Varieties of commuting matrices

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The problem

$\mathbb{F}$ algebraically closed field.

Notation

$$C(d,n) = \{(A_1, A_2, \ldots, A_d) \in M_n(\mathbb{F})^d; A_iA_j = A_jA_i \text{ for each } i \text{ and } j\}$$

$$N(d,n) = \{(A_1, A_2, \ldots, A_d) \in C(d,n); A_1, \ldots, A_d \text{ are nilpotent}\}$$

Both sets are affine varieties in $\mathbb{F}^{dn^2}$.

The problem

For which positive integers d and n are the varieties $C(d,n)$ and $N(d,n)$ irreducible?

Applications

- biology
- physics
- algebraic geometry: Hilbert schemes
- commutative algebra: theory of modules over artinian commutative rings
- linear algebra:
  - stability of invariant subspaces of commuting matrices
  - dimension of commutative matrix subalgebras

Generic and r-regular matrices

Definitions

A matrix $A \in M_n(\mathbb{F})$ is:

- *r-regular*, if each its eigenspace is at most r-dimensional;
- *generic*, if it has n distinct eigenvalues.

Notations

- $R^r(d,n) = \{(A_1, A_2, \ldots, A_d) \in C(d,n)$; some linear combination of $A_1, \ldots, A_d$ is r-regular$\}$
- $R^r_N(d,n) = R^r(d,n) \cap N(d,n)$
- $G(d,n) = \{(A_1, A_2, \ldots, A_d) \in C(d,n)$; some linear combination of $A_1, \ldots, A_d$ is generic$\}$.

Proposition

- *The variety $C(d,n)$ is irreducible if and only if $C(d,n) = \overline{R^1(d,n)} = \overline{G(d,n)}$.*
- *The variety $N(d,n)$ is irreducible if and only if $N(d,n) = \overline{R_N^1(d,n)}$.*

Proof

- The set $G(d,n)$ and all sets $R^r(d,n)$ are open in $C(d,n)$ in the Zariski topology, and $R_N^r(d,n)$ are open in $N(d,n)$.
- A matrix that commutes with 1-regular matrix is a polynomial in that matrix.
- $\overline{R^1(d,n)}$ and $\overline{G(d,n)}$ are irreducible and of dimension $n^2 + (d-1)n$.
- $\overline{R_N^1(d,n)}$ is irreducible and of dimension $n^2 - n + (d-1)(n-1)$.

Corollary

If $N(d,m)$ is irreducible for each $m \le n$, then $C(d,n)$ is irreducible.

Dimension of d-generated commutative matrix algebra

Theorem (Gerstenhaber, 1961)

If the variety $C(d,n)$ is irreducible, then each unital algebra generated by d-tuple of commuting $n \times n$ matrices is at most n-dimensional.

Idea of the proof

- $\dim \mathbb{F}[A_1, A_2, \dots, A_d] \leq n$ is closed condition in the Zariski topology.
- The variety

$$\mathcal{V} = \{(A_1, A_2, \dots, A_d) \in C(d,n); \dim \mathbb{F}[A_1, A_2, \dots, A_d] \leq n\}$$

  contains $G(d,n)$.
- Irreducibility of $C(d,n)$ implies $C(d,n) = \overline{G(d,n)}$, hence $\mathcal{V} = C(d,n)$.

More than 3 matrices

Proposition (Guralnick, 1992)

For $n \geq 4$ the commutative algebra of all $n \times n$ matrices of the form

$$\begin{bmatrix} a_1 & & & & & & & \\ & a_2 & & & & & & \\ & & \ddots & & & & & \\ & & & a_{n-4} & & & & \\ & & & & b & & c & d \\ & & & & & b & e & f \\ & & & & & & b & \\ & & & & & & & b \end{bmatrix}$$

is $(n+1)$-dimensional and generated by 4 matrices as a unital algebra.

Corollary

If $d \geq 4$ and $n \geq 4$, then $C(d,n)$ is reducible.

Proposition (Schur, 1905)

The commutative algebra of all $n \times n$ matrices of the form $\begin{bmatrix} \lambda I & A \\ 0 & \lambda I \end{bmatrix}$ where $\lambda \in \mathbb{F}$, $A \in M_{m\times(n-m)}(\mathbb{F})$, $m = \lfloor \frac{n}{2} \rfloor$ is $(m(n-m)+1)$-dimensional.

