# Algebraic Geometry of Matrices III

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# today

- focus: equivalence of algebra and geometry
- three fundamental theorems in commutative algebra:
  1. Hilbert's basis theorem
  2. Hilbert's nullstellensatz
  3. Noether's normalization lemma[1]
- why they are important for algebraic geometry
- relate to linear algebra/matrix analysis/operator theory

[1] cf. Ke Ye's lecture tomorrow

Algebra $\longleftrightarrow$ Geometry

# let's start closer to home

obvious: $X$ compact Hausdorff, then

$$C(X) := \{f : X \to \mathbb{C} : f \text{ continuous}\}$$

is unital commutative $C^*$-algebra

less obvious: $\mathcal{A}$ unital commutative $C^*$-algebra, then there exists $X$ compact Hausdorff so that $\mathcal{A} \simeq C(X)$

what is $X$: spectrum or, more accurately, maximal spectrum

$$X = \text{Spec}_m(\mathcal{A}) \quad \text{or} \quad \text{Hom}(\mathcal{A}, \mathbb{C})$$

very old: [Mazur, 1938], [Gelfand, 1941], [Gelfand–Naimark, 1943]

# spectrum

$\mathcal{A}$ unital commutative Banach algebra over $\mathbb{C}$

maximal ideal: $\mathfrak{m} \subsetneq \mathcal{A}$ with $\mathcal{A}/\mathfrak{m} \simeq \mathbb{C}$, necessarily closed

character: homomorphism $\varphi : \mathcal{A} \to \mathbb{C}$

spectrum: $\mathrm{Spec}_m(\mathcal{A}) \simeq \mathrm{Hom}(\mathcal{A}, \mathbb{C})$

$$\{\text{maximal ideals in } \mathcal{A}\} \longleftrightarrow \{\text{characters in } \mathcal{A}^*\}$$
$$\ker(\varphi) \longleftarrow\!\!\shortmid \varphi$$
$$\mathfrak{m} \longmapsto \pi_{\mathfrak{m}}$$

where $\pi_{\mathfrak{m}} : \mathcal{A} \to \mathcal{A}/\mathfrak{m}$

why name: because for $a \in \mathcal{A}$,

$$\begin{aligned}\sigma(a) &= \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ not invertible}\}\\ &= \{\varphi(a) \in \mathbb{C} : \varphi \in \mathrm{Hom}(\mathcal{A}, \mathbb{C})\}\\ &= \{\pi_{\mathfrak{m}}(a) \in \mathbb{C} : \mathfrak{m} \in \mathrm{Spec}_m(\mathcal{A})\}\end{aligned}$$

# correspondence

geometry $\to$ algebra: $X \to C(X)$

algebra $\to$ geometry: $\mathcal{A} \to \mathrm{Spec}_m(\mathcal{A})$

$$\text{geometry} \longleftrightarrow \text{algebra}$$

$$\{\text{compact Hausdorff spaces}\} \longleftrightarrow \{\text{abelian unital } C^*\text{-algebras}\}$$

$$\{\text{locally compact Hausdorff spaces}\} \longleftrightarrow \{\text{abelian } C^*\text{-algebras}\}$$

- works for von Neumann algebras too: $(X, \mu) \to L^\infty(X, \mu)$
- inspiration for non-commutative geometry

# furthermore: get dictionary

| **locally compact Hausdorff space** | **commutative $C^*$-algebra** |
|---|---|
| compact | unital |
| 1-point compactification | unitization |
| Stone–Čech compactification | multiplier algebra |
| closed subspace/inclusion | closed ideal/quotient |
| surjection | injection |
| injection | surjection |
| homoemorphism | automorphism |
| Borel measure | positive functional |
| probability measure | state |
| disjoint union | direct sum |
| cartesian product | minimal tensor product |

# many more examples

$$\text{geometry} \longleftrightarrow \text{algebra}$$

$$\{\text{locally compact Hausdorff spaces}\} \longleftrightarrow \{\text{commutative } C^*\text{-algebras}\}$$

$$\{\sigma\text{-finite measure spaces}\} \longleftrightarrow \{\text{commutative von Neumann algebras}\}$$

$$\{\text{vector bundles on } X\} \longleftrightarrow \{\text{fin. gen. projective modules over } C(X)\}$$

$$\{\text{compact Riemann surfaces}\} \longleftrightarrow \{\text{algebraic function fields}\}$$

$$\{\text{affine varieties}\} \longleftrightarrow \{\text{fin. gen. reduced rings over } \overline{\mathbb{F}}\}$$

$$\{\text{affine schemes}\} \longleftrightarrow \{\text{unital commutative rings}\}$$

$$\{\text{quasi-coherent sheaves on } \mathrm{Spec}(R)\} \longleftrightarrow \{\text{modules over } R\}$$

