Isometries and isomorphisms of some spaces of matrices

Lajos Molnár
University of Debrecen, Hungary

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By an isometry we mean a map between metric spaces which preserves the distances among the elements.

Denote by $\mathbb{M}_n$ the algebra of all $n \times n$ complex matrices.

Linear isometries of $\mathbb{M}_n$:

Schur (1925), linear isometries with respect to the spectral norm (operator norm): they are the transformations of the forms

$$A \mapsto UAV, \quad \text{or} \quad A \mapsto UA^{tr}V$$

where U, V are unitaries.

Recall that
algebra *-automorphisms of $\mathbb{M}_n$: maps of the form $A \mapsto UAU^*$, where U is unitary.
algebra *-antiautomorphisms of $\mathbb{M}_n$: maps of the form $A \mapsto UA^{tr}U^*$, where U is unitary.

Linear isometries of C^*-algebras:

Theorem

Kadison (1951) *Let $\mathcal{A}, \mathcal{B}$ be (unital) C^*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ a surjective linear isometry. Then there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and a unitary element $U \in \mathcal{B}$ ($U^*U = UU^* = I$) such that ϕ is of the form*

$$\phi(A) = U \cdot J(A), \qquad A \in \mathcal{A}.$$

Jordan isomorphisms: If $\mathcal{A}, \mathcal{B}$ are complex algebras, then a linear map $J : \mathcal{A} \to \mathcal{B}$ is called a Jordan homomorphism if it satisfies $J(A^2) = J(A)^2$ for any $A \in \mathcal{A}$ or, equivalently, if it satisfies $J(AB + BA) = J(A)J(B) + J(B)J(A)$ for any $A, B \in \mathcal{A}$.

Clearly, every algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ (that is a linear map such that $\phi(AB) = \phi(A)\phi(B)$ holds for any $A, B \in \mathcal{A}$) as well as every algebra antihomomorphism $\psi : \mathcal{A} \to \mathcal{B}$ (that is a linear map $\psi : \mathcal{A} \to \mathcal{B}$ satisfying $\psi(AB) = \psi(B)\psi(A)$, $A, B \in \mathcal{A}$) is a Jordan homomorphism.

A Jordan *-homomorphism $J : \mathcal{A} \to \mathcal{B}$ between *-algebras $\mathcal{A}, \mathcal{B}$ is a Jordan homomorphism which preserves the involution in the sense that it satisfies $J(A^*) = J(A)^*$ for all $A \in \mathcal{A}$.

By a Jordan *-isomorphism we mean a bijective Jordan *-homomorphism.

Russo (1969), linear isometries with respect to the trace norm: they are the transformations of the forms

$$A \mapsto UAV, \quad \text{or} \quad A \mapsto UA^{tr}V$$

where U, V are unitaries.

Li and Tsing (1990), linear isometries with respect to unitarily invariant norms:
If the norm in question is not a scalar multiple of the Frobenius norm (Hilbert-Schmidt norm), then all linear isometries of $\mathbb{M}_n$ are of the above simple form.

There are plenty of other linear isometries wrt the Frobenius norm.

How about linearity of isometries between normed spaces?

Theorem

Mazur-Ulam (1932) *Let X, Y be normed real linear spaces. Every surjective isometry $\phi : X \to Y$ is affine (respects the operation of convex combinations) and hence equals a surjective (real-) linear isometry followed by a translation.*

Consequence: If two normed real linear spaces are isometric (i.e., "isomorphic" as metric spaces), then they are isomorphic as real linear spaces.

Assuming a surjective isometry sends 0 to 0, we have (real-)linearity for free and then in certain cases (cf. Kadison's theorem) we can deduce even stronger algebraic properties (e.g., preservation of Jordan-algebraic structure).

A little detour to transformations preserving operator means.

In what follows H is a complex Hilbert space and $B(H)$ stands for the algebra of all bounded linear operators on H. We denote by $B(H)_+$ the cone of positive semidefinite operators.

Geometric mean of positive operators: Pusz and Woronowicz (1973) and later Ando (1978).

$$A \# B = \max\left\{ X \geq 0 : \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0 \right\}, \quad A, B \in B(H)_+.$$

Properties of the geometric mean:

(G1) $A \# B = B \# A$.

