Matrix norm inequalities and their applications in matrix geometry

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Overview

1. The manifold $\mathbb{P}$ of positive invertible operators on Hilbert space.
2. Convex submanifolds and projection maps onto them.
3. Extending the projection map theorem: several approaches.

Disgression

Why extend?

1. In some settings, the most adequate norm to tackle a problem is not necessarilly the Frobenius norm.

2. Many physical models can only be described with infinite dimensional manifolds, e.g. the famous problem on the motions of an incompressible fluid in a bounded region of space.

3. To learn which parts of the rich structure that is underlying are essential to some geometrical results, and which ones are not! (is it relevant that the objects of $\mathbb{P}$ act on a Hilbert space like $\mathbb{C}^n$? Why not operators acting on a Banach space?)

Disgression

Approach via inequalities

"Finsler spaces are point spaces of an essentially new character which we must approach directly without being prejudiced by euclidean or Riemannian methods......"

H. Busemann, in *The geometry of geodesics*, 1955.

Matrix inequalities

All those wonderful matrix inequalities contain essential geometrical properties and ideas, ideas which can be used in other variety of settings with no underlying associative algebras like $M_n(\mathbb{C})$ or no underlying Hilbert space.

Matrices, operators

Notation In this talk, we will deal with several Banach spaces, starting with $X = \mathbb{C}^n$,

or $H =$infinite dimensional Hilbert space; which are both specific examples with certain additional structure (Riemannian structure).

Layers We will denote $B(X) =$bounded linear operators acting on X (which is also a Banach space).

So eventually we will be dealing with $X = B(E)$ for certain Banach spaces E.

Finsler manifolds

M = differentiable manifold with a given metric

$$\|\cdot\| : TM \to [0, +\infty)$$

Finsler metrics

The metric $\|\cdot\|$ is not necessarily Riemannian.

For instance, the uniform norm for an operator or a matrix T:

$$\|T\|_\infty = \sup\{\|Tx\|_X : \|x\|_X \leq 1\}$$

Lenght of curves and geodesic distance

Rectifiable length and distance

For $\gamma \subset M$, and a given tangent metric

$$L(\gamma) = \int_a^b \|\dot{\gamma}\|_\gamma.$$

$$dist(p,q) = \inf\{L(\gamma) : \ \gamma \textit{ joins } p \textit{ to } q\}.$$

Non-smooth manifolds?

(M,d) metric space, γ continuous,

$$L(\gamma) = \sup\left\{\sum d(\gamma(t_{i+1}),\gamma(t_i))\right\}.$$

Sup is taken over all $\{t_i\}$ partitions of $[a,b]$.

Inner metric space

$$d(p,q) = dist(p,q)$$

Sectional curvature

Riemannian manifold

For orthonormal tangent vectors $x, y \in TM$,

$$Sec(x,y) = \langle R(x,y)x, y\rangle_g$$

where R is the *curvature tensor* of M.

It is a measure of how far away

$$\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y \partial x}$$

is from zero.

Non-Riemannian?

Only if the norm comes from an inner product (Riemannian metric on M), the definition of Sec above makes sense.

Sectional curvature

Geodesics Define geodesics as curves of minimal length on M (for a given metric).

Let $x, y \in T_pM$, consider the unique geodesics α, β starting at p with initial speed x, y respectively,

$$\dot{\alpha}(0) = x, \quad \dot{\beta}(0) = y.$$

Milnor (1963) Consider the quantity

$$s(x,y,t) = \frac{\|tx - ty\|_p - dist(\alpha(t), \beta(t))}{t^2 dist(\alpha(t), \beta(t))}$$

