ICTP, Trieste, Italy; July 2013

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Abstract: There has been longstanding interest in the problem of characterizing normal compressions of normal matrices. Indeed, the Hermitian case is completely solved by the Cauchy interlacing theorem, and its converse (due to Fan and Pall). More recently, the theory of higher–rank numerical ranges has included the solution to the case of scalar compressions. Here we take some steps towards a similar treatment of the general case. We develop some natural necessary conditions on the eigenvalues as well as some convenient sufficient conditions, showing by a study of the 2x2 compressions of 4x4 normals that the necessary conditions are not sufficient. We also give a new proof of the Choi–Kribs–Zyczkowski conjecture for 2x2 compressions by means of a powerful extension of that result.

The CKZ conjecture (theorem) for 2x2 compressions says that diag(a,a) is a compression of normal N if a lies in the intersection L of the Ck, where Ck denotes the convex hull of the eigenvalues of N with the kth eigenvalue omitted. We show that in fact diag(a,b) is a compression whenever a,b both lie in L. This talk is based on joint work with Nishan Mudalige and Rajesh Pereira.

Acknowledgements: We have enjoyed many stimulating discussions of matrix compression, particularly those with M.–D. Choi, C.–K. Li, Y.–T. Poon, N.–S. Sze, and J. F. Queiró. Versions of the material in this paper were developed in [M]. The work of Holbrook and Pereira was supported in part by Discovery Grants from NSERC of Canada.

Given a linear operator T on a complex Hilbert space $\mathbb{H}$, and any orthogonal projection P, we say that $PT|_{P\mathbb{H}}$ is a **compression** of T. If $\mathbb{H} = \mathbb{C}^N$ and T is represented by a matrix $M \in \mathbb{M}_N$ (the $N \times N$ complex matrices), a second matrix C represents a compression of T (or a compression of M) iff there is a unitary matrix U such that C is a NW corner of UMU^*. If C is $k \times k$ we say it is a rank–k compression of M. There is a rich history of results that allow us to identify compressions by means of intrinsic criteria.

A classic example is the Cauchy interlacing theorem [Cau], along with its converse [FP], which may be expressed as follows.

Theorem 1: If $M \in \mathbb{M}_N$ is Hermitian, with eigenvalues

$$a_1 \leq a_2 \leq \cdots \leq a_N,$$

then C is a rank–k compression of M iff C is Hermitian with eigenvalues b_j satisfying

$$a_1 \leq b_1 \leq a_{N-k+1}, a_2 \leq b_2 \leq a_{N-k+2}, \ldots, a_k \leq b_k \leq a_N.$$

In particular, C is a rank $N-1$ compression iff

$$a_1 \leq b_1 \leq a_2 \leq b_2 \leq a_3 \leq ... \leq a_{N-1} \leq b_{N-1} \leq a_N,$$

the classic "interlacing" of eigenvalues.

A much more recent example is provided by the theory of **higher–rank numerical ranges**. The striking development of this theory was motivated originally by problems in quantum information theory. Since the introduction of this concept by Choi, Kribs, and Życzkowski [CKŻ1,CKŻ2] only a few years ago, it has indeed been effectively applied in the area of quantum information (see [CPMSŻ,KPLRdS,LP,LPS1,MMŻ], for example). It has also inspired a remarkable development of its purely mathematical aspects (see, for example, [CHKŻ,CGHK,Wo,LS,LPS2,DGHPŻ]). From this point of view the theory of the higher–rank numerical ranges may be described as a highly successful analysis of **scalar** compressions of arbitrary matrices $M \in \mathbb{M}_N$.

This suggests a more general program: characterize the **normal** (diagonal) compressions of M. Here we present some recent results that may be viewed as a useful beginning for this program.

The rank–k numerical range of M, usually denoted in the literature by $\Lambda_k(M)$, was defined by Choi, Kribs, and Życzkowski as the set of those complex λ such that for some rank–k orthogonal projection P we have

$$PMP = \lambda P.$$

In terms of compressions, we see that $\lambda \in \Lambda_k(M)$ iff λI_k is a (matrix) compression of M. Thus the following fundamental result of Li and Sze [LS] may be placed in the same family as the Cauchy interlacing theorem (and, in fact, the interlacing theorem plays a role in the argument of Li and Sze).

