A Finsler approach to the angular metrics

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Workshop on Matrix Geometries and Applications
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Overview

1. Introduction to angular metrics
2. From angular metrics to short paths in $\mathcal{U}(n)$
3. Further related results

Angular metrics

How can we define distances between subspaces?

Angular metrics

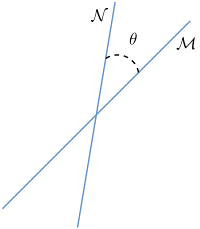
Angular metrics

Let $x, y \in \mathbb{C}^n$, $\mathcal{M} = \text{span}\{x\}$ and $\mathcal{N} = \text{span}\{y\}$. Then

$$\cos\theta = \frac{|\langle x,\, y\rangle|}{\|x\|\,\|y\|} = \|P_{\mathcal{M}} P_{\mathcal{N}}\| = s_1(P_{\mathcal{M}} P_{\mathcal{N}})\,.$$

$\left(\theta \in \left[0, \frac{\pi}{2}\right]\right)$. A natural distance between $\mathcal{M}$ and $\mathcal{N}$ is

$$\rho(\mathcal{M}, \mathcal{N}) = \theta.$$



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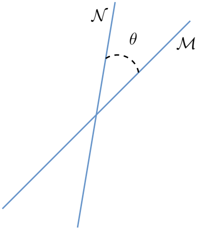

Remark: We can also define:

$$d_{gap}(\mathcal{M}, \mathcal{N}) = \|P_{\mathcal{M}} - P_{\mathcal{N}}\|_{sp} = \sin\theta.$$

However, this metric is not *intrinsic* in the following sense: let (X, d) be a metric space. Given a continuous curve $\alpha : [a, b] \to X$, we can define

$$L_d(\alpha) = \sup_{a=t_0<t_1<\ldots<t_n=b} \sum_{k=1}^{n} d\big(\alpha(t_k), \alpha(t_{k-1})\big), \quad \text{and} \quad \widehat{d}(x, y) = \inf_{\gamma:\, x\leftrightarrow y} L(\gamma).$$

Then, d is called intrinsic if $d = \widehat{d}$. (For d_{gap}, it holds that $\widehat{d}_{gap} = \rho$).

Angular metrics

What happens for subspaces of higher dimensions?

Angular metrics

The Grassmannian, $\mathcal{G}_n$, is the set of all the subspaces of $\mathbb{C}^n$. Given $\mathcal{M} \in \mathcal{G}_n$

$$\mathcal{M} \quad \Longleftrightarrow \quad P_{\mathcal{M}} \quad \Longleftrightarrow \quad S_{\mathcal{M}} = 2P_{\mathcal{M}} - I.$$

- Using these identifications, we can endow $\mathcal{G}_n$ with a topology such that it becomes a compact space.
- The connected components of $\mathcal{G}_n$ are the subsets $\mathcal{G}_{n,m}$ consisting of those subspaces of dimension m.

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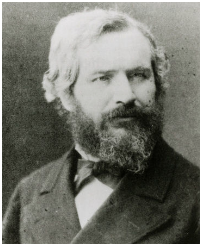
Definition (C. Jordan 1875)

Given two k-dimensional subspaces $\mathcal{M},\ \mathcal{N} \subseteq \mathbb{C}^n$, the principal angles between them are the angles $\theta_i(\mathcal{M},\mathcal{N})$ belonging to $[0,\pi/2]$, $1 \le i \le k$, whose cosinus are the k-biggest singular values of $P_{\mathcal{M}}P_{\mathcal{N}}$. In other words:

$$\cos\ \theta(\mathcal{M},\mathcal{N}) := \left(s_1\left(P_{\mathcal{M}}P_{\mathcal{N}}\right),\ \cdots,\ s_k\left(P_{\mathcal{M}}P_{\mathcal{N}}\right)\right).$$



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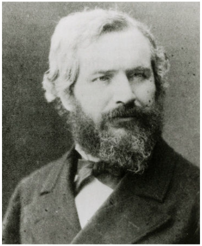

Now, given a norm $\|\cdot\|$ in $\mathbb{R}^k$:

Is the function $d(\mathcal{M},\mathcal{N}) := \|\theta\,(\mathcal{M},\mathcal{N})\,\|$ a metric in each $\mathcal{G}_{n,m}$?

