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Geometry and Spectral Variation

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Suggested Reading

as preparation for John Holbrook's lectures on Geometry and Spectral Variation:

R. Bhatia, "Spectral variation, normal matrices, and Finsler geometry", Mathematical Intelligencer 29 (2007), 41–46

and, for more details:

R. Bhatia, "Matrix Analysis", Springer 1997, chapter VI.1–5 and chapter VII.1–4

Lecture Notes

1. The problem

We consider $n \times n$ complex matrices $A, B, \ldots$, notation: $A, B, \cdots \in \mathbb{M}_n(\mathbb{C})$, or simply $\mathbb{M}_n$. When we change a matrix its eigenvalues also change. Can we quantify this statement by comparing the distance between A and B with the distance between the spectra $\sigma(A)$ and $\sigma(B)$? What about changes in eigenvectors or invariant subspaces? This problem is typical of the questions treated by "matrix analysis" and often goes under the name of the "spectral variation" problem. Evidently we cannot even ask these questions precisely until we decide how to measure the "distances" between matrices and between spectra. There are, in fact, many possibilities. Let's consider some of the most helpful.

2. Matrix norms

Among the many useful norms on $\mathbb{M}_n$ we focus on two: first, the operator norm $\|A\| = \max\{\|Au\| : u \in \mathbb{C}^n, \|u\| = 1\}$; here $\|u\|$ is the Euclidian length of u in $\mathbb{C}^n$. We may think of $\|A\|$ as the Lipschitz constant of the mapping induced on $\mathbb{C}^n$ by A, and it has many attractive structural properties (eg von Neumann's inequality). In most of what follows we will use $\|A - B\|$ as the "distance" between A and B.

Second, the Euclidean norm $\|A\|_2 = (\sum_{i,j} |a_{ij}|^2)^{\frac{1}{2}}$, ie the Euclidian length of A regarded as a vector in $\mathbb{C}^{n^2}$. This is more easily computed and is related to the operator norm by $\|A\| \leq \|A\|_2 \leq \sqrt{n}\|A\|$. The norm $\|A\|_2$ goes by many different names, depending on context: eg, Frobebius norm, Hilbert–Schmidt norm, Schatten–2 norm. In terms of the distance $\|A - B\|_2$, we'll see a particularly satisfying result on spectral variation (Hoffman–Wielandt Theorem).

It is clear that the operator norm is unitarily invariant, ie $\|UAW\| = \|A\|$ for all unitary U, W. A little thought reveals that $\|\cdot\|_2$ is also unitarily invariant.

3. Spectral distances

By the spectrum $\sigma(A)$ of a matrix $A \in \mathbb{M}_n$ we understand a listing (in any order) of the eigenvalues of A repeated according to their multiplicities as roots of the characteristic polynomial $p(\lambda) = \det(\lambda I_n - A)$. Thus the problem of spectral variation for general matrices in $\mathbb{M}_n$ is related to the problem of variation in the roots of a polynomial as the polynomial varies.

A classic result by Ostrowski addresses this problem in terms of the Hausdorff distance between the spectra: given two compact *sets* $S, T \subset \mathbb{C}$, the Hausdorff distance $h(S, T)$ between these sets is

$$h(S, T) = \max\{\max_{s \in S} d(s, T), \max_{t \in T} d(t, S)\},$$

where $d(s, T) = \min_{t \in T} |s - t|$. If p, q are two monic polynomials of degree n and $Z(p)$ is the set of roots (zeros) of p (ignoring multiplicity), then

Ostrowski's result has the form

$$h(Z(p), Z(q)) \leq \Theta(p, q),$$

where $\Theta(p, q)$ is a certain function of the coefficients of p and q. See section VIII.2 of [B1997] for more details.

In problems of spectral variation it is often appropriate to work with the "optimal matching" distance between $\sigma(A)$ and $\sigma(B)$, in which multiplicities make a big difference. In most of what follows we focus on the spectral distance $sd(A, B)$ defined by

$$sd(A, B) = \min_{\pi} \max_{k} |a_k - b_{\pi(k)}|,$$

where π runs over all permutations of the index set $\{1, 2, \ldots, n\}$, $\sigma(A) = \{a_1, \ldots, a_n\}$, and $\sigma(B) = \{b_1, \ldots, b_n\}$. This definition is especially well–suited to the case of normal matrices.