(G2) If $A \leq C$ and $B \leq D$, then $A \# B \leq C \# D$.

(G3) (Transfer property) We have $S(A \# B)S^* = (SAS^*) \# (SBS^*)$ for every invertible bounded linear or conjugate-linear operator S on H.

(G4) Suppose $A_1 \geq A_2 \geq \ldots \geq 0$, $B_1 \geq B_2 \geq \ldots \geq 0$ and $A_n \to A$, $B_n \to B$ strongly. Then we have that $A_n \# B_n \to A \# B$ strongly.

(G5) $A \# B = A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$ if A is invertible.

Transformations on positive semidefinite operators preserving the geometric mean:

Example: By transfer property, the transformation

$$A \longmapsto SAS^*$$

where S is an invertible bounded linear or conjugate-linear operator H is a bijective map of $B(H)_+$ which preserves the geometric mean, i.e., it satisfies

$$\phi(A\#B) = \phi(A)\#\phi(B), \quad A, B \in B(H)_+.$$

Theorem

(PAMS, 2009) *Let H be a complex Hilbert space with* $\dim H \geq 2$. *Suppose that $\phi : B(H)_+ \to B(H)_+$ is a bijective map which satisfies*

$$\phi(A\#B) = \phi(A)\#\phi(B), \quad A, B \in B(H)_+.$$

Then there is an invertible bounded linear or conjugate-linear operator S on H such that ϕ is of the form

$$\phi(A) = SAS^*, \quad A \in B(H)_+.$$

Harmonic mean of positive operators: Ando (1978).

$$A!B = \max\left\{X \geq 0 : \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \geq \begin{bmatrix} X & X \\ X & X \end{bmatrix}\right\}, \quad A, B \in B(H)_+.$$

Harmonic mean preservers have the same structure as that of the geometric mean preservers (LAA, 2009).

General theory of operator means: Kubo-Ando (1980).

Preservers of general symmetric means under certain mild regularity assumptions have also been described (ELA, 2011).

E: a real normed space
K: a nonempty closed convex cone in E with $K \cap (-K) = \{0\}$
$\leq$: partial order induced by K ($x \leq y$ if and only if $y - x \in K$)
$\sim$: equivalence relation on $K \setminus \{0\}$

$$x \sim y \iff \exists r, s > 0 : rx \leq y \leq sx,$$

corresponding equivalence classes are called components.

Thompson metric on a component C:

For $x, y \in C$ denote $M(x/y) = \inf\{t > 0 \ : \ x \leq ty\}$ and set

$$d_T(x, y) = \log \max\{M(x/y), M(y/x)\}.$$

Thompson introduced this quantity in 1963 as a modification of the Hilbert projective metric:

$$d_H(x, y) = \log(M(x/y)M(y/x)).$$

d_H is only a pseudo-metric: $d_H(x, y) = 0$ if and only if $x = ty$ for some $t > 0$.
d_H was originally defined by Hilbert in 1903 for the projective plane.

G. Birkhoff (1957): Hilbert projective metric as a useful tool in solving linear integral equations. Afterwards applications for a variety of algebraic, integral and differential equations.

The modification of d_H to d_T resulted in even more applicability. A modern treatment of the topic can be found in a monograph by Nussbaum (1988).

For any unital C^*-algebra $\mathcal{A}$, the cone $\mathcal{A}_+$ of all positive (semidefinite) elements is a closed cone. The set $\mathcal{A}_+^{-1}$ of all invertible positive elements of $\mathcal{A}$ is a component.

Keep the notation
H: a complex Hilbert space
$B(H)$: C^*-algebra of all bounded linear operators on H.

Thompson metric on $B(H)_+^{-1}$:

$$d_T(A,B) = \| \log A^{-1/2} B A^{-1/2} \|, \quad A,B \in B(H)_+^{-1}.$$

Hilbert projective metric on $B(H)_+^{-1}$:

$$d_H(A,B) = \operatorname{diam} \sigma(\log(A^{-1/2} B A^{-1/2})), \quad A,B \in B(H)_+^{-1}.$$

Problem: What are the corresponding isometries?

