Varieties of commuting matrices

Klemen Šivic
ETH Zürich

Let $C(d,n)$ be the set of all d-tuples of commuting $n \times n$ matrices over an algebraically closed field F of characteristic zero, and let $N(d,n)$ be the subset of $C(d,n)$ consisting of all d-tuples of nilpotent commuting matrices. Both sets can be viewed as affine varieties in F^{dn^2}. It is well-known that both varieties are irreducible for $d = 2$ and each positive integer n, and for $n \leq 3$ and each d, but $C(d,n)$ is reducible for $d, n \geq 4$ and $N(d,n)$ is reducible for $d \geq n \geq 4$. In the case of triples the variety $C(3,n)$ is known to be irreducible for $n \leq 8$ and reducible for $n \geq 29$, and $N(3,n)$ is known to be irreducible for $n \leq 4$ and reducible for $n \geq 13$. Using simultaneous commutative approximation of triples of matrices by triples of 1-regular matrices (i.e. matrices having only one eigenspace for each eigenvalue) we prove that $C(3,n)$ is irreducible for $n \leq 10$ and $N(3,n)$ is irreducible for $n \leq 6$. We also show that the varieties $N(d,n)$ are reducible for $n \geq 4$ and $d \geq 4$, with one exception, $N(4,5)$. We study also a related problem of irreducibility of varieties of pairs of commuting matrices in the centralizers of given matrices. We show that such variety is irreducible if the given matrix is 3-regular, but it can be reducible for 5-regular case. In the nilpotent case the analogous variety is irreducible in 2-regular case, but it can be reducible in 3-regular case.