4. Normal matrices

While much is known about spectral variation for general matrices (see, eg, chapter VIII of [B1997]), we mainly treat the normal case in these lectures, in part because the geometry of matrix spaces plays a greater role there. Recall that a matrix $N \in \mathbb{M}_n$ is called normal if it can be expressed as $N = UDU^*$ where U is unitary and D is diagonal. Because commuting matrices can be put in "Schur" upper–triangular form simultaneously by a unitary similarity, N is normal iff $N^*N = NN^*$. We denote the subset of $\mathbb{M}_n$ consisting of the normal matrices by $\mathbb{N}_n$; we'll see that its geometric structure is rather mysterious.

A landmark result about spectral variation is the Hoffman–Wielandt Theorem (HW1953).
THEOREM: For any $A, B \in \mathbb{N}_n$,

$$sd_2(A, B) \leq \|A - B\|_2,$$

where

$$sd_2(A, B) = \min_{P} \|D_A - PD_BP^*\|_2,$$

P runs over all $n \times n$ permutation matrices, D_A is a diagonal form of A, and D_B is a diagonal form of B.

Note that $sd_2(A, B)$ is the analogue for $\|\cdot\|_2$ of the optimal matching distance $sd(A, B)$ since for normal A, B we may compute that distance as

$$sd(A, B) = \min_P \|D_A - PD_BP^*\|.$$

The Hoffman–Wielandt Theorem has a short and elegant proof.
PROOF: We have $A = UD_AU^*$, $B = WD_BW^*$ where D_A, D_B are diagonal, say $D_A = \text{diag}(a_k)_1^n, D_B = \text{diag}(b_k)_1^n$, and U, W are unitary. By unitary invariance,

$$\|A - B\|_2 = \|D_AU^*W - U^*WD_B\|_2 = \|D_AV - VD_B\|_2,$$

where $V = U^*W$ is unitary. Thus

$$\|A - B\|_2^2 = \sum_{i,j} |a_i - b_j|^2 |v_{i,j}|^2.$$

Now $[|v_{i,j}|^2]$ is a doubly stochastic matrix, ie it has nonnegative entries such that each row and column yields 1 as sum of the entries. Every doubly stochastic matrix is a convex combination of permutation matrices (one version of this famous result is due to Garrett Birkhoff). Thus

$$[|v_{i,j}|^2] = \sum_\pi t_\pi P(\pi),$$

where $t_\pi \geq 0, \sum_\pi t_\pi = 1$, and $P(\pi)$ is the permutation matrix corresponding to permutation π, ie $P_{i,j}(\pi) = 1$ if $j = \pi(i)$, $= 0$ otherwise. Thus

$$\|A - B\|_2^2 = \sum_\pi t_\pi (\sum_{i,j} |a_i - b_j|^2 P_{i,j}(\pi)) = \sum_\pi t_\pi (\sum_i |a_i - b_{\pi(i)}|^2),$$

and it follows that $\min_\pi \sum_i |a_i - b_{\pi(i)}|^2 \leq \|A - B\|_2^2$. Finally,

$$\sum_i |a_i - b_{\pi(i)}|^2 = \|D_A - P(\pi)D_BP^*(\pi)\|_2^2.$$

QED

5. The Wielandt–Mirsky conjecture

Included in [B2007] is a discussion of Hoffman's recollection of his work with Wielandt on the theorem above. It seems that Wielandt's initial aim was to prove the corresponding inequality for the operator norm, ie to show that

$$sd(A, B) \le \|A - B\| \quad \forall A, B \in \mathbb{N}_n. \tag{1}$$

Certainly Mirsky [M1960] clearly stated (1) as a problem or conjecture. It is reasonable then to call (1) the Wielandt–Mirsky conjecture. As it turned out (1) is correct in many cases but not in all, even for $n = 3$. Nevertheless, the conjecture inspired much interesting work, including natural questions about the geometry of $\mathbb{N}_n$ that remain puzzling to this day. This lecture series focuses on that story.

One good reason for considering (1) was an observation of H. Weyl, dating from about 1912: it works for Hermitians.
THEOREM: If A, B are Hermitian matrices with eigenvalues ordered as

$$a_1 \le a_2 \le \cdots \le a_n \text{ and } b_1 \le b_2 \le \cdots \le b_n,$$

then $|a_k - b_k| \le \|A - B\|$, for all k.