Example: From the general definition of d_T and d_H it follows that for any invertible bounded linear or conjugate-linear operator T on H and for any function $\tau : B(H)_+^{-1} \to]0,\infty[$, the transformations

$$A \longmapsto TAT^*, \qquad A \longmapsto TA^{-1}T^*$$

and

$$A \longmapsto \tau(A)TAT^*, \qquad A \longmapsto \tau(A)TA^{-1}T^*$$

are surjective isometries with respect to d_T and d_H, respectively.

Theorem

(PAMS, 2009) *Let* $\phi : B(H)_+^{-1} \to B(H)_+^{-1}$ *be a surjective isometry with respect to the Thompson metric, i.e., assume that* ϕ *is a bijective map satisfying*

$$d_T(\phi(A), \phi(B)) = d_T(A, B), \quad A, B \in B(H)_+^{-1}.$$

Then ϕ *is either of the form*

$$\phi(A) = TAT^*, \quad A \in B(H)_+^{-1}$$

or of the form

$$\phi(A) = TA^{-1}T^*, \quad A \in B(H)_+^{-1}$$

where T *is an invertible bounded linear or conjugate-linear operator on* H*.*

Theorem

Suppose that $\phi : B(H)_+^{-1} \to B(H)_+^{-1}$ is a surjective isometry with respect to the Hilbert projective metric, i.e., ϕ is a bijective map satisfying

$$d_H(\phi(A), \phi(B)) = d_H(A, B), \quad A, B \in B(H)_+^{-1}.$$

Then ϕ is either of the form

$$\phi(A) = \tau(A) TAT^*, \quad A \in B(H)_+^{-1}$$

or of the form

$$\phi(A) = \tau(A) TA^{-1} T^*, \quad A \in B(H)_+^{-1}$$

where T is an invertible bounded linear or conjugate-linear operator on H and $\tau : B(H)_+^{-1} \to]0, \infty[$ is a scalar-valued function.

Comments on the proof relating to Thompson isometries:

Mazur-Ulam theorem states that every surjective isometry between normed real linear spaces is affine. The main point of the proof is to show that every such isometry preserves the arithmetic mean of vectors.

J. Väisälä (2003): a short nice proof for Mazur-Ulam theorem based on the concept of reflections.

Applying a multiplicative analogue of that proof, we could show that every surjective isometry of $B(H)_+^{-1}$ with respect to the Thompson metric preserves the geometric mean of operators.

Combining previous results on preservers (mean preservers, Jordan triple product preservers) we deduce the desired conclusion.

In the case of the Hilbert projective metric similar argument applies.

Question: Is there somewhere a general Mazur-Ulam type result behind all this?

Aim: Produce non-commutative generalizations of Mazur-Ulam theorem; find results on the algebraic behavior of isometries on more general algebraic structures, e.g., on groups.

We obtained: Under certain conditions, surjective isometries between groups (or certain subsets of groups) equipped with metrics compatible with the group operations turn to preserve locally(!) the inverted Jordan triple product $ab^{-1}a$ of elements.

O. Hatori, G. Hirasawa, T. Miura and L. Molnár, *Isometries and maps compatible with inverted Jordan triple products on groups* (Tokyo J. Math., 2012).

Jordan triple product: aba.

Inverted Jordan triple product: $ab^{-1}a$.

Proposition

Let G_1, G_2 be groups equipped with translation and inverse invariant metrics. Let $a, b \in G_1$ be elements for which there exists a constant $K > 1$ such that

$$d_1(bx^{-1}b, x) \geq K d_1(x, b)$$

holds for all $x \in L_{a,b} = \{x \in G_1 : d_1(a, x) = d_1(ba^{-1}b, x) = d_1(a, b)\}$. Then for every surjective isometry $\phi : G_1 \to G_2$ we have

$$\phi(ba^{-1}b) = \phi(b)\phi(a)^{-1}\phi(b).$$

Extensions for certain substructures of groups.

