The Abdus Salam International Centre for Theoretical Physics

2470-1

Advanced School and Workshop on Matrix Geometries and Applications

1 - 12 July 2013

Adjacency preserving maps

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Matrix Geometries and Applications
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1 Introduction

A short biography of L.-K. Hua can be found at http://www-history.mcs.st-and.ac.uk/Biographies/Hua.html. Let us qoute the first sentence: "Loo-Keng Hua was one of the leading mathematicians of his time and one of the two most eminent Chinese mathematicians of his generation, S. S. Chern being the other."

And here is another part from this short article on Hua's life and work: "Thus Hua became interested in matrix algebra and wrote several substantial papers on the geometry of matrices. He had been invited to visit the Institute for Advanced Study in Princeton, but because C. L. Siegel was working there along somewhat similar lines, Hua declined, at first in order to develop his ideas independently. In September 1946, shortly after returning from Russia, Hua did depart for Princeton, bringing with him projects not only in matrix theory but also in functions of several complex variables and in group theory."

In these lecture notes we will explain Hua's theorems on geometry of matrices. We will start with some notation needed to formulate fundamental theorems of geometry of matrices. The exact statements will be given for rectangular matrices and hermitian matrices. These beautiful theorems have many applications, among others in algebra, mathematical physics, and geometry. Some of them will be presented in the second section. The applications motivate the question of possible improvements of Hua's results that were obtained in the period 1945-1951. Later Hua's followers succeeded to remove some technical assumptions appearing in his statements. Substantial improvements have been obtained only in the last few years. New proof techniques were needed. Some of the ideas used already by Hua as well as the recent proof techniques will be discussed in the third section. When trying to find optimal versions of Hua's theorems some natural conjectures turned out to be wrong and some of the recent results have quite surprising conclusions. We will briefly explain some examples showing that relaxing some of the assumptions in classical fundamental theorems of geometry of matrices may lead to conclusions of unexpected form.

Let us start with the notation. We denote by $M_{m\times n}(\mathbb{F})$ the set of all $m \times n$ matrices over a field $\mathbb{F}$. We write shortly $M_{n\times n}(\mathbb{F}) = M_n(\mathbb{F})$. Let H_n be the set of all complex hermitian matrices, $H_n = \{A \in M_n(\mathbb{C}) : A^* = A\}$. We further denote by $S_n(\mathbb{F})$ the set of all $n \times n$ symmetric matrices, $S_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : A^T = A\}$, and by $A_n(\mathbb{F})$ the set of all $n \times n$ skew-symmetric matrices, $A_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : A^T = -A\}$. Note that skew-symmetric matrices are sometimes called alternate matrices.

Let $\mathcal{V}$ be any of the above matrix spaces, $\mathcal{V} \in \{M_{m\times n}(\mathbb{F}), H_n, S_n(\mathbb{F}), A_n(\mathbb{F})\}$ (note that H_n is not a complex vector space, but it is a real vector space). Define the arithmetic distance between two matrices $A, B \in \mathcal{V}$ by

$$d(A, B) = \operatorname{rank}(A - B).$$

The set $\mathcal{V}$ equipped with the distance d is a metric space.

Problem. Verify that d is a metric on $\mathcal{V}$. Hint: the only non-trivial fact that has to be verified is the triangle inequality which follows directly from the triangle inequality for the rank; the easiest way to verify the triangle inequality for the rank function is to identify matrices with linear operators and observe that for any pair of linear operators A, B we have $\operatorname{Im}(A + B) \subset \operatorname{Im} A + \operatorname{Im} B$.

Let $\mathcal{V} \in \{M_{m\times n}(\mathbb{F}), H_n, S_n(\mathbb{F})\}$. Matrices $A, B \in \mathcal{V}$ are said to be adjacent if

$$d(A, B) = \operatorname{rank}(B - A) = 1.$$

Let $\operatorname{char} \mathbb{F} \neq 2$. Matrices $A, B \in A_n(\mathbb{F})$ are adjacent if

$$d(A, B) = 2.$$

Problem. Explain why the definition of adjacency is different in the case when $\mathcal{V} = A_n(\mathbb{F})$.

A map $\phi : \mathcal{V} \to \mathcal{V}$ preserves adjacency in both directions if for every pair $A, B \in \mathcal{V}$ we have

$$d(A, B) = 1 \iff d(\phi(A), \phi(B)) = 1.$$

A map $\phi : \mathcal{V} \to \mathcal{V}$ preserves adjacency if for every pair $A, B \in \mathcal{V}$ we have

$$d(A, B) = 1 \Rightarrow d(\phi(A), \phi(B)) = 1.$$

Fundamental theorems of geometry of rectangular matrices, hermitian matrices, symmetric matrices, skew-symmetric matrices tell that bijective maps

$\phi : \mathcal{V} \to \mathcal{V}$ preserving adjacency in both directions are of very nice simple forms. Here, $\mathcal{V}$ is any of matrix spaces $M_{m\times n}(\mathbb{F})$, H_n, $S_n(\mathbb{F})$, $A_n(\mathbb{F})$.

We will formulate precise statements in the hermitian case and the rectangular case. Thus, we first want to describe the general form of bijective maps on H_n which preserve adjacency in both directions. It is easy to find examples of such maps. Let $S \in H_n$ be any matrix. Clearly, the map $A \mapsto A + S$, $A \in H_n$, is a bijection of H_n onto itself, and obviously, for any $A, B \in H_n$ the matrices A and B are adjacent if and only if $A+S$ and $B+S$ are adjacent. It is also evident that the maps $A \mapsto -A$, $A \in H_n$, and $A \mapsto A^T$, $A \in H_n$, are bijective maps on H_n preserving adjacency in both directions. Here, A^T denotes the transpose of A.

Let $T \in M_n(\mathbb{C})$ be any invertible matrix. Then for every pair $A, B \in H_n$ we have $\mathrm{rank}\,(A - B) = 1 \iff \mathrm{rank}\,(TA - TB) = 1$. The same is true if we multiply by T on the right side. But of course, in general TA need not be a hermitian matrix. However, a complex $n \times n$ matrix A is hermitian if and only if TAT^* is hermitian. Thus, the map $A \mapsto TAT^*$, $A \in H_n$, is another example of a bijective map preserving adjacency in both directions.

Clearly, the composition of bijective maps preserving adjacency in both directions is again a map of such a type. So, we can get further examples by composing the maps we have described in the previous two paragraphs. Hua's theorem tells that in such a way we get all bijective maps on H_n that preserve adjacency in both directions.

Theorem 1.1 *Let $n \geq 2$ be an integer and $\phi : H_n \to H_n$ a bijective map such that for every pair $A, B \in H_n$ the matrices A and B are adjacent if and only if $\phi(A)$ and $\phi(B)$ are adjacent. Then there exist $c \in \{-1, 1\}$, an invertible $n \times n$ complex matrix T, and $S \in H_n$ such that either*

$$\phi(A) = cTAT^* + S, \quad A \in H_n,$$

or

$$\phi(A) = cT\overline{A}T^* + S, \quad A \in H_n.$$

Here, $\overline{A}$ denotes the matrix obtained from A by applying the complex conjugation entrywise,

$$\overline{A} = \overline{\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}} = \begin{bmatrix} \overline{a_{11}} & \dots & \overline{a_{1n}} \\ \vdots & \ddots & \vdots \\ \overline{a_{n1}} & \dots & \overline{a_{nn}} \end{bmatrix}.$$

Note that because A is hermitian we have $\overline{A} = A^T$. Any map $\phi : H_n \to H_n$ that is of one of the two forms appearing in the conclusion of the above theorem is called standard. Clearly, the converse statement is true as well, that is, every standard map is bijective and preserves adjacency in both directions.

The map $\phi : H_n \to H_n$ is a bijective map preserving adjacency in both directions if and only if the same is true for the map $A \mapsto \phi(A) - \phi(0)$, $A \in H_n$. And obviously, ϕ is standard if and only if the map $A \mapsto \phi(A) - \phi(0)$, $A \in H_n$ is standard. Thus, there is no loss of generality if we add the assumption $\phi(0) = 0$ in the above theorem. Then clearly, $S = 0$, and consequently, ϕ is real-linear. It is a remarkable fact that after this harmless normalization the linear character of ϕ is not an assumption but a conclusion.

Let $\mathbb{F}$ be a field. Recall that an automorphism f of the field $\mathbb{F}$ is a bijective function $f : \mathbb{F} \to \mathbb{F}$ which is additive and multiplicative, that is, for every pair $\lambda, \mu \in \mathbb{F}$ we have

$$f(\lambda + \mu) = f(\lambda) + f(\mu)$$

and

$$f(\lambda\mu) = f(\lambda)f(\mu).$$

It is well-known that the identity is the only automorphism of the real field. In the case of the complex field the complex conjugation $f(\lambda) = \overline{\lambda}$, $\lambda \in \mathbb{C}$, is a non-trivial example of an automorphism. There exist other automorphisms of the complex field, but it is non-trivial to describe them (one needs to understand some basic notions from the field theory such as algebraic independence and transcendental basis, and then in the construction of such automorphisms Zorn's lemma is used).