Note: It is not hard to see that, with the eigenvalues ordered in this way, matching a_k with b_k gives an optimal matching, ie $sd(A, B) = \max_k |a_k - b_k|$.

Here is one way to prove Weyl's result.
PROOF: Let $u_1, u_2, \ldots, u_n$ be an orthonormal system of eigenvectors for A, with $Au_k = a_k u_k$, and let $w_1, w_2, \ldots, w_n$ play the corresponding role for B. We may suppose $a_k \ge b_k$. Since the dimensions of $S = \text{span}\{u_k, \ldots, u_n\}$ and $T = \text{span}\{w_1, \ldots, w_k\}$ sum to $n+1$, we can choose a unit vector $u \in S \cap T$. It follows that $(Au, u) \ge a_k$ and $(Bu, u) \le b_k$. Hence $a_k - b_k \le ((A-B)u, u) \le \|A - B\|$. QED

A further recollection of Hoffman is that in working on (1) Wielandt had devised some technique based on continuous paths of matrices. This is tantalizing in view of Bhatia's introduction of the normal path inequality in 1982.

6. Normal paths

Let $\gamma : [a,b] \to \mathbb{N}_n$ be a "path" from A to B, ie γ is continuous, $\gamma(a) = A$, $\gamma(b) = B$ and each $\gamma(t)$ is normal. Bhatia [B1982] (with certain refinements in [BH1985]) noted that

$$sd(A,B) \leq |\gamma|, \tag{2}$$

where $|\gamma|$ is the arclength of γ with respect to the operator norm, ie

$$|\gamma| = \sup\{\sum_k \|\gamma(t_{k+1}) - \gamma(t_k)\| : a = t_0 < t_1 < \cdots < t_m = b\},$$

where the number m of "steps" is unlimited.

To understand (2), first note an elementary fact about normal matrices.
PROPOSITION 6.1: Suppose $N \in \mathbb{N}_n$ and that M is any matrix in $\mathbb{M}_n$. For each eigenvalue μ of M, there is some eigenvalue ν of N such that $|\mu - \nu| \leq \|M - N\|$.

PROOF: Let u be a unit eigenvector of M corresponding to μ, and let $u_1, u_2, \ldots, u_n$ be an orthonormal system of eigenvectors for N such that $Nu_k = \nu_k u_k$. Then

$$\|M-N\|^2 \geq \|(M-N)u\|^2 = \|\mu u - \sum_k \nu_k (u, u_k) u_k\|^2 = \sum_k |\mu - \nu_k|^2 |(u, u_k)|^2.$$

Since $\sum_k |(u, u_k)|^2 = 1$, we must have some k such that $|\mu - \nu_k| \leq \|M - N\|$. QED

To prove (2), suppose for convenience that $[a,b] = [0,1]$ and let γ_r be the initial section of the path that is parametrized on $[0,r]$. Let

$$G = \{r \in [0,1] : sd(A, \gamma(r)) \leq |\gamma_r|\}.$$

We wish to show that $1 \in G$. By continuity of γ and continuity of spectra, G is closed, so that $R = \sup G \in G$. We claim that $R = 1$, for otherwise consider $N = \gamma(R)$; let d be the minimum distance between distinct eigenvalues of N. Using continuity again we can find $r' \in (R, 1]$ such that $sd(N, \gamma(r')) < d/2$. Then each eigenvalue of $\gamma(r')$ must be matched with the closest eigenvalue of N. By Proposition 6.1, we conclude that $sd(N, \gamma(r')) \leq \|N - \gamma(r')\|$, so that

$$sd(A, \gamma(r')) \leq sd(A, N) + sd(N, \gamma(r')) \leq |\gamma_r| + \|N - \gamma(r')\|.$$

This last sum is certainly no greater than $|\gamma_{r'}|$, ie $r' \in G$, a contradiction. QED

7. Consequences of the normal path inequality

To begin with, we have a different proof of Weyl's inequality (section 5): if A, B are Hermitian the straight line segment

$$[A, B] = \{(1-t)A + tB : t \in [0,1]\}$$

consists entirely of Hermitians, ie it is a normal path from A to B. In view of (2),

$$sd(A, B) \leq |[A, B]| = \|A - B\|.$$

We may express a generalization of these ideas in terms of "spectral geometry".
PROPOSITION 7.1: If $A, B \in \mathbb{N}_n$ have spectra lying on parallel straight lines, then $[A, B]$ is a normal path and $sd(A, B) \leq \|A - B\|$.

