}) \in \mathbb{A}^{n\times(n+1)} : A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{C}\}$

repeat eigenvalues: $\{A \in \mathbb{A}^{n\times n} : A$ has repeat eigenvalues$\}$

Zariski open:

full rank: $\{A \in \mathbb{A}^{m\times n} : A$ has full rank$\}$

distinct eigenvalues: $\{A \in \mathbb{A}^{n\times n} : A$ has n distinct eigenvalues$\}$

- write $p_A(x) = \det(xI - A)$

$$\{A \in \mathbb{A}^{n\times n} : \text{repeat eigenvalues}\} = \{A \in \mathbb{A}^{n\times n} : \text{disc}(p_A) = 0\}$$
$$\{A \in \mathbb{A}^{n\times n} : \text{distinct eigenvalues}\} = \{A \in \mathbb{A}^{n\times n} : \text{disc}(p_A) \neq 0\}$$

- note $\text{disc}(p_A)$ is a polynomial in the entries of A
- will use this to prove Cayley-Hamilton theorem later

Irreducibility

reducibility

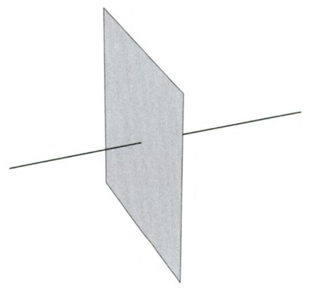
- affine variety V is reducible if $V = V_1 \cup V_2$, $\varnothing \neq V_i \subsetneq V$
- affine variety V is irreducible if it is not reducible
- every *subset* of $\mathbb{A}^n$ can be broken up into nontrivial union of irreducible components

$$S = V_1 \cup \cdots \cup V_k$$

  where $V_i \not\subseteq V_j$ for all $i \neq j$, V_i irreducible and closed in subspace topology of S
- decomposition above unique up to order
- $\mathbb{V}(F)$ irreducible variety if F irreducible polynomial
- non-empty Zariski open subsets of irreducible affine variety are Euclidean dense

example

- a bit like connectedness but not quite: the variety in $\mathbb{A}^3$ below is connected but reducible



- $\mathbb{V}(xy, xz) = \mathbb{V}(y, z) \cup \mathbb{V}(x)$ with irreducible components *yz*-plane and *x*-axis

commuting matrix varieties

- define k-tuples of $n \times n$ commuting matrices

$$\mathcal{C}(k, n) := \{(A_1, \dots, A_k) \in (\mathbb{A}^{n \times n})^k : A_i A_j = A_j A_i\}$$

- as usual, identify $(\mathbb{A}^{n \times n})^k \equiv \mathbb{A}^{kn^2}$
- clearly $\mathcal{C}(k, n)$ is affine variety

question: if $(A_1, \dots, A_k) \in \mathcal{C}(k, n)$, then are $A_1, \dots, A_k$ simultaneously diagaonalizable?

answer: no, only simultaneously triangularizable

question: can we approximate $A_1, \dots, A_k$ by $B_1, \dots, B_k$,

$$\|A_i - B_i\| < \varepsilon, \quad i = 1, \dots, k,$$

where $B_1, \dots, B_k$ simultaneously diagonalizable ?

anwer: yes, if and only if $\mathcal{C}(k, n)$ is irreducible

question: for what values of k and n is $\mathcal{C}(k, n)$ irreducible?

what is known

$k = 2$: $\mathcal{C}(2, n)$ irreducible for all $n \geq 1$ [Motzkin–Taussky, 1955]

$k \geq 4$: $\mathcal{C}(4, n)$ reducible for all $n \geq 4$ [Gerstenhaber, 1961]

$n \leq 3$: $\mathcal{C}(k, n)$ irreducible for all $k \geq 1$ [Gerstenhaber, 1961]