Angular metrics

Recall that $\phi : \mathbb{R}^n \to \mathbb{R}$ is a gauge symmetric function if it satisfies:

- ϕ is a norm;
- $\phi(x_1, \ldots, x_n) = \phi(|x_1|, \ldots, |x_n|)$;
- $\phi(x_1, \ldots, x_n) = \phi(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

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Recall also, that a norm $\| \cdot \|$ is called unitarily invariant if for every $A \in \mathcal{M}_n(\mathbb{C})$ and every pair of unitary matrices $U, V \in \mathcal{U}(n)$

$$\|A\| = \|UAV\|.$$

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Proposition

Given an unitarily invariant norm, there exists a gauge symmetric function ϕ such that:

$$\|A\| = \phi(s(A)).$$

Conversely, every gauge symmetric function defines a unitarily invariant norm, that we will called $\| \cdot \|_\phi$, in this way.

Angular metrics

Theorem (Li-Qiu-Zhang '05)

Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be a gauge symmetric function and $\rho_\phi : \mathcal{G}_{n,m} \times \mathcal{G}_{n,m} \to \mathbb{R}$ the funtion definided by

$$\rho_\phi\left(\mathcal{M},\mathcal{N}\right) = \phi\left(\theta\left(\mathcal{M},\mathcal{N}\right)\right).$$

Then, ρ_ϕ is an unitarily invariant metric.

Proposition

The group $\mathcal{U}(n)$ is a submanifold of $\mathcal{M}_n(\mathbb{C})$, and for each point $U \in \mathcal{U}(n)$

$$T_U\,\mathcal{U}(n) \simeq i\,\mathcal{H}(n).$$

Let $\|\cdot\|$ be a *unitarily invariant* norm in $\mathcal{M}_n(\mathbb{C})$. Given a smooth curve $\alpha : [a,b] \to \mathcal{U}(n)$

$$Length_{\|\,\cdot\,\|}(\alpha) := \int_a^b \|\dot{\alpha}\|\,dt\,,$$

which let us define the distance between two points of $\mathcal{U}(n)$ by:

$$dist_{\|\,\cdot\,\|}(U,V) := \inf\left\{\text{Length}(\gamma) :\ \gamma \ \text{ joins } U \text{ and } V\right\}.$$

With respect to the connection $\nabla_X Y = \frac{1}{2}[X,Y]$, the geodesics are

$$\gamma(t) = Ue^{itX} \qquad (X \in \mathcal{H}(n))$$

Remark: If the unitarily invariant norm is $\|\cdot\|_F$, then $\mathcal{U}(n)$ with the corresponding metric structure is a Riemannian structure.

Proposition

The Grassmann space $\mathcal{G}_n$ is a submanifold of $\mathcal{U}(n)$.

It can be proved that the tangent space at $S = 2P - I \in \mathcal{G}_n$ is:

$$T_S\, \mathcal{G}_n = \left\{ X = \begin{bmatrix} 0 & -\hat{X} \\ \hat{X}^* & 0 \end{bmatrix} \begin{matrix} R(P) \\ R(I-P) \end{matrix} \ : \ \hat{X} \in \mathcal{M}_{k,n-k}(\mathbb{C}) \right\} .$$

Remarks:

- Given $S \in \mathcal{G}_n$ and $X \in T_S\, \mathcal{G}_n$, then by the structure of X

  $$\gamma_{S,\,X}(t) = S\, e^{-tX} = e^{tX/2}\, S\, e^{-tX/2}.$$

  In particular, the geodesic moves inside $\mathcal{G}_n$ all the time.
- The map $S \mapsto -S$, corresponding to the map $\mathcal{M} \mapsto \mathcal{M}^{\perp}$, is an isometric diffeomorphism between $\mathcal{G}_{n,m}$ and $\mathcal{G}_{n,n-m}$. So, from now on we will assume that $m \leq \frac{n}{2}$.