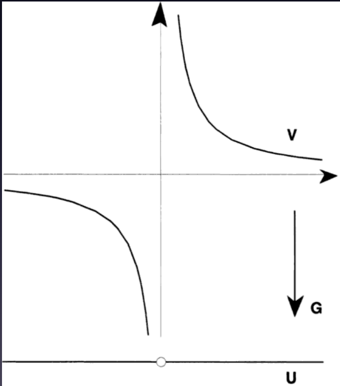
- $X =$ locally compact Hausdorff space
- $R =$ unital commutative ring
- $\overline{\mathbb{F}} =$ algebraically closed field
- fin. gen. $=$ finitely generated

Ideals $\longleftrightarrow$ Varieties

# recall nomenclature

affine variety: Zariski closed subset of $\mathbb{A}^n$, e.g. $\mathbb{V}(xy - 1)$

quasi-affine variety: quasi-projective variety isomorphic to an affine variety, e.g. $\mathbb{A}^1 \backslash \{0\}$



# algebra–geometry correspondence

$$\text{geometry: } \mathbb{A}^n \longleftrightarrow \text{algebra: } \mathbb{C}[x_1, \dots, x_n]$$

$$\{\text{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } \mathbb{C}[x_1, \dots, x_n]\}$$

$$\{\text{irreducible affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{prime ideals in } \mathbb{C}[x_1, \dots, x_n]\}$$

$$\{\text{points in } \mathbb{A}^n\} \longleftrightarrow \{\text{maximal ideals in } \mathbb{C}[x_1, \dots, x_n]\}$$

$$\{\text{regular maps } X \to Y\} \longleftrightarrow \{\text{homomorphisms } \mathbb{C}[Y] \to \mathbb{C}[X]\}$$

- last line: $X$ affine variety, $Y$ quasi-affine variety
- we will study these correspondence next

# glossary

ring: $R$ associative, commutative, unital

reduced: for all $a \in R$, $a^n = 0 \;\Leftrightarrow\; a = 0$

ideal: $\mathfrak{a} \subseteq R$ with $a + rb \in \mathfrak{a}$ for all $a, b \in \mathfrak{a}, r \in R$

- trivial $\{0\}$ or $R$
- maximal $\mathfrak{m} \subsetneq \mathfrak{a} \subseteq R \;\Rightarrow\; \mathfrak{a} = R$
- prime $\mathfrak{p} \subsetneq R$: $ab \in \mathfrak{p} \;\Rightarrow\; a \in \mathfrak{p}$ or $b \in \mathfrak{p}$
- radical $\mathfrak{a} \subseteq R$: $\mathfrak{a} = \sqrt{\mathfrak{a}}$

$$\sqrt{\mathfrak{a}} := \{a \in R : a^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}$$

homomorphism: $f : R \to S$ preserves sums, products, unit

- maximal $\Rightarrow$ prime $\Rightarrow$ radical
- $\mathfrak{m}$ maximal iff $R/\mathfrak{m}$ field
- $\mathfrak{p}$ prime iff $R/\mathfrak{p}$ domain
- $\mathfrak{a}$ radical iff $R/\mathfrak{a}$ reduced

# more glossary

- ideal generated by set $S \subseteq R$ is

$$\begin{aligned}\langle S\rangle &= \bigcap\{I : S \subseteq I,\ I \subseteq R \text{ an ideal}\} \\ &= \text{smallest ideal containing } S \\ &= \{r_1 s_1 + \cdots + r_m s_m : r_i \in R,\ s_i \in S,\ m \in \mathbb{N}\}\end{aligned}$$

- ideal $I$ finitely generated if for some $s_1, \ldots, s_m \in R$,

$$I = \langle s_1, \ldots, s_m\rangle$$

- ring $R$ Noetherian if all its ideals finitely generated
- spectrum and maximal spectrum of $R$ are

$$\begin{aligned}\text{Spec}(R) &:= \{\text{prime ideals of } R\} \\ \text{Spec}_m(R) &:= \{\text{maximal ideals of } R\}\end{aligned}$$

