

# Algebraic Geometry of Matrices IV

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# today

- coordinate ring and pullback
- dimension of affine variety
- relate to linear algebra/matrix analysis/operator theory
- Noether's normalization lemma<sup>1</sup>

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<sup>1</sup>cf. Ke Ye's tutorial at 3:30pm

Rings  $\longleftrightarrow$  Varieties

# algebra–geometry revisited

geometry  $\longleftrightarrow$  algebra

yesterday:

$\{\text{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } \mathbb{C}[x_1, \dots, x_n]\}$

today:

$\{\text{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{fin. gen. reduced rings over } \mathbb{C}\}$

also:

$\{\text{morphisms } X \rightarrow Y\} \longleftrightarrow \{\text{homomorphisms } \mathbb{C}[Y] \rightarrow \mathbb{C}[X]\}$

new terminology later: rings over  $\mathbb{C} = \mathbb{C}$ -algebra

# supplemental glossary

- $\mathfrak{a} \subseteq R$  ideal, then **quotient ring** is

$$R/\mathfrak{a} := \{[r] = r + \mathfrak{a} : r \in R\}$$

- $[r + s] := [r] + [s]$ ,  $[r][s] := [rs]$
- **quotient projection**  $\pi_{\mathfrak{a}} : R \rightarrow R/\mathfrak{a}$ ,  $r \mapsto [r]$  is homomorphism
- $\pi_{\mathfrak{a}}$  yields one-to-one correspondence:

$$\{\text{ideals in } R \text{ containing } \mathfrak{a}\} \longleftrightarrow \{\text{ideals in } R/\mathfrak{a}\}$$

- carries maximal/prime/radical ideals to maximal/prime/radical ideals

# $\mathbb{C}$ -algebra

- $\mathbb{C}$ -algebra  $\mathcal{A}$  is both
  - 1 ring (associative, commutative, unital)
  - 2 vector space over  $\mathbb{C}$
- $\mathbb{C}$ -subalgebras  $J \subseteq \mathcal{A}$  subset that are  $\mathbb{C}$ -algebras:
  - $\mathbb{C}$ -subalgebras **generated by** set  $S \subseteq \mathcal{A}$  is

$$\begin{aligned} [S] &= \bigcap \{ J : S \subseteq J, J \subseteq \mathcal{A} \text{ a } \mathbb{C}\text{-subalgebra} \} \\ &= \text{smallest } \mathbb{C}\text{-subalgebra containing } S \\ &= \{ f(s_1, \dots, s_n) \in \mathcal{A} : f \in \mathbb{C}[x_1, \dots, x_n], s_i \in S \} \end{aligned}$$

- $J$  **finitely generated** if for some  $s_1, \dots, s_m \in \mathcal{A}$ ,

$$J = [s_1, \dots, s_m]$$

- caution: unlike ideals, may not define quotient  $\mathcal{A}/J$
- $\mathbb{C}$ -algebra **homomorphism**  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is both
  - 1 ring homomorphism
  - 2  $\mathbb{C}$ -linear transformation

# examples

- $\mathbb{C}[x, y]$  is  $\mathbb{C}$ -algebra finitely generated by  $x, y$
- quotient ring  $\mathbb{C}[x, y]/\langle x^2 + y^3 \rangle$  also  $\mathbb{C}$ -algebra
- $\mathbb{C}$ -algebra homomorphism  $\varphi : \mathbb{C}[x, y]/\langle x^2 + y^3 \rangle \rightarrow \mathbb{C}[z]$  completely determined by where it sends generators, e.g.

$$\begin{aligned}\varphi(x) &= z^3, & \varphi(y) &= -z^2, \\ \varphi(xy - 3y^2) &= \varphi(x)\varphi(y) - 3\varphi(y)^2 = -z^5 - 3z^4\end{aligned}$$

- complex conjugation

$$\begin{aligned}\mathbb{C}[x] &\rightarrow \mathbb{C}[x] \\ a_0 + a_1x + \cdots + a_nx^n &\mapsto \bar{a}_0 + \bar{a}_1x + \cdots + \bar{a}_nx^n\end{aligned}$$

is ring homomorphism but not  $\mathbb{C}$ -algebra homomorphism

- most important source of examples for us: *coordinate rings* of affine varieties