Theorem (Davis-Kahan ’70)

Let $\mathcal{M}$ and $\mathcal{N}$ two subspaces of $\mathcal{G}_{n,m}$. There exists $X \in T_{S_\mathcal{M}}\mathcal{G}_{n,m}$ such that

$$S_\mathcal{M} e^X = e^{-X/2} S_\mathcal{M} e^{X/2} = S_\mathcal{N} \quad \text{and} \quad \lambda(X) = \{\pm i\,\theta(\mathcal{M},\mathcal{N}), 0, \ldots, 0\}\,.$$

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Let $\gamma(t) = S_\mathcal{M} e^{tX}$ for $t \in [0,1]$. For any unitarily invariant norm $\|\cdot\|$

$$\text{Length}_{\|\cdot\|}(\gamma) = \int_0^1 \|\dot{\gamma}\| dt = \int_0^1 \|S_\mathcal{M} X\, e^{tX}\| dt = \|X\|\,.$$

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If $\|\cdot\|$ is any Schatten norm $\|\cdot\|_p$ then

$$\text{Length}_{\|\cdot\|}(\gamma) = 2\rho_{\|\cdot\|_p}\left(\mathcal{M},\mathcal{N}\right).$$

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More generally, given a g.s.f. ϕ in $\mathbb{R}^m$, if we consider the norm:

$$\|A\|_\phi = \frac{1}{2}\phi\,(\, s_1(A) + s_2(A), \dots, s_{2m-1}(A) + s_{2m}(A)\,)\,,$$

then $\text{Length}_{\|\cdot\|_\phi}(\gamma) = \rho_\phi\,(\mathcal{M},\mathcal{N})$.

Short paths in $\mathcal{U}(n)$

Theorem (A., Larotonda, Varela (maybe '13))

The paths $\gamma(t) = Ue^{itX}$ *with* $X \in \mathcal{H}(n)$ *are short provided* $\|X\|_{sp} \leq \pi$.

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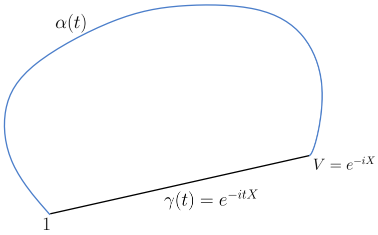
Idea of the proof: Without loss of generality, we can assume that $U = 1$.



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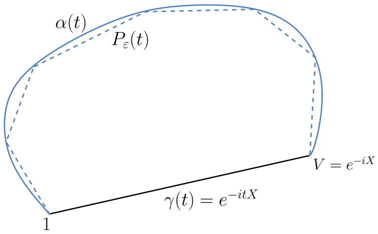

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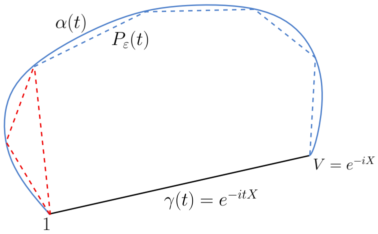

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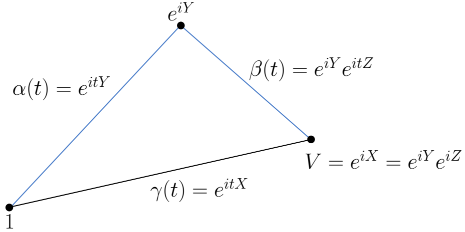
We know that:

- $e^{iX} = e^{iY}e^{iZ}$
- $\|X\|_{sp} \le \pi$

and we want to prove that:

$$\|X\| \le \|Y\| + \|Z\| .$$



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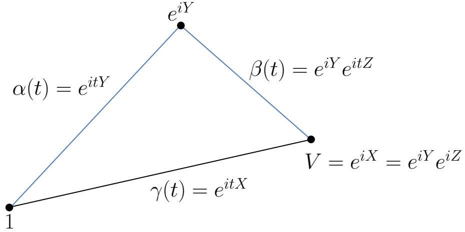

- Suppose for a moment that $e^{iX} = e^{i(Y+Z)}$;
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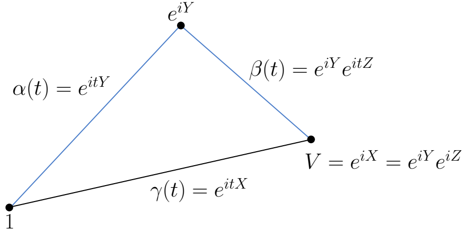

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Thompson's formula

Theorem (R. Thompson '86)

Given $A, B \in \mathcal{H}(n)$, there exist unitary matrices U and V such that:

$$e^{iA}e^{iB} = e^{i(UAU^* + VBV^*)}. \quad (1)$$

History:

- 1962: Given $A, B \in \mathcal{H}(n)$, Horn found a concrete list of inequalities that the eigenvalues of A, B and $A + B$ have to satisfy.

  He conjectured: if $a, b, c \in \mathbb{R}^n$ satisfy these inequalities, then there exist $A, B \in \mathcal{H}(n)$ such that

  $$a = \lambda(A) \quad b = \lambda(B) \quad \text{and} \quad c = \lambda(A + B).$$

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Assume ϕ is strictly convex, and let $U, V, W \in \mathcal{U}(n)$ be such that

$$d_\phi(U,V) = d_\phi(U,W) + d_\phi(W,V), \ \textit{ and } \ d_\infty(U,V) < \pi.$$

Then U, V, W are aligned in $\mathcal{U}(n)$, that is, there exists $t_0 \in [0,1]$ and $Z \in \mathcal{H}(n)$ with $\|Z\|_{sp} < \pi$ such that

$$V = Ue^{iZ}, \qquad \text{while} \quad W = Ue^{it_0 Z}.$$

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Then U, V, W are aligned in $\mathcal{U}(n)$, that is, there exists $t_0 \in [0,1]$ and $Z \in \mathcal{H}(n)$ with $\|Z\|_{sp} < \pi$ such that

$$V = Ue^{iZ}, \qquad \text{while} \quad W = Ue^{it_0 Z}.$$

Remark: In the case of $\mathcal{G}_n$, this result was also proved by Li-Qiu-Zhang.

Uniqueness of short paths
Continuación

Corollary (A., Larotonda, Varela)

Let X be a Hermitian element such that $\|X\|_{sp} < \pi$. Then, $\gamma(t) = Ue^{itX}$ is the unique piecewise C^1 path in $\mathcal{U}(n)$ joining U with $V = Ue^{iX}$.

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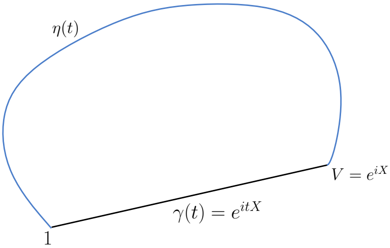

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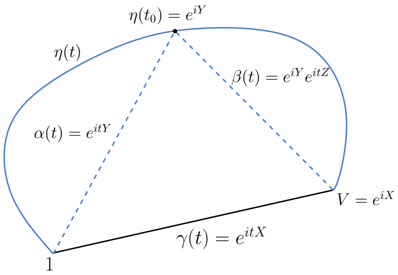