PROOF: If θ if the angle of inclination of the lines we have $A = \alpha I_n + e^{i\theta} H$ and $B = \beta I_n + e^{i\theta} K$ for some $\alpha, \beta \in \mathbb{C}$ and Hermitian H, K. Thus each point on $[A, B]$ is normal:

$$(1-t)A + tB = ((1-t)\alpha + t\beta)I_n + e^{i\theta} H(t),$$

where $H(t) = (1-t)H + tK$ is Hermitian. QED

A more subtle application of the normal path inequality finds "short normal paths" that are *not* straight lines. Again spectral geometry induces a favorable geometry within $\mathbb{N}_n$.
PROPOSITION 7.2: If $A, B \in \mathbb{N}_n$ have spectra lying on concentric circles, then there exists a normal path γ from A to B such that $|\gamma| = \|A - B\|$; hence $sd(A, B) \leq \|A - B\|$.

REMARK: Since, in most cases, the line segment $[A, B]$ will *not* lie in $\mathbb{N}_n$, this phenomenon clearly depends on the *non–Euclidean* geometry induced by the operator norm.

PROOF: We may move the centre c of the concentric circles to the origin (by subtracting cI_n from each of A and B). Then $A = r_0 U$ and $B = r_1 W$ for some $r_0, r_1 \geq 0$ and unitary U, W. Our normal path has the form

$$\gamma(t) = r(t) e^{tK} U \quad (t \in [0,1])$$

where $r : [0,1] \to [0,\infty)$ is a carefully chosen continuous function, $r(0) = r_0$, $r(1) = r_1$, and K is a skew–Hermitian matrix such that $e^K = WU^*$. Evidently $\gamma(0) = A$, $\gamma(1) = B$, and each $\gamma(t)$ is a multiple of a unitary, so that it is normal. With respect to an appropriate orthonormal basis $K = \mathrm{diag}(i\theta_1, \ldots, i\theta_n)$ where

$$|\theta_n| \leq \cdots \leq |\theta_1| \leq \pi. \tag{3}$$

In view of (3),

$$\|A - B\| = \|r_0 I_n - r_1 WU^*\| = \|r_0 I_n - r_1 e^K\| = |r_0 - r_1 e^{i\theta_1}|.$$

Parametrize the straight line from r_0 to $r_1 e^{i\theta_1}$ as $r(t) e^{it\theta_1}$, so that

$$\|A - B\| = \int_0^1 |r'(t) + r(t) i\theta_1|\, dt.$$

On the other hand,

$$|\gamma| = \int_0^1 \|\gamma'(t)\|\, dt = \int_0^1 \|r'(t) + r(t)K\|\, dt.$$

In view of (3), the largest eigenvalue of the diagonal matrix $r'(t) + r(t)K$ is $r'(t) + r(t) i\theta_1$; hence

$$|\gamma| = \int_0^1 |r'(t) + r(t) i\theta_1|\, dt = \|A - B\|.$$

QED

Corollary 7.3: For any pair $A, B \in \mathbb{N}_2$, there is a normal path γ from A to B with length $\|A - B\|$.

Proof: If the eigenvalues a_1, a_2 and b_1, b_2 lie on parallel lines, invoke Proposition 7.1. Otherwise, let c be the point of intersection of the right bisectors of $[a_1, a_2]$ and $[b_1, b_2]$. Then the spectra lie on concentric circles about c, so that Proposition 7.2 applies. QED

Of course, the corollary implies (1) for 2×2 normals A, B, but this may be established more efficiently by an elementary argument.

8. Curvatures

The phenomena discussed in sections 6 and 7 suggest that we define a sort of "metric curvature" for sets of matrices: given a set of normal matrices S, let

$\kappa(S) = \max\{|\gamma|/\|A{-}B\| : \gamma$ is the shortest path from A to B that lies in $S\}$.