# $\mathbb{C}$ -valued functions

- we studied morphisms  $X \rightarrow Y$  between affine varieties
- now we consider special case when  $Y = \mathbb{C}$
- why: to understand a mathematical object, it helps to understand  $\mathbb{C}$ -valued functions on that object
- cf.  $C^*$ -algebra and von Neumann algebra from yesterday
- want the ‘right level’ of regularity:
  - $C^*$ -algebra: **continuous** functions on locally compact Hausdorff space
  - von Neumann algebra:  $L^\infty$ -functions on  $\sigma$ -finite measure space
  - finitely generated reduced  $\mathbb{C}$ -algebra: **polynomial** functions on affine variety
- *polynomial function* means  $f \in \mathbb{C}[x_1, \dots, x_n]$ 
  - defines  $\mathbb{C}$ -valued function  $f : \mathbb{A}^n \rightarrow \mathbb{C}$
  - restricts to  $\mathbb{C}$ -valued function  $f : V \rightarrow \mathbb{C}$  for affine variety  $V \subseteq \mathbb{A}^n$

# coordinate ring

- **coordinate rings** of affine variety  $V \subseteq \mathbb{A}^n$  is

$$\mathbb{C}[V] := \{f : V \rightarrow \mathbb{C} : f \in \mathbb{C}[x_1, \dots, x_n]\}$$

- clearly a  $\mathbb{C}$ -algebra
- $\varphi : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}[V], f \mapsto f|_V$  is homomorphism and since  $\ker(\varphi) = \mathbb{I}(V)$

$$\mathbb{C}[V] \simeq \mathbb{C}[x_1, \dots, x_n]/\mathbb{I}(V)$$

- for any  $V \subseteq \mathbb{A}^n$ ,  $\mathbb{C}[V]$  is always a **finitely generated reduced  $\mathbb{C}$ -algebra** by what we saw yesterday:

$$\{\text{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } \mathbb{C}[x_1, \dots, x_n]\}$$

- i.e.,  $\mathbb{I}(V)$  always a radical ideal

# converse also true

- can show: any finitely generated reduced  $\mathbb{C}$ -algebra is the coordinate ring of some affine variety  $V \subseteq \mathbb{A}^n$
- get correspondence

$$\{\text{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{fin. gen. reduced } \mathbb{C}\text{-algebra}\}$$

- conditions too restrictive: e.g. may have nilpotent elements, infinitely generated,  $\mathbb{Z}$ -module instead of  $\mathbb{C}$ -vector space
- e.g. Fermat's last theorem

$$R = \mathbb{Z}[x, y, z] / \langle x^n + y^n - z^n \rangle$$

- Grothendieck's answer: use affine schemes

$$\{\text{affine schemes}\} \longleftrightarrow \{\text{unital commutative rings}\}$$

# examples

Hilbert nullstellensatz:  $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$

①  $\mathbb{C}[\mathbb{A}^n] = \mathbb{C}[x_1, \dots, x_n]$

②  $V = \mathbb{V}(x^2 + y^2 + z^2) \subseteq \mathbb{A}^3$ ,  $\mathbb{C}[V] \simeq \mathbb{C}[x, y, z]/\langle x^2 + y^2 + z^2 \rangle$

$$x^2 + y^2 + z^2 = 0 \quad \text{in } \mathbb{C}[V]$$

③  $V = \mathbb{V}(xy - 1) \subseteq \mathbb{A}^2$ ,  $\mathbb{C}[V] \simeq \mathbb{C}[x, y]/\langle xy - 1 \rangle$

$$1/x = y \quad \text{in } \mathbb{C}[V]$$

④  $V = \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$ ,  $\mathbb{C}[V] \simeq \mathbb{C}[x, y, z]/\langle x^2 + y^2 - z^2 \rangle$

$$x^3 + 2xy^2 - 2xz^2 + x = 2x(x^2 + y^2 - z^2) + x - x^3 = x - x^3 \quad \text{in } \mathbb{C}[V]$$

moral: arithmetic on  $\mathbb{C}[V]$  is done modulo  $\mathbb{I}(V)$

# pullback

- morphism  $F : V \rightarrow W$  of affine varieties induces unique  $\mathbb{C}$ -algebra homomorphism, called **pullback**,