# Hilbert's basis theorem

usual: $\mathbb{C}[x_1, \ldots, x_n]$ Noetherian

relevant: every $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is finitely generated

general: $R$ Noetherian $\Rightarrow$ $R[x]$ also Noetherian

# Hilbert's nullstellensatz

abstract: $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$

concrete: if $I \subsetneq \mathbb{C}[x_1, \ldots, x_n]$, then there exists $\mathbf{x} \in \mathbb{C}^n$ where $F(\mathbf{x}) = 0$ for all $F \in I$

weak: $F_1, \ldots, F_m \in \mathbb{C}[x_1, \ldots, x_n]$, exactly one holds:

1. $\exists\ \mathbf{x} \in \mathbb{C}^n$: $F_1(\mathbf{x}) = \cdots = F_m(\mathbf{x}) = 0$
2. $\exists\ G_1, \ldots, G_m \in \mathbb{C}[x_1, \ldots, x_n]$:

$$F_1 G_1 + \cdots + F_m G_m = 1$$

strong: $F_1, \ldots, F_m, H \in \mathbb{C}[x_1, \ldots, x_n]$, exactly one holds:

1. $\exists\ \mathbf{x} \in \mathbb{C}^n$: $F_1(\mathbf{x}) = \cdots = F_m(\mathbf{x}) = 0$, $H(\mathbf{x}) \neq 0$
2. $\exists\ G_1, \ldots, G_m \in \mathbb{C}[x_1, \ldots, x_n]$, $p \in \mathbb{Z}_+$:

$$F_1 G_1 + \cdots + F_m G_m = H^p$$

spectral: $\mathfrak{m} \in \mathrm{Spec}_m(\mathbb{C}[x_1, \ldots, x_n])$ iff $\mathfrak{m} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$

# ideals and varieties

- vanishing ideal of $T \subseteq \mathbb{A}^n$ is

$$\mathbb{I}(T) := \{f \in \mathbb{C}[x_1, \ldots, x_n] : f(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in T\}$$

- variety cut out by $S \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is

$$\mathbb{V}(S) := \{(x_1, \ldots, x_n) \in \mathbb{A}^n : f(\mathbf{x}) = 0 \text{ for all } f \in S\}$$

- true for any affine variety $V \subseteq \mathbb{A}^n$ (by Hilbert's basis)

$$\mathbb{V}(\mathbb{I}(V)) = V$$

- true only for radical ideals $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$

$$\mathbb{I}(\mathbb{V}(I)) = I$$

- in general (by Hilbert's nullstellensatz)

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$$

# implications

1. since $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$ and $\mathbb{V}(\mathbb{I}(V)) = V$, get correspondence

$$\{\text{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } \mathbb{C}[x_1, \ldots, x_n]\}$$
$$V \longmapsto \mathbb{I}(V)$$
$$\mathbb{V}(I) \longleftarrow I$$

2. since $\mathfrak{m}$ maximal iff $\mathfrak{m} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$, get

$$\mathbb{A}^n \longleftrightarrow \mathrm{Spec}_m(\mathbb{C}[x_1, \ldots, x_n])$$
$$(a_1, \ldots, a_n) \longleftrightarrow \langle x_1 - a_1, \ldots, x_n - a_n \rangle$$

3. easy: $\mathfrak{p} \in \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n])$ iff $\mathbb{V}(\mathfrak{p})$ is irreducible

# why significant

- a point has no 'internal structure', no intrinsic information
- an ideal is much richer in structure
- points identified with maximal ideals via

$$\mathbb{A}^n \simeq \mathrm{Spec}_m(\mathbb{C}[x_1, \ldots, x_n])$$

- since maximal ideals are prime,

$$\mathbb{A}^n \simeq \mathrm{Spec}_m(\mathbb{C}[x_1, \ldots, x_n]) \subseteq \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n])$$

- $\mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n])$ affine scheme: contains 'generalized points' corresponding to irreducible varieties

$$\begin{array}{ccc} \{\text{affine varieties}\} & \longleftrightarrow & \{\text{finitely generated reduced rings over } \overline{\mathbb{F}}\} \\ \downarrow & & \downarrow \\ \{\text{affine schemes}\} & \longleftrightarrow & \{\text{unital commutative rings}\} \end{array}$$