$k = 3$: $\mathcal{C}(3, n)$ irreducible for all $n \leq 10$, reducible for all $n \geq 29$ [Guralnick, 1992], [Holbrook–Omladič, 2001], [Šivic, 2012]

open: reducibility of $\mathcal{C}(3, n)$ for $11 \leq n \leq 28$

Maps Between Varieties

morphisms

- morphism of affine varieties: polynomial maps
- F morphism if

$$\begin{aligned} \mathbb{A}^n &\xrightarrow{F} \mathbb{A}^m \\ (x_1, \ldots, x_n) &\longmapsto (F_1(x_1, \ldots, x_n), \ldots, F_m(x_1, \ldots, x_n)) \end{aligned}$$

  where $F_1, \ldots, F_m \in \mathbb{C}[x_1, \ldots, x_n]$
- $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ affine algebraic varieties, say $F : V \to W$ morphism if it is restriction of some morphism $\mathbb{A}^n \to \mathbb{A}^m$
- say F isomorphism if (i) bijective; (ii) inverse G is morphism
- $V \simeq W$ isomorphic if there exists $F : V \to W$ isomorphism
- straightforward: morphism of affine varieties continuous in Zariski topology
- caution: morphism need not send affine varieties to affine varieties, i.e., not closed map

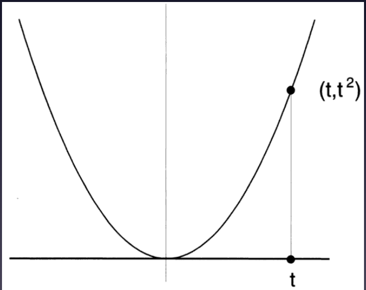
examples

affine map: $F : \mathbb{A}^n \to \mathbb{A}^n$, $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ morphism for $A \in \mathbb{C}^{n\times n}$, $\mathbf{b} \in \mathbb{C}^n$; isomorphism if $A \in \mathrm{GL}_n(\mathbb{C})$

projection: $F : \mathbb{A}^2 \to \mathbb{A}^1$, $(x, y) \mapsto x$ morphism

parabola: $C = \mathbb{V}(y - x^2) = \{(t, t^2) \in \mathbb{A}^2 : t \in \mathbb{A}\} \simeq \mathbb{A}^1$

$$\begin{aligned} \mathbb{A}^1 &\xrightarrow{F} C \\ t &\longmapsto (t, t^2) \end{aligned} \qquad \begin{aligned} C &\xrightarrow{G} \mathbb{A}^1 \\ (x, y) &\longmapsto x \end{aligned}$$



twisted cubic: $\{(t, t^2, t^3) \in \mathbb{A}^3 : t \in \mathbb{A}\} \simeq \mathbb{A}^1$

Cayley-Hamilton

- recall: if $A \in \mathbb{C}^{n \times n}$ and $p_A(x) = \det(xI - A)$, then $p_A(A) = 0$
- $\mathbb{A}^{n \times n} \to \mathbb{A}^{n \times n}$, $A \mapsto p_A(A)$ morphism
- claim that this morphism is identically zero
- if A diagaonlizable then

  $$p_A(A) = Xp_A(\Lambda)X^{-1} = X \operatorname{diag}(p_A(\lambda_1), \ldots, p_A(\lambda_n))X^{-1} = 0$$

  since $p_A(x) = \prod_{i=1}^{n}(x - \lambda_i)$
- earlier: $X = \{A \in \mathbb{A}^{n \times n} : A \text{ has } n \text{ distinct eigenvalues}\}$ Zariski dense in $\mathbb{A}^{n \times n}$
- pitfall: most proofs you find will just declare that we're done since two continuous maps $A \mapsto p_A(A)$ and $A \mapsto 0$ agreeing on a dense set implies they are the same map
- problem: codomain $\mathbb{A}^{n \times n}$ is not Hausdorff!

need irreducibility

- let

$$X = \{A \in \mathbb{A}^{n\times n} : A \text{ has } n \text{ distinct eigenvalues}\}$$
$$Y = \{A \in \mathbb{A}^{n\times n} : p_A(A) = 0\}$$
$$Z = \{A \in \mathbb{A}^{n\times n} : \text{disc}(p_A) = 0\}$$