Corollary (Gerstenhaber, 1961)

If $d \geq n \geq 4$, then $N(d,n)$ is reducible.

Proposition (Kirillov, Neretin, 1984; Guralnick, 1992)

If $n \leq 3$, then $C(d,n)$ and $N(d,n)$ are irreducible for each d.

Pairs

Theorem (Motzkin, Taussky, 1955)

For each positive integer n the variety $C(2,n)$ is irreducible.

Proof

- $(A,B) \in C(2,n)$ arbitrary pair.
- B commutes with some 1-regular matrix C.
- The line $\mathcal{L} = \{(\lambda A + (1-\lambda)C, B); \lambda \in \mathbb{F}\}$ intersects $R^1(2,n)$.
- Line is irreducible, therefore $\mathcal{L} \subseteq \overline{R^1(2,n)}$. In particular, $(A,B) \in \overline{R^1(2,n)}$.

Generalization (Richardson, 1979)

If $\mathfrak{g}$ is a reductive Lie algebra over $\mathbb{F}$ and $\operatorname{char}\mathbb{F} = 0$, then the commuting variety $C(\mathfrak{g}) = \{(x,y) \in \mathfrak{g} \times \mathfrak{g}; [x,y] = 0\}$ is irreducible.

Nilpotent pairs

Theorem (Baranovsky, 2001; Basili, 2003)

If $\operatorname{char}\mathbb{F} = 0$ *or* $\operatorname{char}\mathbb{F} \geq \frac{n}{2}$, *then the variety* $N(2, n)$ *is irreducible.*

Generalization (Premet, 2003)

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{F}$ and assume that $\operatorname{char}\mathbb{F}$ is good for the root system of $\mathfrak{g}$. Then the nilpotent commuting variety

$$C^{nil}(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g}; [x, y] = 0, x, y \,\mathrm{nilpotent}\}$$

is equidimensional and its irreducible components are parametrized by the distinguished nilpotent orbits in $\mathfrak{g}$.

In particular, $N(2, n)$ is irreducible for each n in any characteristic.

Triples

Theorem (Holbrook, Omladič, 2001)

The variety $C(3,n)$ is reducible for $n \geq 29$.

Idea of the proof

The set of all triples $(A,B,C) \in C(3,29)$ with A similar to the matrix

$$\lambda I_{29} + \begin{bmatrix} 0 & I_8 & 0 & 0 \\ 0 & 0 & I_8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for some $\lambda \in \mathbb{F}$ is contained in a proper subvariety of $C(3,29)$ of dimension at least $29^2 + 2 \cdot 29$.

Theorem (Guralnick, Sethuraman, Holbrook, Omladič, Han, K.Š.)

If $\operatorname{char} \mathbb{F} = 0$, then $C(3,n)$ is irreducible for $n \leq 8$.

Nilpotent triples

Theorem (Clark, O'Meara, Vinsonhaler, 2011)

The variety $N(3, n)$ is reducible for $n \geq 13$.

Theorem (Young, 2010)

The variety $N(3, 4)$ is irreducible.

Reductions

Lemma

If $\varphi : C(3,n) \to C(3,n)$ is a polynomial map that maps $R^1(3,n)$ to itself, then $(A,B,C) \in \overline{R^1(3,n)}$ implies $\varphi(A,B,C) \in \overline{R^1(3,n)}$.

Corollaries

- The property that a triple from $C(3,n)$ belongs to $\overline{R^1(3,n)}$ depends only on the algebra generated by the triple.
- While proving that a triple belongs to $\overline{R^1(3,n)}$ we can simultaneously conjugate the matrices in the triple by any invertible matrix.
- We can assume that one of the matrices is in the Jordan canonical form.

The same reductions can be made in the nilpotent case.

Lemma

If $C(3, m)$ is irreducible for each $m < n$ and if all matrices in a triple from $C(3, n)$ commute with a matrix having two distinct eigenvalues, then the triple belongs to $\overline{G(3, n)}$.

Corollary

Only triples of commuting matrices that span a vector space of nilpotent matrices have to be considered.