Equivalent expression

$$Sec_p(x,y) = \lim_{t \to 0^+} 6\, s(x,y,t)$$

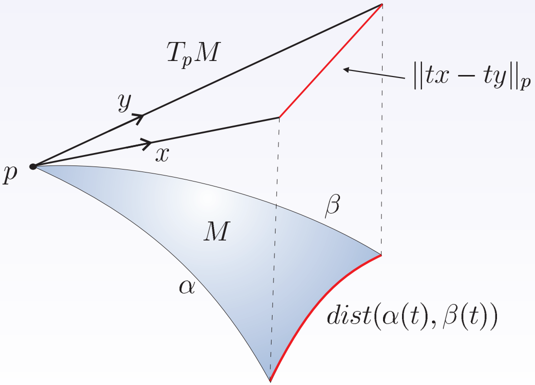
Sectional curvature

The meaning of $s(x,y,t)$

$$s(x,y,t) = \frac{\|tx - ty\|_p - dist(\alpha(t),\beta(t))}{t^2 dist(\alpha(t),\beta(t))}$$

compares divergence of geodesics in M with divergence of flat geodesics (i.e. segments):

Comparing triangles



Non-positive curvature

Triangle comparison inequality

We now focus on the condition $Sec(x, y) \leq 0$, i.e.

$$\|tx - ty\|_p - dist(\alpha(t), \beta(t)) \leq 0$$

or equivalently

$$dist(\alpha(1), \beta(1)) \geq \|x - y\|_p.$$

Exponential map

The exponential map is defined as

$$\exp_p(x) = \alpha(1),$$

where α is the unique geodesic of the connection starting at p with initial speed x.

Non-positive curvature

EMI Then $Sec \leq 0$ if and only if

$$dist(\exp_p(x), \exp_p(y)) \geq \|x - y\|_p,$$

$p \in M$, $x, y \in T_pM$.

This is known as the

Exponential Metric Increasing property

Positive invertible matrices

Famous example, Mostow (1955)

$$\mathbb{P} = \{a > 0 \in B(H),\ i.e.\ a^* = a, \sigma(a) \subset (0, +\infty)\}$$

Geodesics Given $a, b \in \mathbb{P}$, $\sigma(a^{-1}b) = \sigma(b^{\frac{1}{2}} a^{-1} b^{\frac{1}{2}}) \subset (0, +\infty)$.

$a^{-1}b$ has a logarithm, and the one-parameter group

$$\gamma_{a,b}(t) = a \exp(t \log(a^{-1}b))$$

joins a to b, and in fact

$$\gamma_{a,b}(t) = a^{\frac{1}{2}} \left(a^{-1/2} b a^{-1/2}\right)^t a^{\frac{1}{2}} > 0$$

Tangent norms

Gauge norms

A norm $\|\cdot\|_\phi$ on $B(H)$ is *symmetric* if

$$\|xyz\|_\phi \leq \|x\|_\infty \|y\|_\phi \|z\|_\infty.$$

In particular, it is *unitarily invariant*,

$$\|uxv\|_\phi = \|x\|_\phi$$

for u, v unitary operators.

Uniform norm

$$\|x\|_\infty = \sup\{\|x\xi\| : \|\xi\| \leq 1\}$$

Frobenius norm

$$\|x\|_F = \sqrt{Tr(x^*x)}$$

also known as the 2-norm.

Tangent norms on $\mathbb{P}$

Symmetric norms

For $a \in \mathbb{P}$, $v \in T_a\mathbb{P}$, let $\|v\|_a := \|a^{-1/2} v a^{-1/2}\|_\phi$.

Infinitesimal EMI

$x, y \in T_1\mathbb{P}$, $\exp : T_1\mathbb{P} \to \mathbb{P}$, then $D\exp_x(y) \in T_{e^x}\mathbb{P}$.
Remarkably (any ϕ)

$$\|D\exp_x(y)\|_{e^x} \geq \|y\|_\phi = \|y\|_{1\in\mathbb{P}}.$$

This is the *Infinitesimal EMI property*

Frobenius norm: Mostow (1955) - Symmetric norms: Hiai-Kosaki (1999)

It can be written as an inequality involving the logarithmic mean

$$\left\| \int_0^1 e^{(1/2-s)x} y e^{(s-1/2)x} ds \right\|_\phi = \left\| \frac{\sinh(adx)}{adx} y \right\|_\phi \geq \|y\|_\phi,$$

where $adx(y) = [x, y] = xy - yx$.