Proposition 7.2 tells us that $\kappa(\mathbb{C}\mathbb{U}_n) = 1$, where $\mathbb{U}_n$ denotes the unitary group in $\mathbb{M}_n$.

Proposition 7.3 tells us that $\kappa(\mathbb{N}_2) = 1$, and prompts the question: what are the curvatures $\kappa(\mathbb{N}_n)$ for larger n?

As natural as this question may be, the answers seem to be meagre indeed. M.–D. Choi (circa 1985) noted that $\kappa(\mathbb{N}_n) > 1$ for $n \geq 3$. Let $A_n = (J_n + J_n^*)/2$, where J_n is the $n \times n$ Jordan nilpotent, and let $B_n = (J_n - J_n^*)/2$; for example

$$A_3 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, B_3 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Note that A_n is Hermitian and B_n is skew–Hermitian; both are normal and $\|A_n - B_n\| = 1$. However, there cannot be a normal path of length 1 from A_n to B_n since the midpoint C of such a path would satisfy $\|A_n - C\| = \|C - B_n\| = 1/2$; but then each subdiagonal element of C must be 0; thus each subdiagonal element of $A_n - C$ is $1/2$ and since $\|A_n - C\| = 1/2$ we may conclude that $c_{\cdot 1} = \vec{0}$ and that $c_{12} = 1/2$; here is the contradiction: in a normal matrix the length of the 1st column must equal the length of the first row.

A quantitative version of this argument yields a lower bound for $\kappa(\mathbb{N}_n)$ (probably far from sharp).
Proposition 8.1: If $n \geq 3$, a normal path from A_n to B_n must have length

greater than 1.06.

One conclusion is that the normal path inequality (2) *doesn't* always yield the best estimates of spectral variation: since the eigenvalues of A_n are $\{\cos(\frac{k\pi}{n+1})\}_1^n$ and those of B_n are $\{i\cos(\frac{k\pi}{n+1})\}_1^n$, one can verify directly that $sd(A_n, B_n) \leq 1$.

Moreover, we have Sunder's Theorem ([S1982]).
THEOREM 8.2: If A is Hermitian and B is skew–Hermitian, then $sd(A, B) \leq \|A - B\|$.

Sunder's Theorem may be proved by a more incisive application of the ideas in our first proof of Weyl's inequality (see section 5). Again the argument depends on a specific matching between the eigenvalues of A and B; this time we match a_k with b_{n+1-k}, where

$$|a_1| \leq |a_2| \leq \cdots \leq |a_n|, \text{ and } |b_1| \leq |b_2| \leq \cdots \leq |b_n|.$$

9. Mysteries of spectral geometry

While it was clear that the normal path inequality could not establish the Wielandt–Mirsky conjecture in general, it was not until 1992 that the failure of the conjecture itself was discovered. In [H1992] examples of $A, B \in \mathbb{N}_3$ with $sd(A, B) > 1.016\|A - B\|$ were found by means of a carefully engineered computer search. Later Gert Krause described a simple explicit example such that the ratio could be exactly evaluated.

One might think that the following procedure could find such examples: choose two 3×3 diagonal matrices D_A, D_B with random complex diagonal entries; consider $A = D_A$ and $B = UD_BU^*$ where $U \in \mathbb{U}_3$; the spectral geometry is now fixed and we may directly compute $sd(A, B) = sd(D_A, D_B)$; vary U (with modern computing power 10^7 random U can easily be examined) looking for

$$\|D_A - UD_BU^*\| < sd(A, B).$$

In principle, one will eventually find violations of (1), but this "never" happens, unless one cheats by forcing D_A, D_B to have, say, the Gert Krause geometry. In some mysterious way, most choices of spectral geometry are favorable for (1), but we seem to know only a few special cases:

when the spectra lie on parallel lines (Proposition 7.1);
when the spectra lie on concentric circles (Proposition 7.2);
when the spectra lie on perpendicular lines (Theorem 8.2).

Clearly much remains to be explained even in the 3×3 case.

10. Estimates for $\kappa(\mathbb{N}_3)$

For $A, B \in \mathbb{N}_n$ we denote by $sp(A, B)$ the length of a shortest normal path from A to B (geodesic in $\mathbb{N}_n$).

Proposition 8.1 gave 1.06 as a weak lower bound for $\kappa(\mathbb{N}_n)$ $(n \geq 3)$; here, for $n = 3$, we find an upper bound, probably equally weak.