Suppose now that $f : \mathbb{F} \to \mathbb{F}$ is an automorphism and $A = [a_{ij}]$ an $m \times n$ matrix over the field $\mathbb{F}$. We denote by A_f the matrix obtained from A by applying f entrywise,

$$A_f = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}_f = \begin{bmatrix} f(a_{11}) & \dots & f(a_{1n}) \\ \vdots & \ddots & \vdots \\ f(a_{m1}) & \dots & f(a_{mn}) \end{bmatrix}.$$

Problem. Let f be an automorphism of the field $\mathbb{F}$. Show that the map $\phi : M_{m\times n}(\mathbb{F}) \to M_{m\times n}(\mathbb{F})$ defined by $\phi(A) = A_f$ is bijective and preserves adjacency in both directions.

Theorem 1.2 *Let $\mathbb{F}$ be a field, $m, n \geq 2$ integers, and $\phi : M_{m\times n}(\mathbb{F}) \to M_{m\times n}(\mathbb{F})$ a bijective map preserving adjacency in both directions. Then there exist invertible matrices $T \in M_m(\mathbb{F}), S \in M_n(\mathbb{F})$, a matrix $R \in M_{m\times n}(\mathbb{F})$, and an automorphism f of the field $\mathbb{F}$ such that*

$$\phi(A) = TA_fS + R, \quad A \in M_{m\times n}(\mathbb{F}).$$

In the square case $m = n$ we have the additional possibility that

$$\phi(A) = TA_f^TS + R, \quad A \in M_n(\mathbb{F}).$$

The converse is obviously true. As in the hermitian case there is no loss of generality in assuming that $\phi(0) = 0$. Then $R = 0$. We see that every bijective map ϕ preserving adjacency in both directions and satisfying $\phi(0) = 0$ is semilinear (with respect to the field automorphism f), that is

$$\phi(A + B) = \phi(A) + \phi(B), \quad A, B \in M_{m\times n}(\mathbb{F}),$$

and

$$\phi(\lambda A) = f(\lambda)\phi(A), \quad \lambda \in \mathbb{F},\ A \in M_{m\times n}(\mathbb{F}).$$

Both theorems give very nice conclusions under rather weak assumptions. Motivated by numerous applications (see the next section) we can ask if we can do even better? Can we replace the assumption that adjacency is preserved in both directions by the weaker assumption that adjacency is preserved in one direction only and still get the same conclusion? Do we need the bijectivity assumption? Can we characterize adjacency preserving maps acting between spaces of matrices of different sizes?

We have three different problems. Quite surprisingly, it has turned out that in the hermitian case one can answer all three questions simultaneously as the following theorem shows. The rectangular case is much more difficult and we will discuss it later.

Theorem 1.3 *Let m, n be positive integers with $n \geq 2$. Assume that $\phi : H_n \to H_m$ is a map such that the matrices $\phi(A)$ and $\phi(B)$ are adjacent whenever A and B are adjacent, $A, B \in H_n$. Suppose that $\phi(0) = 0$. Then one of the following holds.*

1. *There exist a rank one matrix $R \in H_m$ and a function $\rho : H_n \to \mathbb{R}$ such that $\phi(A) = \rho(A)R$.*

2. *$m \geq n$ and there exist $c \in \{-1, 1\}$ and an invertible $m \times m$ complex matrix T such that either*

$$\phi(A) = cT \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} T^*, \quad A \in H_n,$$

or

$$\phi(A) = cT \begin{bmatrix} \overline{A} & 0 \\ 0 & 0 \end{bmatrix} T^*, \quad A \in H_n.$$

Let $\rho : H_n \to \mathbb{R}$ be any function and $R \in H_m$ a rank one matrix. Then the map $\phi : H_n \to H_m$ defined by $\phi(A) = \rho(A)R$ preserves adjacency if $\rho(A) \neq \rho(B)$ whenever A and B are adjacent. In particular, this happens when ρ is injective. If we take $\rho(A) = \operatorname{tr} A$, where $\operatorname{tr} A$ denotes the trace of A, then ϕ is a continuous (even real-linear) adjacency preserving map. Indeed, if A and B are adjacent, then $A - B$ is a rank one hermitian matrix and every rank one hermitian matrix has a nonzero trace.

2 Applications

We start with the so-called linear preserver problems. Let T and S be $n \times n$ complex matrices. Assume that $\det(TS) = 1$. Using the fact that the determinant is multiplicative ($\det(AB) = \det A \det B$, $A, B \in M_n(\mathbb{C})$) we immediately see that every map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ of the form

$$\phi(A) = TAS, \quad A \in M_n(\mathbb{C}), \tag{1}$$

preserves determinant, that is, $\det \phi(A) = \det A$ for every $A \in M_n(\mathbb{C})$. Since $\det A = \det A^T$ for every $A \in M_n(\mathbb{C})$, every map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ of the form

$$\phi(A) = TA^T S, \quad A \in M_n(\mathbb{C}), \tag{2}$$

preserves determinant as well. This is all trivial. The non-trivial fact that the converse holds true was proved by Frobenius already at the end of the 19th century. His theorem reads as follows. Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map with the property that

$$\det \phi(A) = \det A \tag{3}$$

for every $A \in M_n(\mathbb{C})$. Then there exist matrices $T, S \in M_n(\mathbb{C})$ satisfying $\det(TS) = 1$ such that either we have (1) for all $A \in M_n(\mathbb{C})$, or we have (2) for all $A \in M_n(\mathbb{C})$.

Let us mention two more linear preserver problems. One may ask what is the general form of linear maps $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserving commutativity, that is, having the property that for all $A, B \in M_n(\mathbb{C})$ we have

$$AB = BA \iff \phi(A)\phi(B) = \phi(B)\phi(A)$$

(those familiar with the basic facts from the theory of Lie algebras will observe the connection with Lie homomorphisms: A map ϕ preserves commutativity if and only if it preservers the zero Lie product). If we denote by $W(A)$ the numerical range of A, then we may want to describe all linear maps $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying $W(\phi(A)) = W(A)$, $A \in M_n(\mathbb{C})$.

A lot of research work has been done on this kind of problems and one of the most frequently used proof technique is the reduction of a given linear preserver problem to the problem of characterizing linear maps preserving rank one matrices. The idea is the following. We have a linear map with a certain preserving property. We characterize rank one matrices in terms of this preserving property. Then, as this property is preserved, we conclude that ϕ preserves also rank one matrices, that is, for every $A \in M_n(\mathbb{C})$ we have

$$\operatorname{rank} A = 1 \iff \operatorname{rank} \phi(A) = 1. \tag{4}$$

It is known that if $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear map with the above property, then there exist invertible matrices $T, S \in M_n(\mathbb{C})$ such that either ϕ is of the form (1), or it is of the form (2). Sometimes we need to show that matrices T and S have some additional properties and we are done.

Let us illustrate this idea by skecthing the proof of the Frobenius theorem. Thus, our assumption is that $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear map satisfying (3).

Problem. Prove that if A is an $n \times n$ complex matrix such that for every $B \in M_n(\mathbb{C})$ we have $\det(B + A) = \det B$, then $A = 0$.

Using the above fact we easily see that ϕ is bijective. We want to characterize rank one matrices using the determinant. To get the idea take the rank one matrix E_{11}, that is, the matrix having all entries zero but the $(1,1)$-entry which is equal to 1. Observe that for every matrix $A = [a_{ij}] \in M_n(\mathbb{C})$ we have

$$\det(A + xE_{11}) = \det \begin{bmatrix} a_{11} + x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = p + qx, \quad x \in \mathbb{C},$$

for some complex constants p and q. In other words, for every matrix A, the determinant $\det(A + xE_{11})$ is a polynomial in x of degree at most one. It turns out that this property is characteristic for rank one matrices. Namely, we can show that for a nonzero matrix $R \in M_n(\mathbb{C})$ the following are equivalent:

- $\operatorname{rank} R = 1$,
- for every $A \in M_n(\mathbb{C})$ the polynomial $x \mapsto \det(A + xR)$, $x \in \mathbb{C}$, has degree at most one.

Problem. Prove the above equivalence. Hint: Use the fact that $\operatorname{rank} R = r$ iff there exist invertible matrices $P, Q \in M_n$ such that PRQ is the diagonal matrix whose first r diagonal entries are 1 and the remaining diagonal entries are zero. Note that $\det(A + xR) = \frac{1}{\det(PQ)} \det(PAQ + xPRQ)$.

Now, if $R \in M_n(\mathbb{C})$ is of rank one and A is an arbitrary matrix, then we can find a matrix B such that $\phi(B) = A$, and therefore

$$\det(A + x\phi(R)) = \det(\phi(B + xR)) = \det(B + xR)$$

is a polynomial of degree at most one. By the above equivalence, $\phi(R)$ is of rank one. As ϕ^{-1} is also a linear preserver of determinant, we actually have (4). Thus, we have either (1), or (2). From $\det(TAS) = \det A$ ($\det(TA^TS) = \det A$) it follows directly that $\det(TS) = 1$.