Parenthesis: Riemann-Hilbert manifolds

McAlpin-Grossmann, 60's

The infinitesimal EMI (in Riemannian manifolds M of any dimension)

$$\|D(\exp_p)_x(y)\|_{\exp_p(x)} \geq \|y\|_p$$

is equivalent to Sec ≤ 0 *on M.*

Note that it is in fact the infinitesimal version of $Sec \leq 0$ according to Milnor, i.e.

$$dist(\exp_p(v), \exp_p(w)) \geq \|v - w\|_p.$$

Minimality of one-parameter groups

Theorem: $\gamma_{a,b}$ is short in $\mathbb{P}$

$$dist(a,b) = L(\gamma_{a,b}) = \|\log(a^{-1/2}ba^{-1/2})\|_\phi$$

Why? The IEMI property holds in $\mathbb{P}$ (any ϕ), and this enables comparison of curves in $\mathbb{P}$ with segments (short geodesics on the vector space H).

Convexity of the geodesic distance

Equivalent to $Sec \leq 0$

Two geodesics starting at $p \in M$, with $(M, dist)$ inner metric space:

$$dist(\alpha(t), \beta(t)) \leq t\, dist(\alpha(1), \beta(1)),$$

nonpositive curvature in the sense of Busemann

Matrix/op. inequality in $\mathbb{P}$

Does

$$\| \log(e^{-tx/2} e^{ty} e^{-tx/2}) \|_\phi \leq t \| \log(e^{-x/2} e^{y} e^{-x/2}) \|_\phi$$

hold for any ϕ?

Lawson-Lim (2007)

Proof in a general setting, and in particular for $\mathbb{P}$ with any ϕ.

The action of the full linear group

Action of GL (the group of invertibles)

Consider $I : GL(H) \times \mathbb{P} \to \mathbb{P}$ given by

$$I(g, a) = I_a(g) = g^* a g.$$

This is a transitive action (the orbit of any a is full).

Isometric action

The action is isometric: if $v \in T_a\mathbb{P}$ and $g \in GL(H)$, then

$$\|g^* v g\|_{g^* a g} = \|v\|_a =: \|a^{-1/2} v a^{-1/2}\|_\phi$$

and this holds for any ϕ.

The group of isometries

GL can be regarded as a (sufficiently large part of)

$$Iso(\mathbb{P}).$$

Homogeneous spaces

$\mathbb{P}$ as homogeneous space

From the action $I_1 : GL(H) \to \mathbb{P}$ given by $g \mapsto g^*g$, the isotropy group is

$$\{g \in GL(H) : g^*g = 1\} = \mathcal{U}(H),$$

the group of unitary matrices.

Then $\mathbb{P} \simeq GL/\mathcal{U}(H)$.

Involutive Lie subgroups

$G^* = G \subseteq GL(H)$, consider

$$\mathbb{P}^G = \{g^*g : g \in G\} \subseteq \mathbb{P}.$$

Convex submanifolds

Lie algebra Let $L(G)$ stand for the Lie algebra of G, then

$$L(G) = \mathfrak{p} \oplus \mathfrak{k}$$

where $\mathfrak{p} \subset B(H)_h$ and $\mathfrak{k} = L(\mathcal{U}_G) \subset B(H)_{ah}$.

$T_1\mathbb{P}^G = \mathfrak{p}$ It is easy to check that $\exp(\mathfrak{p}) = \mathbb{P}^G$.

Lie Triple Systems $\mathfrak{p}$ is not a Lie algebra, but $[\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]] \subset \mathfrak{p}$.