Proposition 10.1: For any $A, B \in \mathbb{N}_3$, $sp(A, B) \leq 3\|A - B\|$; hence $\kappa(\mathbb{N}_3) \leq 3$.

Proof: Let the eigenvalues of A and B be $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. By Proposition 6.1, there are $c_k \in \{a_1, a_2, a_3\}$ such that $|b_k - c_k| \leq \|A - B\|$; this is not necessarily a matching (eg, we could have $c_1 = a_1 = c_2$), but there is a normal path of length no more than $\|A - B\|$ from B to $C = U\text{diag}(c)U^*$, where $B = U\text{diag}(b)U^*$ and $U \in \mathbb{U}_3$. This is the path

$$\gamma_1(t) = U((1-t)\text{diag}(b) + t\text{diag}(c))U^* \quad (t \in [0, 1]).$$

In particular, $\|A - C\| \leq \|A - B\| + \|B - C\| \leq 2\|A - B\|$. But the spectrum of C is a subset of the spectrum of A; either both lie on a line or both lie on the circle determined by $\{a_1, a_2, a_3\}$. Invoke Proposition 7.1 or 7.2 to find a normal path γ_2 from C to A with $|\gamma_2| \leq 2\|A - B\|$. Together γ_1 and γ_2 form a normal path from B to A having length no greater than $3\|A - B\|$. QED

To obtain more information about $\kappa(\mathbb{N}_3)$ it would surely be helpful to determine $sp(A_3, B_3)$; the following proposition is a step in that direction. Given $n \times n$ normal A, B, let $spsn(A, B)$ be the length of the shortest normal path from A to B that lies entirely in the span of A, B, and I_n; of course $spsn(A, B) \geq sp(A, B)$.

Proposition 10.2: We can evaluate $spsn(A_3, B_3)$ exactly: $spsn(A_3, B_3) = \sqrt{2}\sin(\frac{\pi}{2\sqrt{2}}) \approx 1.2672$.

PROOF: If $X = \alpha J_3 + \beta J_3^*$ we see that X is normal iff $|\alpha| = |\beta|$; in fact, unless Y itself is normal, the only normal elements in $\mathrm{span}\{Y, Y^*\}$ have the form $\alpha Y + \beta Y^*$ with $|\alpha| = |\beta|$; note that such matrices are essentially Hermitian:

$$\alpha Y + e^{i\theta}\alpha Y^* = e^{i\theta/2}\alpha(Z + Z^*),$$

where $Z = e^{-i\theta/2}Y$. Thus, a normal path in $\mathrm{span}\{A_3, B_3, I_3\}$ has the form

$$\gamma(t) = a(t)J_3 + b(t)J_3^* + c(t)I_3,$$

where $|a(t)| = |b(t)|$, a, b, c are continuous (let's assume smooth, in fact) functions on some parametric interval, with values in $\mathbb{C}$; it will be convenient to use the interval $[0, \pi]$. If γ goes from A_3 to B_3, we have

$$a(0) = \frac{1}{2} = a(\pi), b(0) = \frac{1}{2}, b(\pi) = -\frac{1}{2}, c(0) = 0 = c(\pi).$$

Thus

$$|\gamma| = \int_0^\pi \|a'(t)J_3 + b'(t)J_3^* + c'(t)I_3\|\, dt.$$

Now $\|\alpha J_3 + \beta J_3^* + zI_3\| \geq \|\alpha J_3 + \beta J_3^*\|$ because, eg, if $U = diag(1, -1, 1)$ then

$$-U(\alpha J_3 + \beta J_3^* + zI_3)U^* = \alpha J_3 + \beta J_3^* - zI_3.$$

It follows that to minimize $|\gamma|$ we should take $c(t) \equiv 0$ so that

$$|\gamma| = \int_0^\pi \|a'(t)J_3 + b'(t)J_3^*\|\, dt.$$

By examining the eigenvalues of X^*X we see that $X = \alpha J_3 + \beta J_3^*$ has norm $\sqrt{|\alpha|^2 + |\beta|^2}$; hence

$$|\gamma| = \int_0^\pi \sqrt{|a'(t)|^2 + |b'(t)|^2}\, dt.$$

Since $|a'(t)| \geq ||a(t)|'|$, we decrease $|\gamma|$ by replacing a by $|a|$; then