We suppose that G_i is a group and X_i is a non-empty subset of G_i such that $yx^{-1}y \in X_i$ holds for all $x, y \in X_i$, $i = 1, 2$ (twisted subgroup, Aschbacher).

X_2 is said to be 2-divisible if for every $x \in X_2$ there is $y \in X_2$ such that $y^2 = x$.

X_2 is called 2-torsion free if the unit element e_2 of G_2 belongs to X_2 and for any $x \in X_2$ the equality $x^2 = e_2$ implies $x = e_2$.

Proposition

Assume X_i is equipped with a metric d_i such that $d_i(cx^{-1}c, cy^{-1}c) = d_i(x, y)$ holds for every triple $x, y, c \in X_i$, $i = 1, 2$. Assume X_2 is 2-divisible and 2-torsion free. Let $a, b \in X_1$ be elements for which there exists a constant $K > 1$ such that

$$d_1(bx^{-1}b, x) \geq K d_1(x, b)$$

holds for all $x \in L_{a,b} = \{x \in X_1 : d_1(a, x) = d_1(ba^{-1}b, x) = d_1(a, b)\}$. Then for every surjective isometry $\phi : X_1 \to X_2$ we have

$$\phi(ba^{-1}b) = \phi(b)\phi(a)^{-1}\phi(b).$$

How to get from the local preservation of the inverted Jordan triple product to its global preservation?

Lemma (Simple technical lemma)

For $i = 1, 2$, let G_i be a group and X_i a non-empty subset of G_i such that $yx^{-1}y \in X_i$ holds for every pair $x, y \in X_i$.
Suppose that $\phi : X_1 \to X_2$ is a map, n is a positive integer, and $\{a_k\}_{k=0}^{2^n}$ is a finite sequence in X_1 such that we have

$$a_{k+1}a_k^{-1}a_{k+1} = a_{k+2}$$

and

$$\phi(a_{k+1}a_k^{-1}a_{k+1}) = \phi(a_{k+1})\phi(a_k)^{-1}\phi(a_{k+1})$$

for all $0 \leq k \leq 2^n - 2$. Then it follows that

$$a_{2^{n-1}}a_0^{-1}a_{2^{n-1}} = a_{2^n}$$

and

$$\phi(a_{2^{n-1}}a_0^{-1}a_{2^{n-1}}) = \phi(a_{2^{n-1}})\phi(a_0)^{-1}\phi(a_{2^{n-1}}).$$

With the help of the previous Mazur-Ulam type general results we have determined the surjective isometries of the unitary group of a Hilbert space (we have recently extended the result for the general setting of C^*-algebras).

O. Hatori and L. Molnár, *Isometries of the unitary group* (PAMS, 2012).

Denote by $\mathrm{U}(H)$ the group of all unitary operators on the complex Hilbert space H.

Theorem

Suppose that $\phi : \mathrm{U}(H) \to \mathrm{U}(H)$ is a surjective isometry (wrt the operator norm). Then there exist unitaries $V, W \in \mathrm{U}(H)$ such that ϕ is of one of the following forms:

(1) $\phi(A) = VAW, \quad A \in \mathrm{U}(H)$,

(2) $\phi(A) = VA^*W, \quad A \in \mathrm{U}(H)$,

(3) $\phi(A) = VA^TW, \quad A \in \mathrm{U}(H)$,

(4) $\phi(A) = V\bar{A}W, \quad A \in \mathrm{U}(H)$.

Main steps of the proof: $\phi(I)^{-1}\phi(.)$ is a unital surjective isometry. So we may assume that ϕ is a unital surjective isometry. General Mazur-Ulam type result applies and yields that ϕ locally preserves the inverted Jordan triple product. Application of the technical lemma gives that ϕ globally preserves that product. Next we obtain that ϕ is an isometric Jordan triple isomorphism. Employing algebraic computations and topological considerations we deduce the desired conclusion.

Natural question: How about non-isometric Jordan triple automorphisms?

Infinite dimensional case: joint results with P. Šemrl, (JMAA, 2012).

In the rest of the talk we present new results that have not been published yet.