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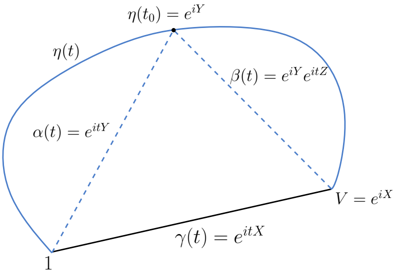
Idea of the proof:

$$\begin{aligned} \|X\| &\leq \|Y\| + \|Z\| \\ &\leq \int_0^{t_0} \|\dot\eta\|_\phi \, dt + \int_{t_0}^1 \|\dot\eta\|_\phi \, dt \\ &= Length(\eta) = \|X\| \end{aligned}$$

$\underset{\text{Previous Thm}}{\Longrightarrow}$

$$\eta(t_0) \in \gamma(t)$$



Unitarily invariant actions

Let $\mathcal{S}$ be the action defined on piecewise C^1 curves $\alpha : [a,b] \to \mathcal{U}(n)$ by

$$\mathcal{S}(\alpha) = \int_a^b \mathcal{L}(\dot{\alpha}(t))\, dt, \tag{2}$$

where $\mathcal{L} : \mathcal{H}(n) \to [0, +\infty)$ is a convex function that satisfies

$$\mathcal{L}(UAV) = \mathcal{L}(A),$$

for every $A \in \mathcal{H}(n)$, and every unitary matrices U and V.

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Examples:

- Any unitarily invariant norm $\| \cdot \|_\phi$;
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Problem

Given $U, V \in \mathcal{U}(n)$, and a fixed interval $[a,b]$, find the piecewise C^1 curves $\gamma : [a,b] \to \mathcal{U}(n)$ such that $\gamma(a) = U$, $\gamma(b) = V$ and γ minimize the action given by (2).

Let $U \in \mathcal{U}(n),\ V = Ue^{iZ},\ Z \in \mathcal{H}(n),$ and γ the path defined on $[a,b]$ by:

$$\gamma(t) = Ue^{i\frac{(t-a)}{b-a}Z}.$$

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Theorem (A., Larotonda, Varela)

If $\|Z\|_{sp} \leq \pi$, then $\gamma(t)$ minimizes $\mathcal{S}$ among piecewise smooth curves $\alpha : [a,b] \to \mathcal{U}(n)$ joining U to V. In particular

$$\inf_{\alpha} \mathcal{S}(\alpha) = (b-a)\mathcal{L}\left(\frac{Z}{b-a}\right).$$

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Theorem (A., Larotonda, Varela)

If $\mathcal{L}$ is strictly convex and $\|Z\|_{sp} < \pi$, then $\gamma(t)$ is the unique minimizer.

Let A and B be two compact operators acting on a Hilbert space $\mathcal{H}$.

Proposition (A., Larotonda, Varela '12)

There exist unitary operators U_k and V_k, such that

$$e^{iA}e^{iB} = \lim_{k\to\infty} e^{i\,U_kAU_k^* + i\,V_kBV_k^*}.$$

Remark: A similar result also holds for embeddable II_1 factors.

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To avoid the limit, we need to embed the operators in a larger Hilbert space:

Theorem (A., Larotonda, Varela '12)

There is an isometry W, and unitary operators U and V such that

$$e^{i\,WAW^*}e^{i\,WBW^*} = e^{i\,U(WAW^*)U^* + i\,V(WBW^*)V^*}.$$

The end


תודה
Dankie
Gracias
Спасибо
شكراً
Merci
Takk
Köszönjük
Terima kasih
Grazie
Dziękujemy
Dėkojame
Ďakujeme
Vielen Dank
Paldies
Kiitos
Täname teid
谢谢
Thank You
Tak
感謝您
Obrigado
Teşekkür Ederiz
감사합니다
Σας ευχαριστούμε
ขอบคุณ
Bedankt
Děkujeme vám
ありがとうございます
Tack