$$(a(t), b(t)) \in [0, \infty) \times \mathbb{C} \equiv [0, \infty) \times \mathbb{R}^2,$$

and $|\gamma|$ is the Euclidian length of a 3–dimensional curve $\sigma(t)$ going from $(\frac{1}{2}, \frac{1}{2}, 0)$ to $(\frac{1}{2}, -\frac{1}{2}, 0)$. The normality of γ requires $\sigma(t)$ to lie on the cone

$$\{(r, r\cos t, r\sin t) : r \geq 0, t \in [0, \pi]\}.$$

We have $\sigma(t) = (r(t), r(t)\cos t, r(t)\sin t)$ for some $r : [0,\pi] \to [0,\infty)$ with $r(0) = \frac{1}{2} = r(\pi)$. Now compute:

$$|\gamma| = |\sigma| = \int_0^\pi (2(r'(t))^2 + r^2(t))^{\frac{1}{2}}\, dt.$$

Let $s(t) = r(\sqrt{2}t)$ and $\theta = t/\sqrt{2}$; then

$$|\gamma| = \int_0^\pi ((s'(t/\sqrt{2}))^2 + s^2(t/\sqrt{2}))^{\frac{1}{2}}\, dt = \sqrt{2}\int_0^{\pi/\sqrt{2}} ((s'(\theta))^2 + s^2(\theta))^{\frac{1}{2}}\, d\theta.$$

But this is $\sqrt{2}|\tau|$ where $\tau(\theta)$ is the polar plane curve $\tau(\theta) = (s(\theta), \theta)$ from $(\frac{1}{2}, 0)$ to $(\frac{1}{2}, \pi/\sqrt{2})$ (polar coordinates). Thus we minimize $|\gamma|$ by taking τ to be the straight line between these points. It has length $\sin(\pi/(2\sqrt{2}))$. QED

11. Spectral variation in $\mathbb{N}_n$

Let $c(n) = \sup\{sd(A,B)/\|A-B\| : A, B \in \mathbb{N}_n\}$. We have seen that $c(2) = 1$ and $c(3) > 1$. The normal path inequality (2) tells us that $c(n) \leq \kappa(n)$ but, while (2) yields sharp estimates in special cases (as in Proposition 7.2), we do not presently know enough about $\kappa(n)$ to make useful conclusions about $c(n)$. It seems that we don't even know whether the constants $\kappa(n)$ are bounded as n increases. Nevertheless, quite different approaches do yield striking information about the constants $c(n)$.

The combined forces of Bhatia, Davis, McIntosh, and Koosis (see [BDM1983] and [BDK1989]) established a uniform bound on $c(n)$.
Theorem 11.1: There is a universal constant c such that $c(n) \leq c$, and $c \approx 2.9$.

Remark: For $n \leq 8$ a better upper bound on $c(n)$ follows from the Hoffman–Wielandt Theorem (section 4): for $A, B \in \mathbb{N}_n$

$$sd(A,B) \leq sd_2(A,B) \leq \|A-B\|_2 \leq \sqrt{n}\|A-B\|,$$

ie $c(n) \leq \sqrt{n}$.

The proof of theorem 11.1 depends on relating spectral variation to a constant c defined by an extremal problem for Fourier transforms and on good estimates for that constant. See chapter VII of [B1997].

12. Estimates for $spsn(A_n, B_n)$

Since $\alpha J_3 + \beta J_3^*$ is a submatrix of $\alpha J_n + \beta J_n^*$ (for $n > 3$), we have

$$\sqrt{|\alpha|^2 + |\beta|^2} = \|\alpha J_3 + \beta J_3^*\| \leq \|\alpha J_n + \beta J_n^*\|,$$

so that the argument of Proposition 10.2 also shows that $spsn(A_n, B_n) \geq \sqrt{2}\sin(\frac{\pi}{2\sqrt{2}}) \approx 1.2672$.