The important observation is that a bijective linear map satisfying (4) preserves adjacency in both directions. Indeed, for an arbitrary pair of matrices $A, B \in M_n(\mathbb{C})$ we have

$$\text{rank}\,(A-B)=1 \iff \text{rank}\,\phi(A-B)=1 \iff \text{rank}\,(\phi(A)-\phi(B))=1.$$

So, the Frobenius theorem as well as many other linear preserver results follow from Hua's fundamental theorem of geometry of matrices. When reducing a linear preserver problem to the problem of rank one preservers and then to Hua's theorem, we end up with a result on maps on matrices with no linearity assumption. Therefore it is not surprising that Hua's theorem has been already proved to be a useful tool in the new research area concerning general (non-linear) preservers.

We continue with applications in mathematical physics. Let M_4 denote the classical Minkowski space of spacetime events,

$$M_4 = \{(x,y,z,t)\,:\,x,y,z,t\in\mathbb{R}\}.$$

Here, the first three coordinates of each spacetime event are spatial coordinates and the last one represents time. Two spacetime events $r_1 = (x_1, y_1, z_1, t_1)$ and $r_2 = (x_2, y_2, z_2, t_2)$ are said to be coherent if the light signal can pass between r_1 and r_2, that is

$$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2=c^2(t_2-t_1)^2.$$

Of course, c denotes the speed of light. A map ϕ defined on the set of spacetime events preserves coherency in both directions if for any two spacetime events r_1 and r_2 the events $\phi(r_1)$ and $\phi(r_2)$ are coherent if and only if r_1 and r_2 are coherent. The problem of characterizing bijective maps on spacetime events preserving coherency in both directions was solved by A.D. Aleksandrov in 1950. It turns out that such maps must be Lorentz transformations up to a scale factor.

We associate to each spacetime event (x, y, z, t) a 2×2 hermitian matrix

$$A=\begin{bmatrix} ct+z & x+iy \\ x-iy & ct-z \end{bmatrix}.$$

Clearly,

$$\det A=\det\begin{bmatrix} ct+z & x+iy \\ x-iy & ct-z \end{bmatrix}=c^2t^2-z^2-x^2-y^2.$$

Hence if we denote by A_1 and A_2 the matrices corresponding to spacetime events $r_1 = (x_1, y_1, z_1, t_1)$ and $r_2 = (x_2, y_2, z_2, t_2)$, then

$$\det(A_2-A_1)=\det\begin{bmatrix} c(t_2-t_1)+(z_2-z_1) & (x_2-x_1)+i(y_2-y_1) \\ (x_2-x_1)-i(y_2-y_1) & c(t_2-t_1)-(z_2-z_1) \end{bmatrix}$$

$$= c^2(t_2 - t_1)^2 - (z_2 - z_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2.$$

Therefore, the spacetime events r_1 and r_2, $r_1 \neq r_2$, are coherent if and only if the corresponding matrices satisfy $\det(A_2 - A_1) = 0$. But this is equivalent to the fact that $A_2 - A_1$ is singular. Obviously, a nonzero singular 2×2 matrix has rank one. We conclude that the spacetime events r_1 and r_2, $r_1 \neq r_2$, are coherent if and only if the corresponding matrices A_1 and A_2 are adjacent.

It follows that if we identify spacetime events with 2 by 2 hermitian matrices as described above, then to each bijective map $\phi : M_4 \to M_4$ preserving coherency in both directions there corresponds a bijective map on H_2 preserving adjacency in both directions. Consequently, Aleksandrov's theorem describing the general form of bijective maps on M_4 preserving coherency in both directions can be equivalently reformulated as the characterization of bijective maps on H_2 preserving adjacency in both directions. Such a characterization is a special case of the much more general Hua's fundamental theorem of geometry of hermitian matrices.

Problem. Use Hua's fundamental theorem of geometry of hermitian matrices to describe the general form of bijective maps $\phi : M_4 \to M_4$ preserving coherency in both directions.

Let H be a complex Hilbert space. Self-adjoint bounded linear operators on H are important in the Hilbert space framework of quantum mechanics as they represent bounded observables. The set $\mathcal{S}(H)$ of all such operators can be equipped with several relations and operations having important physical meanings. It is then of interest to study automorphisms of $\mathcal{S}(H)$ with respect to these realations and/or operations. Such transformations can be viewed as certain kinds of symmetries of the underlying quantum system.

Two of the most studied relations on self-adjoint operators are the usual partial order defined by $A \leq B$ if and only if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in H$, and commutativity (or compatibility in the language of quantum mechanics). In the language of quantum mechanics, the bounded observable A is said to be less or equal to the bounded observable B if the mean value (expectation) of A in any state is less or equal to the mean value of B in the same state. And two bounded observables are compatible if and only if they can be measured jointly. The usual Jordan product $A \circ B = (1/2)(AB + BA)$ and the triple Jordan product $A \circ B = ABA$ are examples of binary operations on the set of all self-adjoint operators that are relevant in mathematical physics.

For the sake of simplicity we will restrict to the finite-dimensional case. When $\dim H = n$ we can identify self-adjoint operators on H with $n \times n$ complex hermitian matrices. To illustrate the use of Hua's fundamental theorem of geometry of hermitian matrices in mathematical physics we will describe the general form of symmetries with respect to the above mentioned partial order. If we write vectors $x \in \mathbb{C}^n$ as $n \times 1$ matrices, then $A \leq B$ if and only if

$x^*Ax \leq x^*Bx$ for every $x \in \mathbb{C}^n$. Alternatively, $A \leq B$ if and only if all the eigenvalues of the hermitian matrix $B - A$ are nonnegative.

We want to describe the general form of bijective maps $\phi : H_n \to H_n$ with the property that for every pair $A, B \in H_n$ we have

$$A \leq B \iff \phi(A) \leq \phi(B).$$

Let us briefly explain the main ideas that are needed to obtain the desired result. If A and B are adjacent, then $B = A + R$ for some rank one matrix $R \in H_n$. Every such matrix is of the form $R = tP$, where t is a nonzero real number and P a projection of rank one. Hence, $B - A = tP$ is either positive (when $t > 0$), or negative. Asume that $A \leq B$ and take any two matrices $C, D \in [A, A + tP] = \{F \in H_n \, : \, A \leq F \leq A + tP\}$. It is easy to verify that $[A, A + tP] = \{A + sP \, : \, 0 \leq s \leq t\}$. It follows that either $C \leq D$, or $D \leq C$, that is, C and D are comparable. To summarize, we have shown that if A and B are adjacent, then A and B are comparable and if C and D are any two matrices from the interval between A and B, then C and D are comparable as well. It turns out that the converse statement is true as well. More precisely, for every pair of matrices $A, B \in H_n$, $A \neq B$, $n \geq 2$, the following are equivalent:

- A and B are adjacent,
- A and B are comparable and any two elements from the operator interval between A and B are comparable.

The main idea of the proof of the converse statement is as follows. Assume that A and B are not adjacent. If they are not comparable, we are done. So, assume they are comparable. Then $A \leq B$ or $B \leq A$. We may consider just one of the two possibilities, say the first one. All we need to do is to find matrices $C, D \in [A, B]$ that are not comparable. To find such matrices one has to use the fact that $\text{rank}\,(B - A) \geq 2$. The problem becomes easier if one observes that the partial order $\leq$ is translation invariant, that is, for any three matrices $M, N, L \in H_n$ we have $M \leq N \iff M + L \leq N + L$. Note that this property yields that there is no loss of generality in assuming that $A = 0$.

Thus, we have a characterization of the adjacency relation expressed in terms of the relation $\leq$. And clearly, as ϕ preserves the order $\leq$ in both directions, it must preserve also the adjacency in both directions. So, we can apply Theorem 1.1 to obtain:

Theorem 2.1 *Let $n \geq 2$ be an integer and $\phi : H_n \to H_n$ a bijective map such that for every pair $A, B \in H_n$ we have $A \leq B \iff \phi(A) \leq \phi(B)$. Then there exist $n \times n$ complex matrices T, S, with T invertible and S hermitian, such that either*

$$\phi(A) = TAT^* + S, \quad A \in H_n,$$

or

$$\phi(A) = T\overline{A}T^* + S, \quad A \in H_n.$$

Problem. We have explained the main ideas of the proof. Give the proof with all the details.

The fundametal theorem of geometry of rectangular matrices can be applied when studying the geometry of Grassmann spaces. Let $G(n,k)$ be the Grassmann space consisting of all k-dimensional subspaces of an n-dimensional vector space V. Thus, $G(n,1)$ is the set of all lines through the origin of V, and $G(n,2)$ is the set of all planes containing the origin of V, ... Two elements of $G(n,k)$ are adjacent if their intersection has dimension $k-1$. Equivalently, their sum is of dimension $k+1$. In other words, the intersection of two adjacent k-dimensional subspaces has codimension one in each of these two subspaces.

In 1949 Chow determined all bijections $\phi : G(n,k) \to G(n,k)$ that preserve adjacency in both directions. Except in the trivial cases $k=1$ and $k=n-1$ every such map is induced by a bijective semilinear transformation $S : V \to V$. In the case when $n = 2k$ we have the additional possibility that such a map is induced by a semilinear transformation from V onto its dual V^*.