Simultaneous commutative approximation

- (A, B, C) any triple of commuting (nilpotent) matrices.
- $X, Y, Z \in M_n(\mathbb{F})$ such matrices that $A + \lambda X$, $B + \lambda Y$ and $C + \lambda Z$ commute for each $\lambda \in \mathbb{F}$ (and they are nilpotent if we are proving irreducibility of $N(3, n)$).
- Suppose that for each λ from some open subset of $\mathbb{F}$ the triple $(A + \lambda X, B + \lambda Y, C + \lambda Z)$ belongs to $\overline{R^1(3, n)}$ (respectively $\overline{R^1_N(3, n)}$).

  Typically this happens if some linear combination of $A + \lambda X$, $B + \lambda Y$ and $C + \lambda Z$ either has higher rank than A, B and C or (in the case of $C(3, n)$) it has two distinct eigenvalues.
- Then $(A, B, C) \in \overline{R^1(3, n)}$ (respectively $\overline{R^1_N(3, n)}$).

Variety of commuting pairs in the centralizer of a matrix

Notations

For $A \in M_n(\mathbb{F})$ let:

- $C(A) = \{B \in M_n(\mathbb{F}); AB = BA\}$: the centralizer of A.
- $N(A) = \{B \in C(A); B\,\mathrm{nilpotent}\}$: the nilpotent centralizer.
- $C_2(A) = \{(B, C) \in C(A) \times C(A); BC = CB\}$.
- $N_2(A) = C_2(A) \cap (N(A) \times N(A))$.

Proposition

If $C_2(A)$ is irreducible for each $n \times n$ matrix A, then $C(3, n)$ is irreducible.

Theorem (Neubauer, Sethuraman, 1999)

If A is 2-regular matrix, then $C_2(A)$ is irreducible.

Pairs of 1-regular matrices in $C_2(A)$

Notation

$R_2(A) = \{(B, C) \in C_2(A); B \text{ or } C \text{ 1-regular } \}$.

Proposition

For each $A \in M_n(\mathbb{F})$ the closure $\overline{R_2(A)}$ is irreducible variety of dimension $\dim C(A) + n$.

Corollary

$C_2(A)$ is irreducible if and only if $\overline{R_2(A)} = C_2(A)$.

Irreducibility of $C_2(A)$ in 3-regular case

Using simultaneous commutative approximation by pairs of 1-regular commuting matrices in the centralizer of A we obtain:

Theorem

If A is 3-regular matrix, then $C_2(A)$ is irreducible.

Reducibility of $C_2(A)$ in 5-regular case

Proposition

For $A = \begin{bmatrix} 0 & I_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ *the variety* $C_2(A)$ *is reducible.*

Proof of the proposition

The variety of all pairs $(\lambda I_{14} + B, \mu I_{14} + C) \in C_2(A)$ with B and C of the form

$$B = \begin{bmatrix} 0 & B_1 & B_2 & B_3 & B_4 & B_5 \\ 0 & 0 & B_1 & B_2 & 0 & B_4 \\ 0 & 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_6 & B_7 & 0 & B_8 \\ 0 & 0 & 0 & B_6 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & C_1 & C_2 & C_3 & C_4 & C_5 \\ 0 & 0 & C_1 & C_2 & 0 & C_4 \\ 0 & 0 & 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_6 & C_7 & 0 & C_8 \\ 0 & 0 & 0 & C_6 & 0 & 0 \end{bmatrix}$$

has dimension at least $72 = 14 + \dim C(A)$.

4-regular cases

Theorem

Let $\operatorname{char}\mathbb{F} = 0$ and let A be a 4-regular matrix whose Jordan canonical form has at most two nonzero Jordan blocks for each eigenvalue. Then $C_2(A)$ is irreducible.

Corollary

Let $\operatorname{char}\mathbb{F} = 0$. If $A \in M_n(\mathbb{F})$ is either 3-regular or 4-regular and its Jordan canonical form has at most two nonzero Jordan blocks for each eigenvalue, then for each pair $(B, C) \in C_2(A)$ the triple (A, B, C) belongs to $\overline{G(3, n)}$.

For general 4-regular matrices A the problem of (ir)reducibility of $C_2(A)$ is still open.