Similarly we obtain *upper* bounds on $spsn(A_n, B_n)$ from the inequalities

$$\|\alpha J_n + \beta J_n^*\| \leq q(n)\sqrt{|\alpha|^2 + |\beta|^2} \text{ where}$$

$$q(n) = \sup\{\|J_n + tJ_n^*\|/\sqrt{1+t^2} : t \in [0,1]\}.$$

We have $spsn(A_n, B_n) \leq q(n)1.2672$ and the constants $q(n)$ can be evaluated numerically (and perhaps exactly, at least for odd n). We have $q(4) \approx 1.1547$, $q(5) \approx 1.2247$, $q(6) \approx 1.2750$, $q(7) \approx 1.3066$, and $q(8) \approx 1.3291$.

In this way we obtain the following estimates.
PROPOSITION 12.1: We have $1.2672 \leq spsn(A_4, B_4) \leq 1.4632$, $1.2672 \leq spsn(A_5, B_5) \leq 1.5519$, $1.2672 \leq spsn(A_6, B_6) \leq 1.6157$, $1.2672 \leq spsn(A_7, B_7) \leq 1.6557$, and $1.2672 \leq spsn(A_8, B_8) \leq 1.6842$.

For larger values of n we obtain more useful estimates for $\|\alpha J_3 + \beta J_3^*\|$ in terms of $|\alpha| + |\beta|$. In fact,

$$(1 - 2/n)^{\frac{1}{2}}(|\alpha| + |\beta|) \leq \|\alpha J_n + \beta J_n^*\| \leq (|\alpha| + |\beta|).$$

The first inequality may be obtained by observing that $((\alpha J_n + \beta J_n^*)u)_k = \alpha u_{k-1} + \beta u_{k+1}$ for $k = 2, \ldots, n-1$ and taking $u_{k+1} = \arg(\alpha/\beta)u_{k-1}$.

PROPOSITION 12.2: For $n > 2$ we have

$$(1 - 2/n)^{\frac{1}{2}}(1 + \cos 1) \leq spsn(A_n, B_n) \leq (1 + \cos 1) \approx 1.5403.$$

PROOF: Consider a normal path $\gamma(t) = a(t)J_n + b(t)J_n^*$ $(t \in [0,1])$ from A_n to B_n; we have $|a(t)| = |b(t)|$, $a(0) = \frac{1}{2} = a(1)$, and $b(0) = \frac{1}{2} = -b(1)$.

Combining

$$|\gamma| = \int_0^1 \|a'(t)J_n + b'(t)J_n^*\| \, dt$$

with the inequality above we obtain

$$(1-2/n)^{\frac{1}{2}} \int_0^1 (|a'(t)| + |b'(t)|) \, dt \leq |\gamma| \leq \int_0^1 (|a'(t)| + |b'(t)|) \, dt,$$

which is the sum of the lengths of paths $a(t)$ and $b(t)$ in the plane. It remains to choose these paths so as to minimize their total length. Suppose that $r = \min |b(t)|$; then the shortest $b(t)$ follows the tangent line from $(\frac{1}{2}, 0)$ to the circle of radius r about $(0,0)$, follows the arc of that circle to the tangent from $(-\frac{1}{2}, 0)$, and follows that tangent to $(-\frac{1}{2}, 0)$. Because $|a(t)| = |b(t)|$ the path $a(t)$ must go from $(\frac{1}{2}, 0)$ to a point on the circle and back. Clearly the shortest such path is the segment from $(\frac{1}{2}, 0)$ to $(r, 0)$ and back. It is a simple exercise in calculus to minimize the total lengths of such paths with respect to r. The result is $1 + \cos 1$. QED

To compare with Proposition 12.1, we now have $spsn(A_4, B_4) \in [1.0892, 1.5403]$, $spsn(A_5, B_5) \in [1.1931, 1.5403]$, $spsn(A_6, B_6) \in [1.2577, 1.5403]$, $spsn(A_7, B_7) \in [1.3018, 1.5403]$, and $spsn(A_8, B_8) \in [1.3399, 1.5403]$.

13. Epilogue

We have seen that spectral geometry affects spectral variation (as in Theorem 8.2) and that geometry in $\mathbb{N}_n$ sometimes yields sharp inequalities for spectral variation (as in Proposition 7.2). We have also seen that geometry in $\mathbb{N}_n$ sometimes comes down to the geometry of curves in $\mathbb{R}^3$ (as in Proposition 10.2) or $\mathbb{R}^2$. Clearly, however, our present understanding of the geometry of $\mathbb{N}_n$ leaves many other questions to be addressed.

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