Theorem (Mostow, 1955) $\mathbb{P}^G = \exp(\mathfrak{p})$ is *geodesically convex*, that is

$$\gamma_{a,b} \subset \mathbb{P}^G$$

whenever $a, b \in \mathbb{P}^G$. Equivalently, $ab^{-1}a$ stays in $\mathbb{P}^G$.

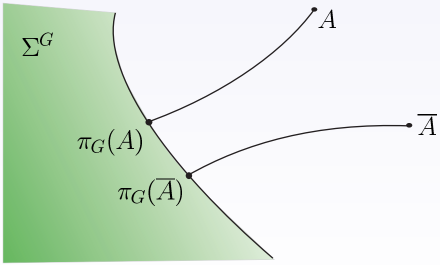
Metric Projections

Theorem: Mostow (1955) $H = \mathbb{C}^n$, $\| \cdot \|_\phi = \| \cdot \|_F$.
Given $a \in \mathbb{P}$, there exists a unique $\pi_G(a) \in \mathbb{P}^G$

$$dist(a, \mathbb{P}^G) = dist(a, \pi_G(a)).$$

Moreover, $\pi_G : \mathbb{P} \to \mathbb{P}^G$ is a contraction:

$$dist(\pi_G(A), \pi_G(\overline{A})) \leq dist(A, \overline{A})$$



Metric Projections

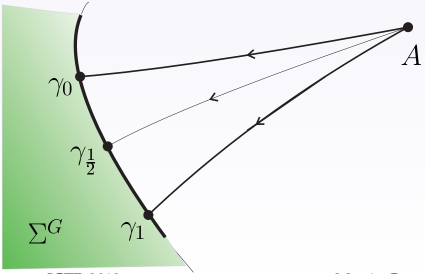
Proof Combine $dist(e^X, e^Y) \geq \|X - Y\|_\phi$ with

$$\frac{1}{4}\|X-Y\|_F^2 = \frac{1}{2}\|X\|_F^2 + \frac{1}{2}\|Y\|_F^2 - \left\|\frac{X+Y}{2}\right\|_F^2$$

to obtain a semi-paralellogram law in $\mathbb{P}$:

$$\frac{1}{4}L(\gamma)^2 \leq \frac{1}{2}dist(A,\gamma_0)^2 + \frac{1}{2}dist(A,\gamma_1)^2 - dist(A,\gamma_{\frac{1}{2}})^2,$$

Uniform Convexity



Metric Projections

Proof $\{B_n\} \subset \mathbb{P}^G$ a minimizing sequence,

$$dist(B_n, A) \to d := dist(\mathbb{P}^G, A)$$

Let $\gamma \in \mathbb{P}$ with $\gamma(0) = B_n$, $\gamma(1) = B_m$. Since $\mathbb{P}^G$ is convex,

$$dist(A, \gamma_{\frac{1}{2}}) \geq d$$

and

$$\frac{1}{4} dist(B_n, B_m)^2 \leq \frac{1}{2} dist(A, B_n)^2 + \frac{1}{2} dist(A, B_m)^2 - d^2$$

Hence $\{B_n\}$ is a Cauchy sequence in $\mathbb{P}^G$.

Another equivalence for $Sec \leq 0$

$CAT(0)$ (or Bruhat-Tits) inequality

$$\frac{1}{4}L(\gamma)^2 \leq \frac{1}{2}dist(x,\gamma_0)^2 + \frac{1}{2}dist(x,\gamma_1)^2 - dist(x,\gamma_{\frac{1}{2}})^2.$$

CAT is for *Cartan, Alexandrov and Toponogov*

Inner metric spaces

$(M, dist)$ verifies $CAT(0)$

nonpositive curvature in the sense of Alexandrov

Hierarchy

Alexandrov $\Rightarrow$ Busemann

But not the other way around.