We will briefly explain the connection with Hua's fundamental theorem of geometry of rectangular matrices. Let m, n be positive integers, $m, n \geq 2$. Then to each point in $G(m+n, m)$, that is, to each m-dimensional subspace U of $\mathbb{F}^{m+n}$, we can associate an $m \times (m+n)$ matrix whose rows are coordinates of the vectors that form a basis of U. Each $m \times (m+n)$ matrix will be written in the block form $[X\ Y]$. Here, X is an $m \times m$ matrix and Y is an $m \times n$ matrix.

Problem. Verify that matrices $[X\ Y]$ and $[X'\ Y']$ are associated to the same subspace U (their rows represent two bases of U) if and only if $[X\ Y] = P[X'\ Y']$ for some invertible $m \times m$ matrix P.

Observe that if $[X\ Y]$ and $[X'\ Y']$ are associated to the same subspace U, then either both X and X' are invertible, or both X and X' are singular.

To each point in a Grassmann space we have associated a (not uniquely determined) matrix $[X\ Y]$. If X is singular, then the corresponding point in the Grassmann space is called a point at infinity. Otherwise, it is called a finite point. Observe that a finite point $[X\ Y]$ can be represented also with the matrix $[I \quad X^{-1}Y]$. The matrix $X^{-1}Y$ in such a representation is uniquely determined by the point in the Grassmann space. So, if U and V are two m-dimensional subspaces that are finite points in the Grassmann space, then they can be represented with two uniquely determined $m \times n$ matrices T and S, respectively.

Problem. Prove that for every pair of m-dimensional subspaces U and V that are finite points in the Grassmann space, uniquely represented by $m \times n$ matrices T and S, respectively, we have: The subspaces U and V are adjacent if and only if the matrices T and S are adjacent.

Using this observation it is possible to deduce Chow's description of bijective maps ϕ on the Grassmann space preserving adjacency in both directions from Hua's structural result for bijective adjacency preserving maps on the set of all $m \times n$ matrices. As one may guess the problem is that we do not know whether ϕ maps finite points to finite points. Of course, this is not true in general. The idea is to choose a basis in the underlying vector space V in such a way that ϕ maps the set of finite points onto itself. Then one can apply Hua's fundamental theorem of geometry of rectangular matrices to obtain the description of the restriction of ϕ to the set of finite points. Once we know that ϕ behaves nicely on finite points it is not difficult to show that we have such a nice behaviour also at points at infinity.

3 Some proof techniques

The classical approach to adjacency preserving maps is based on maximal adjacent sets. Let $\mathcal{V}$ be either a space of hermitian matrices, or the space of all $m \times n$ matrices. A subset $\mathcal{M} \subset \mathcal{V}$ is called an adjacent set if any two matrices $A, B \in \mathcal{V}$, $A \neq B$, are adjacent. A subset $\mathcal{M} \subset \mathcal{V}$ is called a maximal adjacent set if it is an adjacent set, and if $\mathcal{N} \subset \mathcal{V}$ is an adjacent set such that $\mathcal{M} \subset \mathcal{N}$, then $\mathcal{M} = \mathcal{N}$. We first observe that if $\mathcal{M}$ is an adjacent set of matrices and $C \in \mathcal{V}$ is any matrix, then $\mathcal{M} + C = \{A + C \, : \, A \in \mathcal{M}\}$ is an adjacent set as well. Thus, if we want to understand the structure of adjacent sets, we only need to characterize those adjacent sets that contain the zero matrix. Clearly, if $\mathcal{M}$ is an adjacent set containing the zero matrix, then all nonzero members of $\mathcal{M}$ are matrices of rank one.

We will denote by $x, y, \ldots$ $m \times 1$ matrices and by $u^T, v^T, \ldots$ $1 \times n$ matrices. A matrix $A \in M_{m \times n}(\mathbb{F})$ is of rank one if and only if it can be written as $A = xu^T$ for some nonzero column and row matrices x and u^T. For nonzero $x \in M_{m\times 1}(\mathbb{F})$ and $u^T \in M_{1\times n}(\mathbb{F})$ we denote

$$L(x) = \{xv^T \, : \, v^T \in M_{1\times n}(\mathbb{F})\}$$

and

$$R(u^T) = \{yu^T \, : \, y \in M_{m\times 1}(\mathbb{F})\}.$$

For example, if $e_1 \in M_{m\times 1}(\mathbb{F})$ denotes the $m \times 1$ matrix whose all entries are zero, but the first one which is equal to one, then $L(e_1)$ is the set of all matrices having nonzero entries only in the first row. And clearly, $L(x) = PL(e_1)$, where $P \in M_m(\mathbb{F})$ is any invertible matrix satisfying $Pe_1 = x$.

Problem. Prove the following statements:

1. Rank one matrices xu^T and yv^T, $xu^T \neq yv^T$, are adjacent if and only if x and y are linearly dependent or u^T and v^T are linearly dependent.

2. Let $x \in M_{m\times 1}(\mathbb{F})$ and $u^T \in M_{1\times n}(\mathbb{F})$ be nonzero column and row matrices, respectively. Then both $L(x)$ and $R(u^T)$ are maximal adjacent sets. If $\mathcal{M} \subset M_{m\times n}(\mathbb{F})$ is an adjacent set and $0 \in \mathcal{M}$, then either there exists a nonzero $x \in M_{m\times 1}(\mathbb{F})$ such that $\mathcal{M} \subset L(x)$, or there exists a nonzero $u^T \in M_{1\times n}(\mathbb{F})$ such that $\mathcal{M} \subset R(u^T)$.

3. If $\mathcal{M} \subset M_{m\times n}(\mathbb{F})$ is an adjacent set, then there exists $A \in M_{m\times n}(\mathbb{F})$, and either there exists a nonzero $x \in M_{m\times 1}(\mathbb{F})$ such that $\mathcal{M} \subset A + L(x)$, or there exists a nonzero $u^T \in M_{1\times n}(\mathbb{F})$ such that $\mathcal{M} \subset A + R(u^T)$.

If $\phi : M_{m\times n}(\mathbb{F}) \to M_{m\times n}(\mathbb{F})$ is a bijective map preserving adjacency in both directions, then for every subset $\mathcal{M} \subset M_{m\times n}(\mathbb{F})$ the set $\phi(\mathcal{M})$ is a maximal adjacent set if and only if $\mathcal{M}$ is a maximal adjacent set. Hence, if we additionally assume that $\phi(0) = 0$, then for each nonzero $x \in M_{m\times 1}(\mathbb{F})$

- either there exists a nonzero $y \in M_{m\times 1}(\mathbb{F})$ such that $\phi(L(x)) = L(y)$, or
- there exists a nonzero $v^T \in M_{1\times n}(\mathbb{F})$ such that $\phi(L(x)) = R(v^T)$;

and for each nonzero $u^T \in M_{1\times n}(\mathbb{F})$

- either there exists a nonzero $y \in M_{m\times 1}(\mathbb{F})$ such that $\phi(R(u^T)) = L(y)$, or
- there exists a nonzero $v^T \in M_{1\times n}(\mathbb{F})$ such that $\phi(R(u^T)) = R(v^T)$.

Problem. Let $x, y \in M_{m\times 1}(\mathbb{F})$ and $u^T, v^T \in M_{1\times n}(\mathbb{F})$ be nonzero column and row matrices, respectively. Show that $L(x) \cap L(y)$ is either a singleton $\{0\}$, or $L(x) = L(y)$. When do we have the first possibility and when the second one? Formulate and verify an analogous statement for the sets $R(u^T)$ and $R(v^T)$. Describe the set $L(x) \cap R(u^T)$. Conclude that the intersection of two maximal adjacent sets containing zero is either a singleton $\{0\}$ or these two sets are equal (this happens if and only if both maximal adjacent sets are of type L or both are of type R; here the notions of type L and type R should be self-explanatory); or the intersection is not a singleton and is a proper subset of both maximal adjacent sets (this happens when one of them if of type L, while the other one is of type R).

It follows easily that

- either sets of type L are mapped into sets of type L and sets of type R are mapped into sets of type R; or
- sets of type L are mapped into sets of type R and sets of type R are mapped into sets of type L.

Let us consider the first possibility only. After replacing the map ϕ by the map $A \mapsto P\phi(A)$, where P is a suitable invertible matrix, we may assume with no loss of generality that $\phi(L(e_1)) = L(e_1)$. Clearly, $L(e_1)$ can be identified with $\mathbb{F}^n$.

Denote by E_{ij}, $1 \le i \le m$, $1 \le j \le n$, the matrix units, that is, E_{ij} is the matrix with the (i,j)-entry equal to one and all other entries equal to zero. We further denote by $f_2^T \in M_{1\times n}(\mathbb{F})$ the row matrix whose all entries are zero but the second entry that is equal to one.

Observation. $L(e_1) \cap (E_{11} + R(f_2^T)) = \{E_{11} + \lambda E_{12} \,:\, \lambda \in \mathbb{F}\}$.

The set $\{E_{11}+\lambda E_{12} \,:\, \lambda \in \mathbb{F}\} \subset L(e_1) \equiv \mathbb{F}^n$ geometrically represents the line through the point E_{11} with the direction vector E_{12}. In the above observation it is represented as an intersection of two maximal adjacent sets.