We denote by $\mathbb{M}_n$ the algebra of all $n \times n$ complex matrices and by $\mathbb{U}_n$ the group of its unitary elements.

Theorem

Let $\phi : \mathbb{U}_n \to \mathbb{U}_n$ be a continuous map which is a Jordan triple endomorphism, i.e., assume that ϕ satisfies

$$\phi(VWV) = \phi(V)\phi(W)\phi(V), \quad V, W \in \mathbb{U}_n.$$

Then there exist a unitary matrix $U \in \mathbb{U}_n$, an integer k, a number $c \in \{-1, 1\}$, and a set $\{P_1, \ldots, P_n\}$ of mutually orthogonal rank-one projections in $\mathbb{M}_n$, a set $\{k_1, \ldots, k_n\}$ of integers and a set $\{c_1, \ldots, c_n\} \subset \{-1, 1\}$ such that ϕ is of one of the following forms:

(1) $\phi(V) = c(\det V)^k UVU^{-1}, \quad V \in \mathbb{U}_n;$

(2) $\phi(V) = c(\det V)^k UV^{-1}U^{-1}, \quad V \in \mathbb{U}_n;$

(3) $\phi(V) = c(\det V)^k UV^{tr}U^{-1}, \quad V \in \mathbb{U}_n;$

(4) $\phi(V) = c(\det V)^k U\overline{V}U^{-1}, \quad V \in \mathbb{U}_n;$

(5) $\phi(V) = \sum_{j=1}^{n} c_j(\det V)^{k_j} P_j, \quad V \in \mathbb{U}_n.$

Key ideas and main steps of the proof:

ϕ is Lipschitz continuous.

For each self-adjoint matrix $H \in \mathbb{H}_n$, the map $t \mapsto \phi(\exp(itH))$ is a continuous one-parameter unitary group, Stone theorem applies. There is a function $f : \mathbb{H}_n \to \mathbb{H}_n$ such that

$$\phi(\exp(itH)) = \exp(itf(H)), \quad t \in \mathbb{R}, H \in \mathbb{H}_n.$$

f is a linear map and it preserves commutativity. The structure of such transformations is known for $n \geq 3$, we apply the corresponding result.

The case $n = 2$ is treated separately and differently.

The structure of all continuous Jordan triple automorphisms of $\mathbb{U}_n$ is as follows.

Corollary

Let $\phi : \mathbb{U}_n \to \mathbb{U}_n$ be a continuous Jordan triple automorphism, i.e., a continuous bijective map which satisfies

$$\phi(VWV) = \phi(V)\phi(W)\phi(V), \quad V, W \in \mathbb{U}_n.$$

Then there exist a unitary matrix $U \in \mathbb{U}_n$ and a number $c \in \{-1, 1\}$ such that ϕ is of one of the following forms:

(1) $\phi(V) = cUVU^{-1}, \quad V \in \mathbb{U}_n$;

(2) $\phi(V) = cUV^{-1}U^{-1}, \quad V \in \mathbb{U}_n$;

(3) $\phi(V) = cUV^{tr}U^{-1}, \quad V \in \mathbb{U}_n$;

(4) $\phi(V) = cU\overline{V}U^{-1}, \quad V \in \mathbb{U}_n$.

As a consequence we determine the structure of all isometries of the unitary group $\mathbb{U}_n$ with respect to any unitarily invariant norm given on $\mathbb{M}_n$.

Theorem

Let $N(.)$ be a unitarily invariant norm on $\mathbb{M}_n$. If $\phi : \mathbb{U}_n \to \mathbb{U}_n$ is an isometry, i.e., ϕ is a map which satisfies

$$N(\phi(V) - \phi(W)) = N(V - W), \quad V, W \in \mathbb{U}_n,$$

then there exists a pair $U, U' \in \mathbb{U}_n$ of unitary matrices such that ϕ is of one of the following forms:

(1) $\phi(V) = UVU', \quad V \in \mathbb{U}_n;$
(2) $\phi(V) = UV^{-1}U', \quad V \in \mathbb{U}_n;$
(3) $\phi(V) = UV^{tr}U', \quad V \in \mathbb{U}_n;$
(4) $\phi(V) = U\overline{V}U', \quad V \in \mathbb{U}_n.$

Proof: $\phi(I)^{-1}\phi(.)$ is a unital surjective isometry. General result applies and yields that it locally preserves the inverted Jordan triple product. Application of the technical lemma gives that it globally preserves that product. Next we obtain that it is an isometric (wrt to N) Jordan triple isomorphism hence it is continuous relative to the operator norm. The previous result applies.