The set $D_2(A)$

Notations

For a nilpotent $n \times n$ matrix A let:

- $D(A) = \{B \in N(A); \dim \mathbb{F}[A, B] = n\}$.
- $D_2(A) = \{(B, C) \in N_2(A); \dim \mathbb{F}[A, B] = n\}$.

Lemma

The sets $D(A)$ and $D_2(A)$ are Zariski open in $N(A)$ and $N_2(A)$, respectively.

Lemma (Basili, 2003)

The variety $N(A)$ is irreducible and of dimension $\dim C(A) - \dim \ker A$.

Proposition

$\overline{D_2(A)}$ is irreducible and of dimension $\dim C(A) - \dim\ker A + n - 1$.

Proof

- Let A, B be nilpotent $n \times n$ matrices with $\dim \mathbb{F}[A, B] = n$.
- If C commutes with A and B, then $C \in \mathbb{F}[A, B]$ by the theorem of Neubauer and Saltman.
- If the Jordan canonical form of A has Jordan blocks of orders $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$, then

$$\{A^i B^j; 0 \leq j \leq k-1, 0 \leq i \leq n_{j+1} - 1\}$$

  is a basis of $\mathbb{F}[A, B]$ by the theorem of Barria and Halmos.

- The polynomial map $\varphi\colon D(A) \times \mathbb{F}^{n_1-1} \times \mathbb{F}^{n_2} \times \cdots \times \mathbb{F}^{n_k} \to D_2(A)$ defined by

$$\varphi(B,(c_{21},\ldots,c_{n_1 1}),(c_{12},\ldots,c_{n_2 2})\ldots,(c_{1k},\ldots,c_{n_k k})) =$$

$$= (B, \sum_{i=2}^{n_1} c_{i1}A^{i-1} + \sum_{j=2}^{k}\sum_{i=1}^{n_j} c_{ij}A^{i-1}B^{j-1})$$

is bijective and a birational equivalence.

Corollary

$N_2(A)$ is irreducible if and only if $\overline{D_2(A)} = N_2(A)$.

Irreducibility of $N_2(A)$ in 2-regular case

Theorem

If A is 2-regular nilpotent matrix, then $N_2(A)$ is irreducible.

Proof in the case when the Jordan blocks of A are of the same size

- $A = \begin{bmatrix} 0 & I_2 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & 0 & I_2 \\ & & & 0 \end{bmatrix}$.
- $B = \begin{bmatrix} B_1 & B_2 & \cdots & B_k \\ & \ddots & \ddots & \vdots \\ & & B_1 & B_2 \\ & & & B_1 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & C_2 & \cdots & C_k \\ & \ddots & \ddots & \vdots \\ & & C_1 & C_2 \\ & & & C_1 \end{bmatrix}$.

- If $B_1 \neq 0$, then $\dim(\ker A \cap \ker B) = 1$ or $\dim(\ker A^T \cap \ker B^T) = 1$, and $\dim \mathbb{F}[A, B] = n$ by the theorem of Košir.
- Let $B_1 = C_1 = 0$ and assume that B_2 is not scalar. Then $B_2 C_2 = C_2 B_2$, therefore $C_2 = \alpha I_2 + \beta B_2$ for some $\alpha, \beta \in \mathbb{F}$. We can add polynomials in A and B to C, so we can assume that $C_2 = 0$. Similarly $C_3 = \cdots = C_{k-1} = 0$.
- The set of all $n \times n$ matrices that are not generic is $(n^2 - 1)$-dimensional variety defined by $\det p'_X(X) = 0$, where p'_X is the derivative of the characteristic polynomial of X.
- There exist $\mu, \nu \in \mathbb{F}$ such that $\mu B_2 - \nu C_k$ is not generic.

- If $\nu = 0$, then we can assume that B_2 is nilpotent. Then $X = \begin{bmatrix} B_2 & B_3 & \cdots & B_k & 0 \\ & B_2 & B_3 & \cdots & B_k \\ & & \ddots & \ddots & \vdots \\ & & & B_2 & B_3 \\ & & & & B_2 \end{bmatrix}$ is nilpotent, it commutes with B and $(B, C + \lambda X) \in D_2(A)$ for $\lambda \neq 0$.
- If $\nu \neq 0$, then we can subtract $\frac{\mu}{\nu} A^{k-2} B$ from C, so we can assume that C_k is not generic and we can proceed similarly as above.