More generally, let $w, z \in \mathbb{F}^n$ be two vectors with $z \neq 0$. Then the set $\{w + \lambda z \,:\, \lambda \in \mathbb{F}\}$ is called the line through w with the direction vector z. We need here the fundamental theorem of affine geometry: If $\xi : \mathbb{F}^n \to \mathbb{F}^n$ is a bijective map satisfying $\xi(0) = 0$ and mapping each line in $\mathbb{F}^n$ onto some line in $\mathbb{F}^n$, then ξ is of the form

$$\xi(w_1, \ldots, w_n) = M(f(w_1), \ldots, f(w_n))$$

where $M : \mathbb{F}^n \to \mathbb{F}^n$ is an invertible linear transformation and $f : \mathbb{F} \to \mathbb{F}$ is an automorphism of the underlying field $\mathbb{F}$.

Having in mind the above observation it is easy to believe (and not very difficult to prove) that the restriction of ϕ to $L(e_1)$ satisfies all the assumptions of the fundamental theorem of affine geometry. Hence, ϕ maps $L(e_1)$ onto $L(e_1)$, all other sets of type L on the sets of the same type, all sets of type R on the sets of the same type, its restriction to $L(e_1)$ is of a very simple nice form, and so is true for all restrictions of ϕ to the sets of type L or R. With these ideas one has all the main tools needed to prove the classical Hua's fundamental theorem of geometry of rectangular matrices. On the way to the complete proof one needs to deal with quite a few non-trivial problems, but these are just technical problems, while all the essential ideas have been explained above.

Let us now turn to hermitian matrices.

Problem. Prove the following statements:

1. Each rank one matrix in H_n can be written as tP where t is a nonzero real number and P is a rank one projection (hermitian idempotent matrix).

2. Let $P, Q \in H_n$ be rank one projections. Rank one hermitian matrices tP and sQ, $tP \neq sQ$, are adjacent if and only if $P = Q$.

3. Let $P \in H_n$ be a rank one projection. Then $\{tP \,:\, t \in \mathbb{R}\}$ is a maximal adjacent set. If $\mathcal{M} \subset H_n$ is an adjacent set and $0 \in \mathcal{M}$, then there exists a projection P of rank one such that $\mathcal{M} \subset \{tP \,:\, t \in \mathbb{R}\}$.

4. Let $A, P \in H_n$ with P being a rank one projection. Then $\{A+tP \,:\, t \in \mathbb{R}\}$ is a maximal adjacent set. If $\mathcal{M} \subset H_n$ is an adjacent set, then there exist $A \in H_n$ and a projection P of rank one such that $\mathcal{M} \subset \{A+tP \,:\, t \in \mathbb{R}\}$.

Geometrically, the set $\{A + tP\,;\, t \in \mathbb{R}\}$ is a line through the point A with the direction vector P.

As above, if $\phi : H_n \to H_n$ is a bijective map preserving adjacency in both directions, then for every subset $\mathcal{M} \subset H_n$ the set $\phi(\mathcal{M})$ is a maximal adjacent set if and only if $\mathcal{M}$ is a maximal adjacent set. Hence, ϕ maps lines whose direction vectors are rank one projections onto the lines of the same type. Note that we cannot apply the fundamental theorem of the affine geometry since we do not know how ϕ maps all the other lines.

Actually, the point-line geometry that we have (where points are hermitian $n \times n$ matrices and lines are maximal adjacent sets) has some unusual properties. In particular, there are no triangles in this geometry, that is, there do not exist three pairwise distinct lines p, q, r such that each two of them intersect, but the intersection of all three of them is empty. Indeed, assume on the contrary that we have such lines p, q, r and denote the points of intersections with A, B, C:

Figure 1

As lines are maximal adjacent sets and since we know that any translation maps maximal adjacent sets to maximal adjacent sets, we may assume with no loss of generality that $A = 0$:

Figure 2

But then the point B (which belongs to the line p through the matrix 0) is adjacent to 0, and therefore B is a nonzero matrix of rank one. Thus, $B = tP$, and similarly, $C = sQ$ for some nonzero real numbers t, s and some rank one projections P and Q. As B and C belong to the line r, they are adjacent. As we already know this yields that $P = Q$, and consequently, $p = q$, a contradiction.

On one hand, such a “no trinagle” property looks rather strange, but on the other hand such “exotic” geometries have been studied a lot and pure geometrical results can be applied to get the desired conclusion of Hua’s theorem.

The next idea we would like to explain is the "reduction to small pieces". We will explain this idea in the hermitian case. So, assume that $\phi : H_n \to H_n$ is a bijective map preserving adjacency in both directions and satisfying $\phi(0) = 0$. Then rank one matrices (these are exactly matrices adjacent to 0) are mapped into rank one matrices. Actually, for $A \in H_n$ we have: the matrix A is of rank one if and only if the matrix $\phi(A)$ is of rank one. Now, if A is of rank two, then we can find a rank one $B \in H_n$ that is adjacent to A. It follows that $\phi(A)$ is adjacent to $\phi(B)$ which is of rank one. Thus, $\phi(A)$ is either the zero matrix, or it is of rank one, or it is of rank two. The first possibility cannnot occur because of $\phi(0) = 0$ and the bijectivity assumption. The second possibility would imply that $\operatorname{rank} A = 1$, again a contradiction.

To conclude, we have $\operatorname{rank} \phi(A) = 2 \iff \operatorname{rank} A = 2$. It is well-known that if $A \in H_n$ is a matrix of rank two, then there exists an invertible complex matrix T such that TAT^* is equal to either $E_{11}+E_{22}$, or $-E_{11}-E_{22}$, or $E_{11}-E_{22}$. There are now several possibilities concerning A and $\phi(A)$ and we will consider just one of them. So, assume that $TAT^* = E_{11}+E_{22} = S\phi(A)S^*$ for some invertible matrices S and T. After composing ϕ with the two congruence transformations we may, and we will assume that $\phi(E_{11} + E_{22}) = E_{11} + E_{22}$. The next step is to show that then every hermitian matrix of the form

$$\begin{bmatrix} * & * & 0 & \dots & 0 \\ * & * & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{5}$$

is mapped into a matrix of the same type.

Assume for a moment that we have already verified the above statement. Then we can restrict our attention to the restriction of ϕ to the subset of hermitian matrices having nonzero entries only in the upper left 2×2 corner. The set of all such matrices can be identified with H_2, the set of all 2×2 hermitian matrices.

Thus we need to solve two problems. The first one is to describe the general form of bijective maps on H_2 preserving adjacency in both directions and the second one is to get the global picture once we know that the behaviour of ϕ on small 2×2 pieces is nice. The second part is technically complicated, but it does not require any deep original ideas. For the first probem we can apply elementary linear algebra techniques hoping that this low dimensional case is much easier to treat than the general case - this turns out to be true. But we have another possibility. Namely, we have shown before that this problem is equivalent to the problem of describing the general form of bijective maps on the Minkowski space of all spacetime events preserving coherency in both directions. And because this problem has been studied a lot due to its importance in the relativity theory, we can apply the results that have been obtained in this equivalent setting.

Hence, it remains to explain the main ideas needed to verify the statement that if $\phi(E_{11} + E_{22}) = E_{11} + E_{22}$, then the set of matrices of the form (5) is invariant under ϕ.

Problem. Assume that $P \in H_n$ is a matrix that is adjacent to both the zero matrix and $E_{11} + E_{22}$. Prove that then

$$P = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix},$$

where Q is a 2×2 projection of rank one. The converse is trivial.

It follows that projections of rank one having nonzero entries only in the upper left 2×2 corner are mapped into rank one projections of the same type.

Problem. Assume that $A \in H_n$ is a rank one matrix that is not a projection of rank one. Prove that then A is a matrix of the form (5) if and only if A is adjacent to some projection of rank one having nonzero entries only in the upper left 2×2 corner.

It then follows immediately that all rank one matrices of the form (5) are mapped into matrices of the same form. And then it is not difficult to verify the same conclusion for rank two matrices.

This completes the explanation of what we had in mind when speaking of "reduction to small pieces".

The next idea that we would like to explain is to reduce the problem of describing the general form of adjacency preserving maps to the problem of characterizing order-preserving maps on idempotent matrices. We denote by $P_n(\mathbb{F}) \subset M_n(\mathbb{F})$ the subset of all idempotents, that is, $P_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : A^2 = A\}$.

Problem. Prove that $A \in M_n(\mathbb{F})$ is an idempotent if and only if there exists an invertible matrix T such that

$$A = T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

Here, I_r is the $r \times r$ identity matrix.

Note that $r = \operatorname{rank} A$. When $r = 0$ or $r = n$ (A is the zero matrix or A is the identity matrix) we speak of trivial idempotents. In the language of operators the above statement reads as follows: An endomorphism of the vector space $\mathbb{F}^n$ is idempotent if and only if $\mathbb{F}^n$ is the direct sum of the image of A and the kernel of A and A acts like the identity on its image.

Recall that for two idempotents $P, Q \in P_n(\mathbb{F})$ we write $P \leq Q$ if and only if $PQ = QP = P$.