In the next theorem we determine the isometries of $\mathbb{U}_n$ with respect to the following recently defined class of interesting metrics introduced by Chau, Li, Poon and Sze, and considered also by Antezana, Larotonda and Varela.

First remark the following. To any $V \in \mathbb{U}_n$ there corresponds a unique self-adjoint matrix $H \in \mathbb{H}_n$ with spectrum in $]-\pi, \pi]$ such that $V = \exp(iH)$. For temporary use, we call this self-adjoint matrix H the angular matrix of V. Now, given a unitarily invariant norm $N(.)$ on $\mathbb{M}_n$, for any pair $V, W \in \mathbb{U}_n$ of unitary matrices pick the angular matrix H of VW^{-1} and define $d_N(V, W) = N(H)$. d_N is a true metric on $\mathbb{U}_n$.

Theorem

Let $N(.)$ be a unitarily invariant norm on $\mathbb{M}_n$. The structure of the isometries of $\mathbb{U}_n$ with respect to the metric d_N is exactly the same as in the previous theorem.

The idea of the proof is basically the same as above.

Abe, Akiyama and Hatori: Isometries of the special orthogonal group (LAA, 2013).

Isometries of the general linear group? Hatori, Watanabe (Studia Math., 2012): Isometries between groups of invertible elements in C^*-algebras.

We now turn to the space $\mathbb{P}_n$ of all positive definite elements in $\mathbb{M}_n$. We begin with the description of continuous Jordan triple endomorphisms.

Theorem

Assume $n \geq 3$. Let $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a continuous map which is a Jordan triple endomorphism, i.e., ϕ is a continuous map which satisfies

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \quad A, B \in \mathbb{P}_n.$$

Then there exist a unitary matrix $U \in \mathbb{U}_n$, a real number c, a set $\{P_1, \ldots, P_n\}$ of mutually orthogonal rank-one projections in $\mathbb{M}_n$, and a set $\{c_1, \ldots, c_n\}$ of real numbers such that ϕ is of one of the following forms:

(1) $\phi(A) = (\det A)^c UAU^*, \quad A \in \mathbb{P}_n$;
(2) $\phi(A) = (\det A)^c UA^{-1}U^*, \ A \in \mathbb{P}_n$;
(3) $\phi(A) = (\det A)^c UA^{tr}U^*, \ A \in \mathbb{P}_n$;
(4) $\phi(A) = (\det A)^c U{A^{tr}}^{-1}U^*, \ A \in \mathbb{P}_n$;
(5) $\phi(A) = \sum_{j=1}^{n} (\det A)^{c_j} P_j, \ A \in \mathbb{P}_n$.

Main steps in the proof:

ϕ is Lipschitz continuous in a neighborhood of I.

We define $f : \mathbb{H}_n \to \mathbb{H}_n$ by

$$f(T) = \log \phi(e^T), \quad T \in \mathbb{H}_n.$$

Clearly,

$$\phi(A) = e^{f(\log A)}, \quad A \in \mathbb{P}_n.$$

We show that f is linear and preserves commutativity. We use the structural result of such transformations that holds in the case $n \geq 3$.

Question: What happens if $n = 2$?

As for the form of the continuous Jordan triple automorphisms of $\mathbb{P}_n$, we immediately have the following.

Corollary

Assume $n \geq 3$. Let $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a continuous Jordan triple automorphism, i.e., a continuous bijective map which satisfies

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \quad A, B \in \mathbb{P}_n.$$

Then there exist a unitary matrix $U \in \mathbb{U}_n$ and a number $c \neq -1/n$ such ϕ is of one of the following forms:

(1) $\phi(A) = (\det A)^c UAU^*, \ A \in \mathbb{P}_n$;

(2) $\phi(A) = (\det A)^c UA^{-1}U^*, \ A \in \mathbb{P}_n$;

(3) $\phi(A) = (\det A)^c UA^{tr}U^*, \ A \in \mathbb{P}_n$;

(4) $\phi(A) = (\det A)^c U{A^{tr}}^{-1}U^*, \ A \in \mathbb{P}_n$.