$$F^* : \mathbb{C}[W] \rightarrow \mathbb{C}[V], \quad g \mapsto g \circ F$$

- converse also true: any  $\mathbb{C}$ -algebra homomorphism  $\varphi : \mathbb{C}[W] \rightarrow \mathbb{C}[V]$  induces unique morphism  $\varphi^* : V \rightarrow W$
- get correspondence

$$\begin{aligned} \{\text{morphisms } X \rightarrow Y\} &\longleftrightarrow \{\text{homomorphisms } \mathbb{C}[Y] \rightarrow \mathbb{C}[X]\} \\ F &\longmapsto F^* \\ \varphi^* &\longleftarrow \varphi \end{aligned}$$

- cf. smooth maps  $f : M \rightarrow N$  of differential manifolds induces pullback  $f^*$  from 1-forms on  $N$  to 1-forms on  $M$

# pullbacks are useful

- pullback  $F^* : \mathbb{C}[W] \rightarrow \mathbb{C}[V]$  injective iff  $F$  is **dominant**, i.e., image  $F(V)$  is dense in  $W$
- pullback  $F^* : \mathbb{C}[W] \rightarrow \mathbb{C}[V]$  surjective iff  $F$  defines isomorphism between  $V$  and some affine subvariety of  $W$
- how these may be applied: Ke Ye's talk next week

# examples

## 1 morphism

$$F : \mathbb{A}^3 \rightarrow \mathbb{A}^2, \quad (x, y, z) \mapsto (x^2y, x - z)$$

induces pullback

$$F^* : \mathbb{C}[u, v] \rightarrow \mathbb{C}[x, y, z], \quad u \mapsto x^2y, \quad v \mapsto x - z$$

completely determined by where it sends generators, e.g.

$$\varphi(u^2 + 5v^3) = (x^2y)^2 + 5(x - z)^3$$

## 2 linear morphism

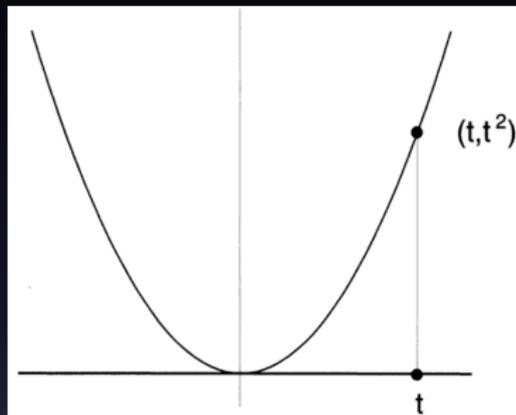
$$F : \mathbb{A}^n \rightarrow \mathbb{A}^m, \quad \mathbf{x} \mapsto A\mathbf{x}$$

for some  $A \in \mathbb{C}^{m \times n}$  has pullback

$$F^* : \mathbb{A}^m \rightarrow \mathbb{A}^n, \quad \mathbf{y} \mapsto A^T \mathbf{y}$$

# an earlier example

- parabola  $C = \mathbb{V}(y - x^2) = \{(t, t^2) \in \mathbb{A}^2 : t \in \mathbb{A}\} \simeq \mathbb{A}^1$



- morphism is isomorphism of affine varieties

$$F : \mathbb{A}^1 \rightarrow C, \quad t \mapsto (t, t^2)$$

- pullback  $F^* : \mathbb{C}[C] \rightarrow \mathbb{C}[\mathbb{A}^1]$  surjective with zero kernel

$$\mathbb{C}[x, y]/\langle y - x^2 \rangle \rightarrow \mathbb{C}[t], \quad x \mapsto t, \quad y \mapsto t^2$$

i.e., isomorphism of  $\mathbb{C}$ -algebras

# exercises for the audience

- 1 if  $F = (F_1, \dots, F_n) : \mathbb{A}^n \rightarrow \mathbb{A}^n$  is an isomorphism of affine varieties, then the **Jacobian determinant**,

$$\det \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix} \in \mathbb{C}^\times$$

- 2 show that the converse is also true<sup>2</sup>

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<sup>2</sup>just kidding: this is the Jacobian conjecture