Problem. Let $P, Q \in P_n(\mathbb{F})$. Prove that $P \leq Q$ if and only if there exists an invertible matrix T such that

$$P = T \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} T^{-1} \quad \text{and} \quad Q = T \begin{bmatrix} I_r & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & 0 \end{bmatrix} T^{-1}.$$

Note that $r = \operatorname{rank} P$ and $r + s = \operatorname{rank} Q$. Some bordering zeroes may be absent when $P = 0$ or $Q = I$ or $P = Q$. In the language of operators we have: $P \leq Q$ if and only if $\operatorname{Im} P \subset \operatorname{Im} Q$ and $\operatorname{Ker} Q \subset \operatorname{Ker} P$.

We consider now an adjacency preserving map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$. We assume that $\phi(0) = 0$ and $\phi(I) = I$. We claim that then ϕ maps idempotents into idempotents and that the restriction of ϕ to the set of idempotents preserves order. Indeed, let P and Q be idempotents with $P \leq Q$. Then P and Q are simultaneously similar to diagonal idempotents, and therefore, we can find a chain of idempotents

$$0 = P_0 \leq P_1 \leq P_2 \leq \ldots \leq P_{n-1} \leq P_n = I$$

such that $\operatorname{rank} P_k = k$, $k = 0, 1, \ldots, n$, P_k and P_{k+1} are adjacent, $k = 0, 1, \ldots, n-1$, and P and Q are members of this chain (let the similarity transformation that brings P and Q to the diagonal form be induced by T, that is, $TPT^{-1} = E_{11} + \ldots + E_{pp}$ and $TQT^{-1} = E_{11} + \ldots + E_{pp} + E_{p+1,p+1} + \ldots + E_{qq}$. Then $TP_1T^{-1} = E_{11}$, $TP_2T^{-1} = E_{11} + E_{22}$,...). We denote $\phi(P_k) = Q_k$, $k = 0, 1, \ldots, n$. We know that $Q_0 = 0$ and $Q_n = I$. Now, Q_1 is adjacent to 0, and therefore, Q_1 is of rank one. Since Q_2 is adjacent to Q_1, it is of rank at most two. Proceeding in the same way we conclude that $\operatorname{rank} Q_k \leq k$, $k = 0, 1, \ldots, n$. In particular, Q_{n-1} is a matrix of rank at most $n - 1$ adjacent to I.

Problem. Let $P \in P_n(\mathbb{F})$ and $A \in M_n(\mathbb{F})$ satisfy

$$\operatorname{rank} P = \operatorname{rank} A + \operatorname{rank} (P - A).$$

Prove that then A is an idempotent and $A \leq P$. Hint: Observe that from $\operatorname{Im} P \subset \operatorname{Im} A + \operatorname{Im} (P - A)$ and the above rank additivity assumption one gets $\operatorname{Im} P = \operatorname{Im} A \oplus \operatorname{Im} (P - A)$. For $x \in \operatorname{Im} A \subset \operatorname{Im} P$ we have $x = Px = Ax + (P - A)x$, and therefore $Ax = x$ and $(P - A)x = 0$.

Hence, Q_{n-1} must be an idempotent of rank $n-1$. Further, Q_{n-2} is a matrix of rank at most $n - 2$ that is adjacent to Q_{n-1} which is an idempotent of rank $n - 1$. It follows that Q_{n-2} is an idempotent of rank $n - 2$ satisfying $Q_{n-2} \leq$

Q_{n-1}. Continuing in this way we conclude that all the Q_k's are idempotents with $Q_k \leq Q_{k+1}$, $k = 0, 1, \ldots, n-1$. In particular, $\phi(P)$ and $\phi(Q)$ are idempotents satisfying $\phi(P) \leq \phi(Q)$, as desired.

Thus, the following idea can be used to study bijective maps ϕ on $M_{m\times n}(\mathbb{F})$ preserving adjacency in both directions. Assume that $m \geq n$. With no loss of generality we may assume that $\phi(0) = 0$. It can be proved that there exists a matrix of rank n that is mapped into a matrix of rank n. This can be further reduced to the case when $E_{11} + \ldots + E_{nn}$ is mapped into itself which yields that matrices having nonzero entries only in the upper $n \times n$ block are mapped onto itself. Restricting to such matrices we have the situation as above and therefore idempotents are mapped into idempotents and order is preserved in both directions. This can be then used to show that either idempotents with the same image are mapped into idempotents with the same image, or idempotents with the same image are mapped into idempotents with the same kernel. Therefore we have an induced map on the set of subspaces of $\mathbb{F}^n$ and it is natural to apply projective geometry. Applying fundamental theorem of projective geometry we conclude that the restriction of the map ϕ to the set of idempotents is of the nice form and then one has to extend this nice behaviour to all matrices.

If we start with the weaker assumption that adjacency is preserved in one direction only and we do not assume bijectivity, the study becomes much more difficult and besides standard forms one has to study also certain degenerate forms. Still, the idea to reduce the study of adjacency preserving maps to order preserving maps on idempotents is very efficient. The reason is that idempotents in $M_n(\mathbb{F})$ are exactly the points that are "between" 0 and I with respect to the arithmetic distance, that is, if $A \in M_n(\mathbb{F})$ and

$$n = d(0, I) = d(0, A) + d(A, I)$$

then A has to be an idempotent. The converse is obviously true.

4 Surprising examples

Quantum effects play an important role in mathematical foundations of quantum mechanics. They correspond to yes-no measurements that can be unsharp. In the Hilbert space formalism they are represented by linear bounded self-adjoint operators A acting on a Hilbert space H satisfying $0 \leq A \leq I$. Such operators are usually called (Hilbert space) effects. The Hilbert space effect algebra $E(H)$ is the set of all effects. Symmetries of the effect algebra $E(H)$ are bijective maps $\phi : E(H) \to E(H)$ which preserve certain operations and/or relations defined on $E(H)$ that are important in various aspects of quantum theory. For the sake of simplicity we will restrict our attention to the finite-dimensional case. If $\dim H = n$, then we will write $E_n = E(H)$ and after identifying operators with matrices we can consider E_n as the set of all $n \times n$ complex hermitian matrices whose all eigenvalues belong to the unit interval $[0, 1]$.

In the second section we have explained that the study of various symmetries on H_n can be reduced to the fundamental theorem of geometry of hermitian matrices. One can use exactly the same ideas to reduce the problem of description of the general form of various symmetries of E_n to the problem of describing the general form of bijective maps $\phi : E_n \to E_n$ which preserve adjacency in both directions. But it turns out that the study of adjacency preserving maps on E_n is far more complicated than the corresponding problem on H_n. Namely, in the paper: L. Molnár and E. Kovács, An extension of a characterization of the automorphisms of Hilbert space effect algebras, *Rep. Math. Phys.* **52** (2003), 141-149, the authors claim that for any fixed invertible matrix $T \in E_n$, the transformation

$$A \mapsto \left(\frac{T^2}{2I - T^2}\right)^{-1/2} \left((I - T^2 + T(I + A)^{-1}T)^{-1} - I\right) \left(\frac{T^2}{2I - T^2}\right)^{-1/2} \tag{6}$$

is a bijective map of E_n onto itself which preserves order in both directions. No proof of this statement is given. We will later on see that the authors were right and that such maps also preserve adjacency in both directions (this is not surprising if we recall the connection between maps preserving order in both directions and maps preserving adjacency in both directions).

This happens ocasionally in mathematics. We have a certain problem and we have an idea how to solve it (in our case this is the problem of describing the general form of bijective maps on E_n which preserve adjacency in both directions, and then consequently we should be able to get various characterizations of symmetries of E_n as rather easy consequences; we further have at least two ideas how to approach this problem: either by reducing the problem to small pieces, that is, to the case when $n = 2$, and then using the results on coherency preserving maps on subsets of Minkowski space, or to use maximal adjacent sets which are intersections of lines in H_n with E_n and then applying geometrical techniques). But we have no idea how the result should look like - in our case we want to have a "nice" description of bijective preservers of adjacency on E_n, but we do not know what "nice" should mean: this "nice" description should cover maps appearing in example (6).

For quite some time it was believed that the example (6) simply tells that there is no nice structural result for bijective maps on E_n preserving adjacency in both directions. This was changed once the following explanation of the above example had been given.

Let $A, B \in H_n$ be positive invertible matrices. Then A and B are adjacent if and only if A^{-1} and B^{-1} are adjacent. Indeed,

$$B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$$

is of rank one if and only if $A - B$ is a rank one operator. For two hermitian matrices A, B such that $B - A$ is positive invertible, we set $[A, B] = \{C \in H_n : A \le C \le B\}$. In particular, $E_n = [0, I]$.

Problem. Let $A, B \in H_n$ be such that $A \leq B$, and both A and $B - A$ are positive invertible matrices. Prove that the map $X \to X^{-1}$ is a bijection of $[A, B]$ onto $[B^{-1}, A^{-1}]$. Hint: Show first that for any two positive invertible matrices C, D we have $C \leq D \iff I \leq C^{-1/2} D C^{-1/2}$ and if C and D commute then $C \leq D \iff D^{-1} \leq C^{-1}$.