Let N be a unitary invariant norm on $\mathbb{M}_n$. Define

$$\delta_N(A, B) = N(\log A^{-1/2} B A^{-1/2}), \quad A, B \in \mathbb{P}_n.$$

Differential geometrical background:
The set $\mathbb{P}_n$ of positive definite matrices is an open subset of the space $\mathbb{H}_n$, hence it is a differentiable manifold which can naturally be equipped a Riemannian structure in the following way. For any $A \in \mathbb{P}_n$, the tangent space at A can be identified with $\mathbb{H}_n$ on which we define an inner product by

$$\langle X, Y \rangle_A = \mathrm{tr}(A^{-1/2} X A^{-1} Y A^{-1/2}), \quad X, Y \in \mathbb{H}_n.$$

The corresponding norm is given by

$$\|X\|_A = \|A^{-1/2} X A^{-1/2}\|_{HS}, \quad X \in \mathbb{H}_n.$$

Here $\|.\|_{HS}$ stands for the Hilbert-Schmidt norm (Frobenius norm) which is defined by $\|T\|_{HS}^2 = \mathrm{tr}(T^* T)$, $T \in \mathbb{M}_n$. In that way we obtain a Riemannian manifold. The geodesic distance $\delta_R(A, B)$ between the points $A, B \in \mathbb{P}_n$ in this space is given by

$$\delta_{\|.\|_{HS}}(A, B) = \|\log A^{-1/2} B A^{-1/2}\|_{HS}.$$

Now, replace the Hilbert-Schmidt norm by any unitarily invariant norm N on $\mathbb{M}_n$. The formula

$$\delta_N(A, B) = N(\log A^{-1/2} B A^{-1/2}), \quad A, B \in \mathbb{P}_n$$

defines a true metric on $\mathbb{P}_n$ with similar differential geometrical content. (In the much more general context of C^*-algebras we refer to works of Corach and his coauthors.)

The surjective isometries of $\mathbb{P}_n$ wrt δ_N are the following.

Theorem

Suppose $n \geq 2$. Let N be a unitarily invariant norm on $\mathbb{M}_n$ and $\phi : \mathbb{P}_n \to \mathbb{P}_n$ a surjective isometry relative to the metric δ_N. Assume $n \geq 3$ and N is not a scalar multiple of the Hilbert-Schmidt norm (Frobenius norm). If $n \neq 4$, then there exists an invertible matrix $T \in \mathbb{M}_n$ such that ϕ is of one of the following forms:

(1) $\phi(A) = TAT^*, \quad A \in \mathbb{P}_n;$

(2) $\phi(A) = TA^{-1}T^*, \ A \in \mathbb{P}_n;$

(3) $\phi(A) = TA^{tr}T^*, \quad A \in \mathbb{P}_n;$

(4) $\phi(A) = T{A^{tr}}^{-1}T^*, \ A \in \mathbb{P}_n.$

If $n = 4$, then beside (1)-(4) the following additional possibilities can occur:

(5) $\phi(A) = (\det A)^{-2/n}TAT^*, \ A \in \mathbb{P}_n;$

(6) $\phi(A) = (\det A)^{2/n}TA^{-1}T^*, \ A \in \mathbb{P}_n;$

(7) $\phi(A) = (\det A)^{-2/n}TA^{tr}T^*, \ A \in \mathbb{P}_n;$

(8) $\phi(A) = (\det A)^{2/n}T{A^{tr}}^{-1}T^*, \ A \in \mathbb{P}_n.$

In the case where $n \geq 3$ and N is a scalar multiple of the Hilbert-Schmidt norm, ϕ is of one of the forms (1)-(8). Finally, if $n = 2$, then ϕ can necessarily be written in one of the forms (1)-(4).