Theorem 2: Given $M \in \mathbb{M}_N$, let $\lambda_j(\theta)$ be an enumeration of the eigenvalues of the (Hermitian)

$$\mathrm{Re}(e^{i\theta}M) = (e^{i\theta}M + e^{-i\theta}M^*)/2$$

such that

$$\lambda_1(\theta) \leq \lambda_2(\theta) \leq \cdots \leq \lambda_N(\theta).$$

For each real θ, let the half–plane $H(M, \theta)$ be defined by

$$H(M, \theta) = e^{i\theta}\{z : \mathrm{Re}(z) \leq \lambda_{N-k+1}(-\theta)\}.$$

Then

$$\Lambda_k(M) = \bigcap \{H(M, \theta) : \theta \in [0, 2\pi]\}. \tag{1}$$

Note that the Li-Sze theorem identifies $\Lambda_N(M)$ for an **arbitrary** matrix M. If we specialize to the case of **normal** M we can describe $\Lambda_N(M)$ nicely in terms of the eigenvalues of M. The result is the following corollary, which first established the CKŻ conjecture/theorem.

Corollary 3: Let $z_1, \ldots, z_N$ be the eigenvalues of normal $M \in \mathbb{M}_N(\mathbb{C})$. Then

$$\Lambda_k(M) = \Omega_k(M), \tag{2}$$

where

$$\Omega_k(M) = \bigcap_{\#(J)=N-k+1} \operatorname{conv}\{z_j : j \in J\}. \tag{3}$$

Note that for $k = 2$ equation (3) becomes

$$\Omega_2(M) = \bigcap_{j=1}^{N} \operatorname{conv}\{z_i : i \neq j\}.$$

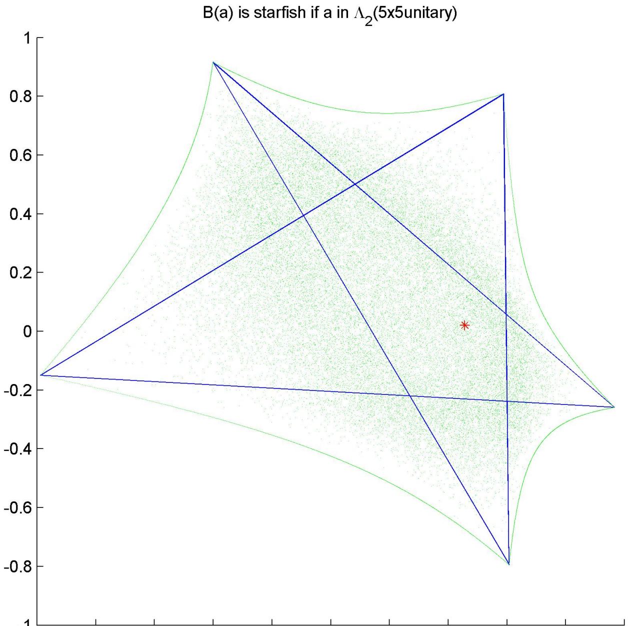
Whereas Corollary 3 says that when $a \in \Omega_2(M)$ we have $\operatorname{diag}(a, a)$ as a compression of M, it is actually the case that for any **pair** $a, b \in \Omega_2(M)$ the normal $\operatorname{diag}(a, b)$ is a compression of M. The following proposition is stated for unitary M, but these can stand in for more general normal M.

Proposition 4: Let M be normal in $\mathbb{M}_N$ and such that the eigenvalues $z_1, \ldots, z_N$ are distinct and each is an extreme point of $W(M)$ (eg M unitary). Then $a, b \in \Omega_2(M)$ implies that $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is a compression of M.

It turns out that $N = 5$ is the difficult case for this proposition, and the following figure shows what happens in this case. We see a red point a in the inner pentacle $\Omega_2(M)$ and a green cloud of points mapping out a “starfish” $B(a)$ which covers the inner pentacle, and more. In general, for any a in the numerical range $W(M) = \mathrm{conv}\{z_1, \ldots, z_N\}$, we use the notation

$$B(a) = \{b \in \mathbb{C} : \mathrm{diag}(a, b) \text{ is a compression of } M\}.$$

in what follows we give a brief account of the reasons behind this stronger form of the CKŻ conjecture/theorem.


B(a) is starfish if a in Λ_2(5x5unitary)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8

We have the following simple **necessary** condition for compressions.