Counterexample: $\mathbb{P}$ with the uniform norm $\|\cdot\|_\infty$

Projections to closed convex subsets

Essential for in Mostows's approach

The parallelogram law for Hilbert spaces.

Porta-Recht decompositions (1994)

Consider $\mathbb{P}$ with the uniform norm, and a convex submanifold as before (A is a subalgebra of $B(H)$)

$$\mathbb{P}^G = \{gg^* : g \in GL(A)\}.$$

With the uniform norm $\|\cdot\|_\infty$, strong convexity properties are not available (no semi-parallelogram law). However...

Theorem

Each $a \in \mathbb{P}$ is reached by a unique *normal* geodesic with foot in $\mathbb{P}^G$. Moreover, the projection map $\pi_A : \mathbb{P} \to \mathbb{P}^G$ is smooth.

What is essential here?

Do *all* these count?

There is an interplay between norms, geodesic convexity, associative algebra structure, topology, dimension.

First naive approach

Study a straightforward extension of Mostow's:

$$HS = \{a \in B(H) : a \text{ is compact },\ \sigma(a) \in \ell^2\}$$

$$\mathbb{P} = \{gg^* : g \in GL(H),\ g - 1 \in HS\} = \{e^X : X \in HS_h\},$$

a typical tangent vector $v \in T_a\mathbb{P}$ is an operator in HS_h, and

$$\|v\|_a = \|a^{-1/2} v a^{1/2}\|_2$$

as before, where

$$\|X\|_2 = \sqrt{Tr|X|^2} = \sqrt{\sum \langle |X|e_i, |X|e_i\rangle}.$$

Projections to closed convex subsets

L (2007) Expected theorems on minimality of geodesics, strict convexity of the geodesic distance, and

- Projection theorem to convex closed submanifolds. The projection map π here is not only a contraction, but also smooth and in fact real analytic.

The technique of the proof is a combination of the ideas mentioned before due to Mostow, Porta and Recht (mixed proof).

Second naive approach

p-uniform convexity

The p-norm $\|X\|_p = \sqrt[p]{Tr|X|^p}$ obeys Clarkson's inequality ($p \geq 2$)

$$\frac{1}{2^p}\|X - Y\|_p^p \leq \frac{1}{2}\|X\|_p^p + \frac{1}{2}\|Y\|_p^p - \left\|\frac{X+Y}{2}\right\|_p^p$$

and there is a similar one for $1 < p \leq 2$.

p-Alexandrov spaces

Combining Clarkson with the EMI,

$$\frac{1}{2^p}L(\gamma)^p \leq \frac{1}{2}dist(A,\gamma_0)^p + \frac{1}{2}dist(A,\gamma_1)^p - dist(A,\gamma_{\frac{1}{2}})^p,$$

Conde-L (2010)

Metric projections to convex submanifolds (here the algebraic underlying structure is not relevant, it does not play any role).

The Lie group approach

Abstract setting

G a Lie group, $L(G)$ its Lie algebra, σ an involutive ($\sigma^2 = id$) automorphism of G. Consider the homogeneous space

$$M = G/K$$

where K is the fixed point set of σ (and, in fact, a Lie subgroup).

The example to have in mind here is, of course $G = B(H)$, $\sigma(g) = (g^*)^{-1}$, $K = U(H)$, so $M = \mathbb{P}$.

Helgason's book or Neeb's paper

σ induces a direct sum decomposition

$$L(G) = \mathfrak{p} \oplus L(K)$$

and we identify $TM \simeq \mathfrak{p}$.

The Lie group approach

An adequate Ad_K invariant norm on $\mathfrak{p}$ makes of M a space of nonpositive curvature, in the following sense:

Finsler manifolds of semi-negative curvature

K.-H. Neeb (2003) adopted the infinitesimal EMI

$$\|D(\exp_p)_X(Y)\|_{\exp_p(X)} \geq \|Y\|_p$$

as the *definition* of non-positive curvature. By the discussion above, our $\mathbb{P}$ is such a manifold (any ϕ).