Let $A, B \in H_n$ be such that $B - A$ is a positive invertible matrix and let $T \in H_n$ be any hermitian matrix. Then the translation map $X \mapsto X + T$ is a bijective map of $[A, B]$ onto $[A + T, B + T]$ preserving adjacency in both directions. Further, if T is any invertible $n \times n$ complex matrix, then the transformation $X \mapsto TXT^*$ is a bijective map of $[A, B]$ onto $[TAT^*, TBT^*]$ preserving adjacency in both directions. And finally, if $A, B \in H_n$ with both A and $B - A$ being positive invertible, then the bijective map $X \mapsto X^{-1}$ of $[A, B]$ onto $[B^{-1}, A^{-1}]$ preserves adjacency in both directions as well. The map (6) is a product of such maps. Indeed, the map ξ defined as a product of maps:

$$A \mapsto I + A \mapsto (I + A)^{-1} \mapsto T(I + A)^{-1}T \mapsto$$
$$I - T^2 + T(I + A)^{-1}T \mapsto (I - T^2 + T(I + A)^{-1}T)^{-1} \mapsto$$
$$(I - T^2 + T(I + A)^{-1}T)^{-1} - I,$$

is a bijective map of E_n onto $[\xi(0), \xi(I)]$ preserving adjacency in both directions. Clearly, $\xi(0) = 0$ and $\xi(I) = T^2(2I - T^2)^{-1}$. Composing ξ with the suitable congruence transformation we obtain the map (6).

Once we understand the above example, the problem becomes much easier, and indeed, the following result has been proved recently. In order to formulate it we need two more notions. If $T \in M_n(\mathbb{C})$, then $\|T\|$ denotes the operator norm of $\|T\|$, that is, the square root of the largest eigenvalue of TT^*. Let p be any real number satisfying $p < 1$. Define a real function $f_p : [0, 1] \to [0, 1]$ by

$$f_p(x) = \frac{x}{px + (1 - p)}, \quad x \in [0, 1].$$

If $A \in E_n$, then there exists a unitary matrix U such that $A = UDU^*$, where D is a diagonal matrix $D = \operatorname{diag}(t_1, \ldots, t_n)$. Here, the diagonal entries $t_1, \ldots, t_n$ are eigenvalues of A, and therefore, $t_j \in [0, 1]$, $j = 1, \ldots, n$. The matrix $f_p(A)$ is defined by

$$f_p(A) = U \operatorname{diag}(f_p(t_1), \ldots, f_p(t_n)) U^*.$$

It is not difficult to verify that this definition is unambiguous (the unitary matrix U is not uniquely determined and the diagonal matrix D is determined up to a permutation of diagonal entries).

Theorem 4.1 *Let $n \geq 3$. Assume that $\phi : E_n \to E_n$ is a bijective map preserving adjacency in both directions. Then there exist real numbers $p, q \in (-\infty, 1)$ and an invertible $n \times n$ complex matrix T with $\|T\| \leq 1$ such that either*

$$\phi(A) = f_q\left((f_p(TT^*))^{-1/2} f_p(TAT^*) \left(f_p(TT^*)\right)^{-1/2} \right), \quad A \in E_n,$$

or

$$\phi(A) = f_q\left((f_p(TT^*))^{-1/2}\, f_p(T(I-A)T^*)\, (f_p(TT^*))^{-1/2}\right), \quad A \in E_n,$$

or

$$\phi(A) = f_q\left((f_p(TT^*))^{-1/2}\, f_p(T\overline{A}T^*)\, (f_p(TT^*))^{-1/2}\right), \quad A \in E_n,$$

or

$$\phi(A) = f_q\left((f_p(TT^*))^{-1/2}\, f_p(T(I-\overline{A})T^*)\, (f_p(TT^*))^{-1/2}\right), \quad A \in E_n.$$

One of the first problems that we were dealing with was the question wheather we can omit the bijectivity assumption in Theorem 1.2 and still get the same conclusion with the only difference that f is not an automorphism but just an endomorphism of the underlying field $\mathbb{F}$. Much to our surprise it turned out that the answer depends on the underlying field. In particular, the answer is positive in the case of real matrices and negative in the case of complex matrices. As we will see in the following examples the main reason for this difference between real and complex matrices is the existence of nonzero nonsurjective endomorphisms of the complex field. The proof that such endomorphisms exist is too complicated to be included here. It is well-known that the only nonzero endomorphism of the real field is the identity.

Let $f : \mathbb{C} \to \mathbb{C}$ be a nonzero nonsurjective endomorphism of the complex field, c a complex number that is not contained in the range of f, and define a map $\phi : M_{m\times n}(\mathbb{C}) \to M_{m\times n}(\mathbb{C})$ by

$$\phi\left(\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-2,1} & a_{m-2,2} & \dots & a_{m-2,n} \\ a_{m-1,1} & a_{m-1,2} & \dots & a_{m-1,n} \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}\right)$$

$$= \begin{bmatrix} f(a_{11}) & f(a_{12}) & \dots & f(a_{1n}) \\ \vdots & \vdots & \ddots & \vdots \\ f(a_{m-2,1}) & f(a_{m-2,2}) & \dots & f(a_{m-2,n}) \\ f(a_{m-1,1}) + cf(a_{m1}) & f(a_{m-1,2}) + cf(a_{m2}) & \dots & f(a_{m-1,n}) + cf(a_{mn}) \\ 0 & 0 & \dots & 0 \end{bmatrix}. \tag{7}$$

Problem. Prove that

- ϕ is additive, that is, $\phi(A+B) = \phi(A) + \phi(B)$, $A, B \in M_{m\times n}(\mathbb{C})$,
- ϕ preserves rank one, that is, if $A \in M_{m\times n}(\mathbb{C})$ is of rank one, then $\phi(A)$ is of rank one, and consequently, ϕ preserves adjacency,

- ϕ does not preserve adjacency in both directions.

The map ϕ is a composition of two maps: we have first applied the endomorphism f entrywise and then we have replaced the last row by zero and the $(m-1)$-st row by the sum of the $(m-1)$-st row and the m-th row multiplied by c. We could do the same with columns instead of rows. Of course, we could make the example more complicated by adding the scalar multiples of the m-th row to other rows as well.

Thus, we have an example of an adjacency preserving map ϕ on $M_{m\times n}(\mathbb{C})$ which maps $m\times n$ matrices into the set of $m\times n$ matrices with the last row equal to zero. If we compose this map with a similar map where the roles of columns and rows are interchanged we arrive at an adjacency preserving map ϕ on $M_{m\times n}(\mathbb{C})$ whose range is contained in the set of $m\times n$ matrices with both the last row and the last column equal to zero. We can now go on and compose the obtained map with yet another map of this type to get an adjacency preserving map ϕ on $M_{m\times n}(\mathbb{C})$ whose range is contained in the set of $m\times n$ matrices with the last two rows and the last column equal to zero. After finitely many steps we end up with and adjacency preserving map $\phi : M_{m\times n}(\mathbb{C}) \to M_{m\times n}(\mathbb{C})$ with the property that each matrix A is mapped into a matrix of the form

$$\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

where $*$ stands for some $p\times q$ matrix. Here, p and q are arbitrary positive integers, $p\le m$ and $q\le n$.

Even more, it is possible to modify the above example in such a way that we get a map preserving adjacency in both directions. The details are too complicated to be included in these lecture notes.

Let us now present another example illustrating that the problem of the optimal version of Hua's fundamental theorem of geometry of rectangular matrices is far more complicated than the corresponding problem for hermitian matrices. For $A\in M_{m\times n}(\mathbb{C})$ we denote by A^{1c} and A^{1r} the first column and the first row of A, respectively. Hence, A_f^{1c} is the $m\times 1$ matrix obtained in the following way: we take the first column of A and apply f entrywise. We define a map $\phi : M_{m\times n}(\mathbb{C}) \to M_{m\times n}(\mathbb{C})$ by

$$\phi(A) = A_f - \frac{c}{1+cf(a_{11})} A_f^{1c}\, A_f^{1r}, \quad A\in M_{m\times n}(\mathbb{C}). \tag{9}$$

Problem. Prove that ϕ preserves adjacency.

It is clear that in the above example the first row and the first column can be replaced by other columns and rows. And then, as a compositum of adjacency preserving maps preserves adjacency, we may combine such maps and those described in the previous example to get adjacency preserving maps that seem to be too complicated to be described nicely.

But we have not exhausted the list of surpring examples yet. We will next show that if $M \in M_m(\mathbb{C})$, $N \in M_{m\times n}(\mathbb{C})$, $L \in M_{n\times m}(\mathbb{C})$, and $K \in M_n(\mathbb{C})$ are matrices such that

$$E = \begin{bmatrix} M & N \\ L & K \end{bmatrix} \in M_{m+n}(\mathbb{C})$$

is invertible and for every $A \in M_{m\times n}(\mathbb{C})$ the matrix $M + A_f L$ is invertible then the map $\phi : M_{m\times n}(\mathbb{C}) \to M_{m\times n}(\mathbb{C})$ defined by

$$\phi(A) = (M + A_f L)^{-1}(N + A_f K) \tag{10}$$

preserves adjacency in both directions.