- Suppose that B_2 is a scalar. There exists C_l that is not a scalar.

Then $X = \begin{bmatrix} 0 & C_l & \cdot & C_k & 0 & \cdots & 0 \\ & \ddots & \ddots & & \ddots & \ddots & \vdots \\ & & 0 & C_l & \cdots & C_k & 0 \\ & & & 0 & C_l & \cdots & C_k \\ & & & & \ddots & \ddots & \vdots \\ & & & & & 0 & C_l \\ & & & & & & 0 \end{bmatrix}$ is nilpotent, it commutes with C and $(B + \lambda X, C) \in \overline{D_2(A)}$ for $\lambda \neq 0$.

Corollary

If $(A, B, C) \in N(3, n)$ and A is 2-regular, then $(A, B, C) \in \overline{R_N^1(3, n)}$.

Reducibility of $N_2(A)$ in 3-regular case

Proposition

For $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ *the variety* $N_2(A)$ *is reducible.*

Only one nonzero Jordan block

Proposition

Let $k < n$ and assume that each triple of commuting nilpotent $n \times n$ matrices whose linear span contains a matrix of rank at least k belongs to $\overline{R_N^1(3,n)}$. If $(A, B, C) \in N(3,n)$ and the Jordan canonical form of A has one Jordan block of order k and $n-k$ zero Jordan blocks, then $(A, B, C) \in \overline{R_N^1(3,n)}$.

Remark

The same result for triples from $C(3,n)$ was proved by Holbrook and Omladič.

Remark

Additional assumptions have no influence on the irreducibility of $N(3,n)$.

More zero than nonzero Jordan blocks

Proposition

Let $k < n$ and assume that $C(3, m)$ is irreducible for each $m < n$ and that each triple of commuting $n \times n$ matrices whose linear span contains a matrix of rank at least k belongs to $\overline{G(3, n)}$. If $(A, B, C) \in C(3, n)$, $\operatorname{rank} A = k - 1$ and the Jordan canonical form of A has more zero than nonzero Jordan blocks, then $(A, B, C) \in \overline{G(3, n)}$.

Algebra with radical of square zero

Theorem (Holbrook, Omladič, 2001)

If commuting $n \times n$ matrices A, B and C generate an algebra with radical of square zero, then $(A, B, C) \in \overline{G(3, n)}$.

Generalizations

- If commuting $n \times n$ matrices A, B and C generate an algebra with radical of square zero, then $(B, C) \in \overline{R_2(A)}$.
- Let $l > 0$, $m \geq 0$ and $n = 2l + m$, and assume that each triple of commuting nilpotent $n \times n$ matrices whose linear span contains either a matrix of rank at least $l + 1$ or a matrix with nonzero square belongs to $\overline{R_N^1(3, n)}$. If $(A, B, C) \in N(3, n)$ and the Jordan canonical form of A has l Jordan blocks of order 2 and m zero Jordan blocks, then $(A, B, C) \in \overline{R_N^1(3, n)}$.

Only one Jordan block of order exceeding 2

If A, B and C generate vector space of nilpotent matrices, then the square of the algebra $\mathbb{F}[A, B, C]$ is zero if and only if the Jordan canonical form of each linear combination of A, B and C has Jordan blocks of orders at most 2 only.

Theorem

Let $k \geq 3$, $l \geq 0$ and $n \geq k + 2l$, and assume that $C(3, m)$ is irreducible for each $m < n$ and that each triple of commuting $n \times n$ matrices whose linear span contains either a matrix of rank at least $k + l$ or a matrix with square of rank at least $k - 1$ belongs to $\overline{G(3, n)}$. If $(A, B, C) \in C(3, n)$ and the Jordan canonical form of A has one Jordan block of order k, l Jordan blocks of order 2 and $n - 2l - k$ zero Jordan blocks, then $(A, B, C) \in \overline{G(3, n)}$.