# Dimension

# dimension

- important notion for graphs, commutative rings, vector spaces, manifolds, metric spaces, topological spaces
- many ways to define  $\dim(V)$  of affine variety  $V \subseteq \mathbb{A}^n$ 
  - 1 largest  $d$  so that there exists

$$V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0$$

where  $V_i$  irreducible subvarieties of  $V$  for all  $i = 1, \dots, d$

- 2 largest  $d$  so that there exists

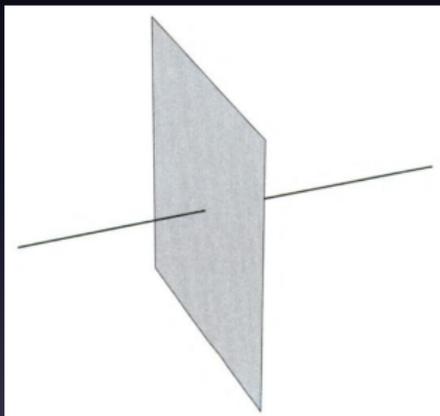
$$\mathfrak{p}_d \supsetneq \mathfrak{p}_{d-1} \supsetneq \cdots \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_0$$

where  $\mathfrak{p}_i$  prime ideals of  $\mathbb{C}[V]$  for all  $i = 1, \dots, d$

- second way: **Krull dimension** of a commutative ring
- several other ways:
  - transcendental degree of  $\mathbb{C}[V]$
  - maximal dimensions of tangent space at smooth points
  - number of general hyperplanes needed to intersect  $V$

# examples

- $\dim(\mathbb{A}^1) = 1$  since  $\{\text{line}\} \supsetneq \{\text{point}\}$
- $\dim(\mathbb{A}^n) = n$
- $\dim(\mathbb{V}(xy, xz)) = 2$



irreducible components  $\mathbb{V}(y, z), \mathbb{V}(x)$  different dimensions

- **dimension near a point**  $\dim_p(V)$  is largest  $d$  so that

$$V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0 = \{p\}$$

# subvariety and dimension

- dimension of irreducible variety is same at all points
- every variety contains dense Zariski-open subset of smooth points
- dimension of variety same as dimension of complex manifold of smooth points
- if  $V \subseteq W$ , then  $\dim(V) \leq \dim(W)$
- if  $V \subseteq W$  where  $W$  irreducible, then

$$\dim(V) = \dim(W) \quad \Rightarrow \quad V = W$$

# more examples

Grassmann variety: dimension same as Grassmann manifold

$$\dim(\text{Gr}(n, k)) = k(n - k)$$

commuting matrix varieties: if  $\mathcal{C}(k, n)$  irreducible, then

$$\dim(\mathcal{C}(k, n)) = n^2 + (k - 1)n$$

nilpotent matrices:  $\mathcal{N}(n) := \{X \in \mathbb{A}^{n \times n} : A^k = 0 \text{ for some } k \in \mathbb{N}\}$  is irreducible and

$$\dim(\mathcal{N}(n)) = n^2 - n$$

# morphism and dimension

- $V$  and  $W$  vector spaces
  - $F : V \rightarrow W$  surjective linear map, then  $\dim(V) \geq \dim(W)$
  - $F : V \rightarrow W$  surjective linear map, then for all  $w \in W$

$$\dim(F^{-1}(w)) = \dim(V) - \dim(W)$$

rank-nullity theorem:  $\text{nullity}(F) = \dim(V) - \text{rank}(F)$

- $V \subseteq \mathbb{A}^n$  and  $W \subseteq \mathbb{A}^m$  affine varieties
  - $F : V \rightarrow W$  surjective morphism, then  $\dim(V) \geq \dim(W)$
  - $F : V \rightarrow W$  surjective morphism, then for all  $w \in W$ ,

$$\dim(F^{-1}(w)) \geq \dim(V) - \dim(W)$$

and for generic  $w \in W$ ,

$$\dim(F^{-1}(w)) = \dim(V) - \dim(W)$$

'rank-nullity theorem for morphisms'

- why: *linear transformations on vector spaces*  
= *linear morphisms on linear affine varieties*

# Acknowledgment

# SOURCES

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## web:

- Mathematics Stack Exchange
- MathOverflow
- Terry Tao’s Blog
- Wikipedia

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