- now $X \subseteq Y$ by previous slide
- but $X = \{A \in \mathbb{A}^{n\times n} : \text{disc}(p_A) \neq 0\}$ by earlier slide
- so we must have $Y \cup Z = \mathbb{A}^{n\times n}$
- since $\mathbb{A}^{n\times n}$ irreducible, either $Y = \varnothing$ or $Z = \varnothing$
- $Y \supseteq X \neq \varnothing$, so $Z = \varnothing$ and $Y = \mathbb{A}^{n\times n}$

Questions From Yesterday

unresolved questions

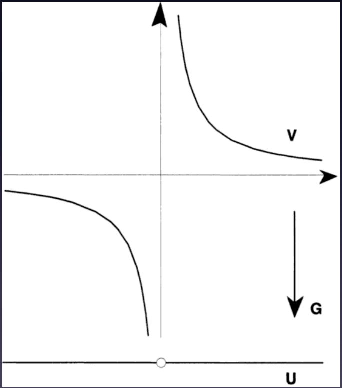
1. how to define affine variety intrinsically?
2. what is the ‘actual definition’ of an affine variety that we kept alluding to?
3. why is $\mathbb{A}^1 \backslash \{0\}$ an affine variety?
4. why is $GL_n(\mathbb{C})$ an affine variety?

same answer to all four questions

special example

hyperbola: $C = \mathbb{V}(xy-1) = \{(t, t^{-1}) : t \neq 0\} \simeq \mathbb{A}^1 \backslash \{0\}$

$$\begin{array}{rclcrcl} \mathbb{A}^1\backslash\{0\} & \xrightarrow{F} & C & \qquad & C & \xrightarrow{G} & \mathbb{A}^1\backslash\{0\} \\ t & \longmapsto & (t, t^{-1}) & & (x,y) & \longmapsto & x \end{array}$$



(there's a slight problem)

new defintion

- redefine affine variety to be any object that is isomorphic to a Zariski closed subset of $\mathbb{A}^n$
- advantage: does not depend on embedding, i.e., intrinsic
- what we called 'affine variety' should instead have been called Zariski closed sets
- $\mathbb{A}^1 \backslash \{0\} \simeq \mathbb{V}(xy - 1)$ and $\mathbb{V}(xy - 1)$ Zariski closed in $\mathbb{A}^2$, so $\mathbb{A}^1 \backslash \{0\}$ affine variety

(there's a slight problem again)

general linear group

- likewise $\mathrm{GL}_n(\mathbb{C}) \simeq \mathbb{V}(\det(X)y - 1)$

$$\begin{aligned} \mathrm{GL}_n(\mathbb{C}) &\xrightarrow{F} \{(X, y) \in \mathbb{A}^{n^2+1} : \det(X)y = 1\} \\ X &\longmapsto (X, \det(X)^{-1}) \end{aligned}$$

  has inverse

$$\begin{aligned} \{(X, y) \in \mathbb{A}^{n^2+1} : \det(X)y = 1\} &\xrightarrow{G} \mathrm{GL}_n(\mathbb{C}) \\ (X, y) &\longmapsto X \end{aligned}$$

- $\mathrm{GL}_n(\mathbb{C})$ affine variety since $\mathbb{V}(\det(X)y - 1)$ closed in $\mathbb{A}^{n^2+1}$

(there's a slight problem yet again)

resolution of slight problems

- problems:
  1. $t \mapsto (t, t^{-1})$ and $X \mapsto (X, \det(X)^{-1})$ are not morphisms of affine varieties as t^{-1} and $\det(X)^{-1}$ are not polynomials
  2. we didn't specify what we meant by 'any object'
- resolution:

  object = quasi-projective variety
  morphism = morphism of quasi-projective varieties
- from now on:

  affine variety: Zariski closed subset of $\mathbb{A}^n$, e.g. $\mathbb{V}(xy - 1)$
  quasi-affine variety: quasi-projective variety isomorphic to an affine variety, e.g. $\mathbb{A}^1 \backslash \{0\}$