Sketch of the proof for $k \geq 4$

- $$A = \begin{bmatrix} J_k & 0 & 0 & 0 \\ 0 & 0 & I_l & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & \mathbf{e_1 a}^T & \mathbf{e_2 a}^T + \mathbf{e_1 b}^T & \mathbf{e_1 c}^T \\ \mathbf{d e_{k-1}}^T + \mathbf{f e_k}^T & D & E & F \\ \mathbf{d e_k}^T & 0 & D & 0 \\ \mathbf{g e_k}^T & 0 & G & H \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & \mathbf{e_1 a'}^T & \mathbf{e_2 a'}^T + \mathbf{e_1 b'}^T & \mathbf{e_1 c'}^T \\ \mathbf{d' e_{k-1}}^T + \mathbf{f' e_k}^T & D' & E' & F' \\ \mathbf{d' e_k}^T & 0 & D' & 0 \\ \mathbf{g' e_k}^T & 0 & G' & H' \end{bmatrix}.$$

- We can assume $\mathrm{rank}\,(\alpha A + \beta B + \gamma C) \le k + l - 1$ and $\mathrm{rank}\,(\alpha A + \beta B + \gamma C)^2 \le k - 2$ for all $\alpha, \beta, \gamma \in \mathbb{F}$, therefore $D = D' = 0$, $H = H' = 0$, and $(\beta F + \gamma F')(\beta G + \gamma G') = 0$ and $(\beta G + \gamma G')(\beta F + \gamma F') = 0$ for all $\beta, \gamma \in \mathbb{F}$.
- The only nontrivial case to consider is $G = G' = 0$.
- It turns out that we can assume that $\mathbf{a}$ and $\mathbf{a}'$ are linearly independent, therefore by conjugation we can get $\mathbf{b} = \mathbf{b}' = 0$ and $\mathbf{c} = \mathbf{c}' = 0$. Similarly, $\mathbf{f} = \mathbf{f}' = 0$.

- It suffices to prove the existence of the matrices $X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Z & 0 & 0 \\ 0 & 0 & Z & 0 \\ 0 & 0 & U & T \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Z' & 0 & 0 \\ 0 & 0 & Z' & 0 \\ 0 & 0 & U' & T' \end{bmatrix}$, not both of them zero, such that $XY = YX$ and $BY + XC = CX + YB$.
- The equations $XY = YX$ and $BY + XC = CX + YB$ are equivalent to

$$\mathbf{a}^T Z' = \mathbf{a}'^T Z, \quad Z\mathbf{d}' = Z'\mathbf{d}, \tag{1}$$

$$ZE' + EZ' + FU' = Z'E + E'Z + F'U, \tag{2}$$

$$ZF' + FT' = Z'F + F'T, \tag{3}$$

$$U\mathbf{d}' + T\mathbf{g}' = U'\mathbf{d} + T'\mathbf{g}, \tag{4}$$

$$ZZ' = Z'Z, \tag{5}$$

$$UZ' + TU' = U'Z + T'U, \quad TT' = T'T. \tag{6}$$

- If A has $p = n - k - 2l > 0$ zero Jordan blocks, then the dimension of

$$\{(Z, Z', U, U', T, T') \in M_l(\mathbb{F})^2 \times M_{p\times l}(\mathbb{F})^2 \times M_p(\mathbb{F})^2;$$

$$\mathbf{a}^T Z' = \mathbf{a}'^T Z, Z\mathbf{d}' = Z'\mathbf{d}, U\mathbf{d}' + T\mathbf{g}' = U'\mathbf{d} + T'\mathbf{g}\}$$

is at least $2l^2 + 2pl + 2p^2 - 2l - p$, and the dimension of

$$\{(Z, Z', U, U', T, T') \in M_l(\mathbb{F})^2 \times M_{p\times l}(\mathbb{F})^2 \times M_p(\mathbb{F})^2;$$

$$ZZ' = Z'Z, UZ' + TU' = U'Z + T'U, TT' = T'T,$$

$$ZE' + EZ' + FU' = Z'E + E'Z + F'U, ZF' + FT' = Z'F + F'T\}$$

is at least $p^2 + p + 2l$, therefore their intersection has dimension at least p^2 and it contains some nonzero point.