To sketch the proof of this statement it is useful to recall the connection between the adjacency relation in the set of matrices and the adjacency relation in Grassmann spaces. It is clear that $A, B \in M_{m\times n}(\mathbb{C})$ are adjacent matrices if and only if A_f and B_f are adjacent. Equivalently, the row spaces of matrices $[\, I \quad A_f \,]$ and $[\, I \quad B_f \,]$ are adjacent. Now, since E is invertible, the row spaces of matrices

$$[\, I \quad A_f \,] \begin{bmatrix} M & N \\ L & K \end{bmatrix} = [\, M + A_f L \quad N + A_f K \,]$$

and

$$[\, M + B_f L \quad N + B_f K \,]$$

are adjacent if and only if the matrices A and B are adjacent. We know that the row space of the matrix $[\, M + A_f L \quad N + A_f K \,]$ is the same as the row space of the matrix

$$(M + A_f L)^{-1} [\, M + A_f L \quad N + A_f K \,] = [\, I \quad (M + A_f L)^{-1}(N + A_f K) \,].$$

Hence, we conclude that the row spaces of matrices

$$[\, I \quad (M + A_f L)^{-1}(N + A_f K) \,]$$

and

$$[\, I \quad (M + B_f L)^{-1}(N + B_f K) \,]$$

are adjacent, and consequently, $\phi(A)$ and $\phi(B)$ are adjacent if and only if A and B are adjacent, as desired.

The last two examples are not unrelated.

Problem. Show that each map of the form (9) can be expressed as a map of the form (10). Hint: Choose $M = I$, $N = 0$, and $K = I$.

Let $\mathbb{F}$ be an infinite field. Assume that n is odd, $m \geq n$, and $p, q \geq n + 1$. Let $M^j_{m\times n}(\mathbb{F})$, $j = 1, \ldots, n$, denote the set of all $m \times n$ matrices of rank j. We define $\phi : M_{m\times n}(\mathbb{F}) \to M_{p\times q}(\mathbb{F})$ in the following way. Set $\phi(0) = 0$. Let φ_j be a map from $M^j_{m\times n}(\mathbb{F})$ into $\mathbb{F}$, $j = 1, \ldots, n$, with the propety that $\varphi_j(A) \neq \varphi_j(B)$ whenever $A, B \in M^j_{m\times n}(\mathbb{F})$ are adjacent. In particular, this property is satisfied when φ_j is injective. Set

$$\phi(A) = \begin{bmatrix} 1 & \varphi_1(A) & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

for every $A \in M^1_{m\times n}(\mathbb{F})$ and

$$\phi(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & \varphi_2(A) & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

for every $A \in M^2_{m\times n}(\mathbb{F})$. We continue in a similar way. For every $A \in M^3_{m\times n}(\mathbb{F})$ we set $\phi(A) = E_{11} + E_{22} + E_{33} + \varphi_3(A)E_{34}$, for every $A \in M^4_{m\times n}(\mathbb{F})$ we set $\phi(A) = E_{11}+E_{22}+E_{33}+E_{44}+\varphi_4(A)E_{54}$,..., and finally, for every $A \in M^n_{m\times n}(\mathbb{F})$ we set $\phi(A) = E_{11} + E_{22} + \ldots + E_{nn} + \varphi_n(A)E_{n,n+1}$.

Problem. Prove that ϕ preserves adjacency.

All these examples show that the problem of obtaining the optimal version of Hua's fundamental theorem in the case of rectangular matrices is much more difficult than the corresponding problem in the case of complex hermitian matrices. Still, it is possible to give a satisfactory answer. For the details see the fifth and the seventh paper in the list of the references at the end of these lecture notes. Roughly speaking, we say that each map of the form (10) composed with a standard embedding of $m \times n$ matrices into $p \times q$ matrices possibly composed with the transposition is a standard map. If we have an adjacency preserving map from the set of all $n \times n$ matrices into the set of all $p \times q$ matrices, we would like to conclude that such a map is either standard or degenerate. Here, degenerate maps are maps having a similar behaviour as the map given in our last example. Of course, such a result cannot be true because of the example (7). To avoid troubles with this kind of maps we need to have an additional assumption. The most natural is the following one: because every adjacency preserving map is a contraction with respect to the arithmetic distance, the distance between any two elements in the range of an adjacency preserving map acting between the set of $n \times n$ and the set of $p \times q$ matrices is at most n. We need to assume that this maximal possible distance is attained at one pair of matrices and then we get the desired result. Note that the domain is the set of square matrices. Once again, the example (7) shows that such a result cannot be extended to the nonsquare case. The problem becomes much easier if every nonzero endomorphism of the underlying field is automatically surjective (the filed of real numbers, the field of rational numbers, and each finite field are examples of such fields). Namely, the construction of examples (7), (9), and (10) was based on the existence of nonzero nonsurjective endomorphisms of the complex field. As already mentioned, the precise formulation of these results can be found in the references. It should be mentioned here that to the best of our knowledge the problem of finding the optimal version of Hua's fundamental theorem of geometry of skew-symmetric matrices is completely open, while only one partial result is known in the symmetric case (see the second paper in the references at the end of these lecture notes).

5 Matrices over division rings

So far we have worked with matrices over fields. Hua's fundamental theorems of geometry of matrices were formulated and proved for matrices over (not necessarily commutative) division rings. In the last section we will recall the definition of the rank in the noncommutative case and briefly emphasize some differences that appear when working in this more general setting. Following the standard notation used by mathematicians working in abstract algebra we will use a different notation for the transpose of a matrix than it is common in linear algebra: the transpose will be denoted by the lower case t appearing as a left superscript.

Let A be an $m \times n$ matrix with entries in a division ring $\mathbb{D}$. We will always consider $\mathbb{D}^n$, the set of all $1 \times n$ matrices, as a left vector space over $\mathbb{D}$. Correspondingly, we have the right vector space of all $m \times 1$ matrices ${}^t\mathbb{D}^m$. The row space of A is defined to be the left vector subspace of $\mathbb{D}^n$ generated by the rows of A, and the row rank of A is defined to be the dimension of this subspace. Similarly, the column rank of A is the dimension of the right vector space generated by the columns of A. This space is called the column space of A. These two ranks are equal for every matrix over $\mathbb{D}$ and this common value is called the rank of a matrix. If $\operatorname{rank} A = r$, then there exist invertible matrices $T \in M_m(\mathbb{D})$ and $S \in M_n(\mathbb{D})$ such that

$$TAS = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{11}$$

Here, I_r denotes the $r \times r$ identity matrix and the zeroes stand for zero matrices of the appropriate sizes.

As in the commutative case the set of matrices $M_{m\times n}(\mathbb{D})$ equipped with the distance d defined by

$$d(A,B) = \operatorname{rank}(A-B), \quad A,B \in M_{m\times n}(\mathbb{D}),$$

is a metric space and two matrices A and B are adjacent if $d(A,B)=1$.

Let $a \in \mathbb{D}^n$ and ${}^tb \in {}^t\mathbb{D}^m$ be any nonzero vectors. Then ${}^tba = ({}^tb)a$ is a matrix of rank one. Every matrix of rank one can be written in this form. It is easy to verify that two rank one matrices ${}^tba$ and ${}^tdc$, ${}^tba \neq {}^tdc$, are adjacent if and only if a and c are linearly dependent or ${}^tb$ and ${}^td$ are linearly dependent.

Let us now point the main difference between the commutative and the non-commutative case.

Problem. Find an example showing that the rank of a matrix A need not be equal to the rank of its transpose ${}^tA$.

Recall that $\tau : \mathbb{D} \to \mathbb{D}$ is called an anti-endomorphism if it is additive and anti-multiplicative, that is, $\tau(\lambda\mu) = \tau(\mu)\tau(\lambda)$, $\lambda, \mu \in \mathbb{D}$.

We have the following analogue of the invariance of the rank under the transposition in the commutative case.

Problem. Let $A \in M_{m\times n}(\mathbb{D})$ and let τ be a non-zero anti-endomorphism of $\mathbb{D}$. Prove that then $\operatorname{rank} A = \operatorname{rank} {}^t(A_\tau)$.

It is trivial to see that the transposition is not necessarily anti-multiplicative in the noncommutative case. However, we have the following analogue.

Problem. Let $A,B \in M_n(\mathbb{D})$ and let τ be a non-zero anti-endomorphism of $\mathbb{D}$. Prove that then ${}^t(AB)_\tau = {}^tB_\tau\, {}^tA_\tau$.

A short list of some recent papers and a book treating these problems - further references can be found therein:

- W.-l. Huang and P. Šemrl, *Adjacency preserving maps on hermitian matrices*, Canad. J. Math. **60** (2008), 1050-1066.
- W.-l. Huang, R. Höfer, and Z.-X. Wan, *Adjacency preserving mappings of symmetric and Hermitian matrices*, Aequationes Math. **67** (2004), 132-139.
- W.-l. Huang and Z.-X. Wan, *Adjacency preserving mappings of rectangular matrices*, Beiträge Algebra Geom. **45** (2004), 435-446.

- P. Šemrl, *Comparability preserving maps on Hilbert space effect algebras*, Comm. Math. Phys. **313** (2012), 375-384.
- P. Šemrl, *The optimal version of Hua's fundamental theorem of geometry of rectangular matrices*, to appear in Mem. Amer. Math. Soc.
- P. Šemrl, *Symmetries of Hilbert space effect algebras*, to appear in J. Lond. Math. Soc.
- P. Šemrl, *Hua's fundamental theorem of geometry of rectangular matrices over EAS division rings*, preprint.
- Z.-X. Wan, Geometry of matrices, World Scientific, Singapore, 1996.