Key steps in the proof:

Consider $\phi(I)^{-1/2}\phi(.)\phi(I)^{-1/2}$. This is a unital isometry, and a version of our general Mazur-Ulam type results can be applied. We have it is a unital map which preserves the inverted Jordan triple product globally.

The above transformation is necessarily a continuous Jordan triple automorphism. Assume $n \geq 3$. We use our structural result concerning those morphisms, see the previous corollary.

The question that remains: what is the constant c? We apply elements of a beautiful and deep result of Đoković, Li and Rodman providing the complete description of linear isometries of symmetric gauge functions.

The above works if $n \geq 3$. The case $n = 2$ is treated separately and rather differently.

For any pair $A, B \in \mathbb{P}_n$ of positive definite matrices, their symmetric Stein divergence is defined by

$$S(A, B) = \log\det\left(\frac{A+B}{2}\right) - \frac{1}{2}\log\det(AB).$$

Actually, it is just the Jensen-Shannon symmetrization of the divergence called Stein's loss. Sra has recently proved that

$$d_S(A, B) = \sqrt{S(A, B)}, \quad A, B \in \mathbb{P}_n$$

is a true metric on $\mathbb{P}_n$.

Immediate question: What are the corresponding surjective isometries?

Theorem

Assume $n \geq 2$. Let $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a surjective isometry relative to the metric d_S. Then there is an invertible matrix $T \in \mathbb{M}_n$ such that ϕ is of one of the following forms:

1. $\phi(A) = TAT^*, \quad A \in \mathbb{P}_n;$
2. $\phi(A) = TA^{-1}T^*, \; A \in \mathbb{P}_n;$
3. $\phi(A) = TA^{tr}T^*, \; A \in \mathbb{P}_n;$
4. $\phi(A) = TA^{tr^{-1}}T^*, \; A \in \mathbb{P}_n.$

Key steps of the proof:

Considering $\phi(I)^{-1/2}\phi(.)\phi(I)^{-1/2}$ we can assume that our original isometry ϕ is unital. One of our general Mazur-Ulam type results can be applied to verify that it necessarily preserves the inverted Jordan triple product and, since it is unital, we deduce that ϕ is a Jordan triple automorphism. In addition, we prove that it is continuous.

As in the proof of the first theorem in the present section, this gives rise to a bijective linear transformation $f : \mathbb{H}_n \to \mathbb{H}_n$ which satisfies

$$\phi(A) = e^{f(\log A)}, \quad A \in \mathbb{P}_n$$

and preserves commutativity. We can assume $f(I) = cI$ holds with a positive scalar c.

The isometric property of ϕ is implies the equality

$$\operatorname{tr}\left[\log\left(\frac{e^{f(\log A)} + e^{f(\log B)}}{2}\right) - \frac{1}{2}(f(\log A) + f(\log B))\right] =$$
$$\operatorname{tr}\left[\log\left(\frac{A+B}{2}\right) - \frac{1}{2}(\log A + \log B)\right].$$

We apply differential calculus concerning matrix valued functions $A \mapsto g(A)$ of a matrix variable, where g is a real function on an open interval in $\mathbb{R}$. We deduce $c = 1$ and

$$\phi(A) = (f^{-1})^*(A), \quad A \in \mathbb{P}_n.$$

This means that the linear bijection $F = (f^{-1})^*$ on $\mathbb{H}_n$ is an extension of ϕ. F is a Jordan triple automorphism on $\mathbb{P}_n$ and by continuity it is a Jordan triple map also on the set of positive semidefinite matrices. In particular, F maps projections to projections.

It follows that F is a linear Jordan *-automorphism of $\mathbb{H}_n$. The structure of those transformations is well known. We have a unitary matrix $U \in \mathbb{U}_n$ such that either

$$F(A) = UAU^*, \quad A \in \mathbb{H}_n$$

or

$$F(A) = UA^{tr}U^*, \quad A \in \mathbb{H}_n$$

and this completes the proof.

Conclusion: General Mazur-Ulam type results prove very useful in determining the structure of isometry groups of noncommutative matrix groups and related objects.

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