In the abstract setting, $L(G)$ is not an associative algebra. Hence, majorization techniques, continuous functional calculus techniques are out of reach.

The Lie group approach

However, $L(G)$ is a Lie algebra, acting on itself via $ad_x(y) = [x, y]$.
And moreover, $B(L(G))$ is a Banach algebra, so analytic functional calculus is in reach. We can produce inequalities.

L (2008) We needed a series of well-known inequalities for matrix norms (or at least, some adequate replacements) in order to tackle the study of the homogeneous space, and in particular to control projections (or even existence) to convex closed submanifolds.

The Lie group approach

Let $H \subset G$ be a Lie subgroup, involutive (i.e. $\sigma(H) \subset H$). Then, if we recall $M = G/K$, let

$$C = H/(H \cap K).$$

Theorems: Conde-L (2010)

$C \subset M$ is a geodesically convex submanifold, and there is a reductive structure

$$TC \oplus S = TM.$$

S acts as a natural *normal bundle* to TC, and

$$NB = \{\exp_p(v) : p \in C, v \in S\}$$

called the *normal bundle* fills M.

The Lie group approach

Corollaries

Each $a \in M$ is reached by a unique geodesic with normal speed $v \in S$ and *foot* $p \in C$.

The normal bundle is isomorphic to M. We obtain *Corach-Porta-Recht* factorizations of M:

$$a = \exp_p(v), \quad M \simeq TC \oplus S.$$

Sketch of proof

Consider the map $\Theta : NB \to NB$ given by

$$\Theta : \exp_p(v) \mapsto \exp_p(2v), \qquad p \in C, v \in S.$$

This map is expansive, i.e.

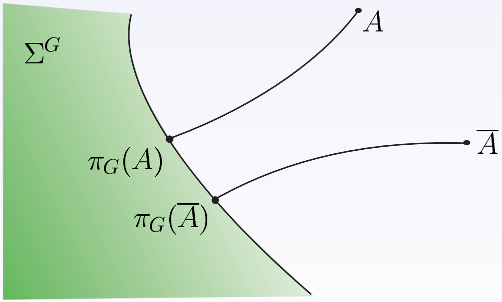
$$\|D\Theta_Z(W)\| \geq \|W\|$$

for any Z.

The Lie group approach

Then Θ is a local iso $\Rightarrow$ NB open.

Moreover $\pi_C : NB \to C$ given by taking the foot (goes in the opposite direction of Θ) is contractive$\Rightarrow$ NB is closed.



An inequality

CPR The inequality $\|D\Theta_Z(W)\| \geq \|W\|$ is essentially

$$\|\cosh(adZ)W\| \geq \|W\|,$$

which in turn can be rewritten as

$$\|e^Z We^{-Z} + e^{-Z} We^Z\| \geq 2\|W\|,$$

Corach-Porta-Recht inequality .

As shown by Kittaneh (1994), in the setting of Hilbert space operators it is equivalent to the arithmetic-geometric mean inequality:

$$\|AA^*X + XBB^*\| \geq 2\|A^*XB\|.$$

Application: complexification

Via polar decomposition, $GL(H)$ can be though of as the complexification of the real Lie group $U(H)$, in a very precise sense:

Complexif. A complexification of a real manifold M is a complex Banach manifold Y endowed with an anti-holomorphic involutive diffeomorphism σ such that the fixed point submanifold $Y_0 = \{y \in Y : \sigma(y) = y\}$ is a strong deformation retract of Y, and also Y_0 is diffeomorphic to X.

Theorem: Miglioli (2013) Let G be a complex Lie group with involution, K its (real!) fixed point subgroup, and H an involutive Lie subgroup as before. Assume that $M = G/K$ is of semi-negative curvature.
Then G/H is a complexification of $K/K \cap H$.

Thank you all for your kind attention!

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