- If A has no zero Jordan blocks, then using the theorem on fibres of polynomial map we show that the projection from

$$\{(Z, Z', W, W', \mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}') \in M_l(\mathbb{F})^4 \times (\mathbb{F}^l)^4; Z \text{generic and invertible},$$
$$ZZ' = Z'Z, ZW' + WZ' = Z'W + W'Z,$$
$$\mathbf{x}^T Z' = \mathbf{x}'^T Z, Z\mathbf{y}' = Z'\mathbf{y}, \mathbf{x}^T \mathbf{y}' = \mathbf{x}'^T \mathbf{y}\}$$

to

$$\{(W, W', \mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}') \in M_l(\mathbb{F})^2 \times (\mathbb{F}^l)^4; \mathbf{x}^T \mathbf{y}' = \mathbf{x}'^T \mathbf{y}\},$$

is dominant. The first set is of dimension $2l^2 + 4l$ and the second one of the dimension $2l^2 + 4l - 1$, therefore the set of all pairs $(Z, Z') \in (M_l(\mathbb{F}))^2$ satisfying $ZZ' = Z'Z$, $\mathbf{a}^T Z' = \mathbf{a}'^T Z$, $Z\mathbf{d}' = Z'\mathbf{d}$ and $ZE' + EZ' = Z'E + E'Z$ has dimension at least 1 by the theorem on fibres.

Irreducibility of $C(3, n)$ for $n \leq 10$

Let $\mathrm{char}\, \mathbb{F} = 0$. The following cases of sizes of Jordan blocks remain to be considered:

- $n = 9$: $3, 3, 2, 1$.
- $n = 10$: $4, 3, 2, 1$, $3, 3, 3, 1$, $3, 3, 2, 2$, $3, 3, 2, 1, 1$.

Using simultaneous commutative approximation we prove that such triples belong to $\overline{G(3, n)}$.

Theorem

For $n \leq 10$ the variety $C(3, n)$ is irreducible.

Corollary

Each unital algebra generated by three commuting $n \times n$ matrices is at most n-dimensional for $n \leq 10$.

Irreducibility of $N(3,n)$ for $n \leq 6$

Proposition

If $(A,B,C) \in N(3,6)$ and the Jordan canonical form of A has Jordan blocks of orders 3, 2 and 1, then $(A,B,C) \in \overline{R_N^1(3,n)}$.

Theorem

For $n \leq 6$ the variety $N(3,n)$ is irreducible.

Reducibility of $N(d,n)$ for $d,n \geq 4$ in almost all cases

We already know that $N(d,n)$ is reducible for $d \geq n \geq 4$.

Theorem

If $d \geq 5$ and $n \geq 4$ or if $d = 4$ and $n \geq 6$, then $N(d,n)$ is reducible.

Proof

- $m = \lfloor \frac{n}{2} \rfloor$.
- Let $\mathcal{V}$ be the variety of all d-tuples of the form
$$\left(\begin{bmatrix} 0 & I_m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_2 & C_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & B_d & C_d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$
where $B_2, \ldots, B_d \in M_m(\mathbb{F})$ and $C_2, \ldots, C_d \in M_{m\times(n-2m)}(\mathbb{F})$.
- $\mathcal{V}$ is a subvariety of $N(d,n)$, it is irreducible and of dimension $(d-1)m(n-m)$.

- Define a polynomial map $\varphi\colon SL_n(\mathbb{F}) \times \mathcal{V} \to N(d,n)$ by $\varphi(P,(A_1,A_2,\dots,A_d)) = (P^{-1}A_1P, P^{-1}A_2P, \dots, P^{-1}A_dP)$.
- All fibres are birationally equivalent to $C(A_1) \cap SL_n(\mathbb{F})$, where $A_1 = \begin{bmatrix} 0 & I_m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, therefore they are of dimension $m^2 + (n-m)^2 - 1$.
- Let $\mathcal{W}$ be the closure of the image of the map φ. Then $\dim \mathcal{W} = (d+1)m(n-m)$ by the theorem on fibres.
- If $d \geq 5$ and $n \geq 4$ or if $d = 4$ and $n \geq 6$, then $\dim \mathcal{W} = (d+1)m(n-m) \geq (d+n-1)(n-1) = \dim \overline{R_N^1(d,n)}$.
- $\mathcal{W}$ is a proper subvariety of $N(d,n)$, so $N(d,n)$ is reducible.

Irreducibility of $N(4,5)$

Theorem

$N(4,5)$ is irreducible.

Proof

- For any $(A_1, A_2, A_3, A_4) \in N(4,5)$ we have to show $(A_1, A_2, A_3, A_4) \in \overline{R^1_N(4,5)}$.
- The only nontrivial case is when A_1, A_2, A_3, A_4 are of rank 2 and square zero.
- $A_1 = \begin{bmatrix} 0 & I_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A_i = \begin{bmatrix} 0 & B_i & \mathbf{a_i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for $i = 2, 3, 4$ where $B_i \in M_2(\mathbb{F})$, $\mathbf{a_i} \in \mathbb{F}^2$.

- Let $\mathcal{U}$ be the set of all quadruples $(X_1, X_2, X_3, X_4) \in N(4,5)$ such that X_1 is 1-regular and for each $i \leq 4$, $X_i = \begin{bmatrix} Y_i & Z_i \\ 0 & W_i \end{bmatrix}$ for some strictly upper triangular $Y_i \in M_2(\mathbb{F})$, some $Z_i \in M_{2\times 3}(\mathbb{F})$ and some $W_i \in M_3(\mathbb{F})$.
- If $(X_1, X_2, X_3, X_4) \in \mathcal{U}$, then X_2, X_3, X_4 are polynomials in X_1, therefore $\overline{\mathcal{U}}$ is birationally equivalent to $\mathbb{F} \times N_3 \times M_{2\times 3}(\mathbb{F}) \times (\mathbb{F}_3[t])^3$, where N_n denotes the variety of all nilpotent $n \times n$ matrices.
- It is known that N_n is irreducible and of dimension $n^2 - n$ for each n, therefore $\overline{\mathcal{U}}$ is irreducible and of dimension 25.

- Let $\pi\colon \overline{\mathcal{U}} \to M_{2\times 3}(\mathbb{F})^4$ be the projection defined by

$$\pi\left(\begin{bmatrix} Y_1 & Z_1 \\ 0 & W_1 \end{bmatrix}, \begin{bmatrix} Y_2 & Z_2 \\ 0 & W_2 \end{bmatrix}, \begin{bmatrix} Y_3 & Z_3 \\ 0 & W_3 \end{bmatrix}, \begin{bmatrix} Y_4 & Z_4 \\ 0 & W_4 \end{bmatrix}\right) =$$

$$= (Z_1, Z_2, Z_3, Z_4).$$

- Assume that π is not dominant. Then $\overline{\pi(\overline{\mathcal{U}})}$ is a proper subvariety of $M_{2\times 3}(\mathbb{F})^4$, therefore $\dim \overline{\pi(\overline{\mathcal{U}})} \leq 23$.
- By the theorem on fibres all fibres of π have dimension at least $\dim \overline{\mathcal{U}} - \dim \overline{\pi(\overline{\mathcal{U}})} \geq 2$.
- If

$$Z_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, Z_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$Z_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Z_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\dim \pi^{-1}(Z_1, Z_2, Z_3, Z_4) = 1$, a contradiction.

- Therefore $\pi(\mathcal{U})$ is dense in $M_{2\times 3}(\mathbb{F})^4$.
- If $(Z_1, Z_2, Z_3, Z_4) \in \pi(\mathcal{U})$ is arbitrary quadruple, then there exist strictly upper triangular matrices $Y_1, Y_2, Y_3, Y_4 \in M_2(\mathbb{F})$ and nilpotent matrices $W_1, W_2, W_3, W_4 \in M_3(\mathbb{F})$ such that

$$\left(\begin{bmatrix} Y_1 & Z_1 \\ 0 & W_1 \end{bmatrix}, \begin{bmatrix} Y_2 & Z_2 \\ 0 & W_2 \end{bmatrix}, \begin{bmatrix} Y_3 & Z_3 \\ 0 & W_3 \end{bmatrix}, \begin{bmatrix} Y_4 & Z_4 \\ 0 & W_4 \end{bmatrix} \right)$$

belongs to $R_N^1(4,5)$.
- Identify $M_{2\times 3}(\mathbb{F})$ with the upper right corners of 5×5 matrices of the form $\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$. Then $\pi(\mathcal{U}) \subseteq \overline{R_N^1(4,5)}$, therefore $M_{2\times 3}(\mathbb{F})^4 = \overline{\pi(\mathcal{U})} \subseteq \overline{R_N^1(4,5)}$.
- In particular, $(A_1, A_2, A_3, A_4) \in \overline{R(4,5)}$.