Proposition 5: For every $M \in \mathbb{M}_N$, if $k \leq N$, C is a rank-k compression of M, and Q is a compression of rank $N-k+1$, then

$$W(C) \cap W(Q) \neq \emptyset.$$

In particular, if $M \in \mathbb{M}_N$ is normal with eigenvalues $z_1, \ldots, z_N$, and the rank–k compression C is normal with eigenvalues $c_1, \ldots, c_k$, then for every index set J having $\#(J) = N-k+1$

$$\operatorname{conv}\{c_1, \ldots, c_k\} \cap \operatorname{conv}\{z_j : j \in J\} \neq \emptyset.$$

Specializing further, if $\operatorname{diag}(a, b)$ is a compression, then the line segment $[a, b]$ must meet $\operatorname{conv}\{z_j : j \in J\}$ whenever $\#(J) = N-1$.

Proof: Let S and T be the subspaces corresponding to compressions C and Q. Since the dimensions add to more than N, S and T must intersect non–trivially; let u be a unit vector in $S \cap T$. Then

$$(Mu, u) = (Mu, P_S u) = (P_S Mu, u) = (Cu, u) \in W(C),$$

and similarly $(Mu, u) \in W(Q)$. QED

There is a also a simple **sufficient** condition for normal compressions.

Proposition 6: If $M \in \mathbb{M}_N$ is normal with eigenvalues $z_1, \ldots, z_N$ then $c_1, \ldots, c_k \in \mathbb{C}$ are eigenvalues of a normal compression C of M provided that there exists a **partition** $J_1, \ldots, J_k$ of $\{1, 2, \ldots, N\}$ such that for each $i = 1, \ldots, k$

$$c_i \in \mathrm{conv}\{z_j : j \in J_i\}.$$

Proof: This is by means of a direct construction of an appropriate orthonormal basis.

Note that if $b \in B(a)$ we have orthonormal u, w such that

$$(Mu, u) = a, (Mw, w) = b, \text{ and } (Mu, w) = (Mw, u) = 0.$$

Thus $a = \sum_1^N |u_j|^2 z_j$, a convex combination. Let Δ_N denote the N–dimensional simplex, ie $\text{conv}\{e_1, \dots, e_N\}$; then $|u|^2$ (where the operations are performed componentwise) belongs to

$$C(a) = \{t \in \Delta_N : a = \sum_1^N t_j z_j\}.$$

By exchanging complex arguments between the components of u and w we may assume that $u \geq 0$; then the possible u lie in $\{\sqrt{t} : t \in C(a)\}$. The conditions on $w \in \mathbb{C}^N$ are then given by

$$\|w\| = 1, w \perp u, w \perp z \circ u, \text{ and } w \perp \bar{z} \circ u,$$

where $\circ$ indicates Schur (componentwise) multiplication, so that

$$z \circ u = (z_1 u_1, \dots, z_N u_N)',$$

with $'$ indicating transpose.

We may thus describe $B(a)$ as follows.
Proposition 7: Given $a \in W(M)(= \mathrm{conv}\{z_1, \ldots, z_N\})$,

$$B(a) = \bigcup_{t \in C(a)} B(a, t),$$

where

$$B(a, t) = \{\sum_1^N |w_j|^2 z_j : \|w\| = 1, w \perp \sqrt{t}, z \circ \sqrt{t}, \bar{z} \circ \sqrt{t}\}.$$

The structure of the convex set $C(a)$ is determined by its extreme points.

Lemma 8: The extreme points of $C(a)$ are those $t \in C(a)$ such that at most three $t_k > 0$.

The complexity of $B(a, t)$ increases with the number of nonzero t_k. For example, if only one $t_k > 0$, then $t_k = 1$ and $a = z_k$. Here the simple sufficient condition of Proposition 6 is also necessary:

$$B(a, t) = \mathrm{conv}\{z_j : j \neq k\}.$$

We see this as follows. Evidently, with $u = \sqrt{t} = e_k$, u, w are orthonormal exactly when $w = \sum_{j \neq k} \alpha_j e_j$ with $\sum_{j \neq k} |\alpha_j|^2 = 1$; then

$$b = (Nw, w) = \sum_{j \neq k} |\alpha_j|^2 z_j \in \mathrm{conv}\{z_j : j \neq k\},$$

and any $b \in \mathrm{conv}\{z_j : j \neq k\}$ can be obtained in this way.

The same sort of simplification occurs if only two or three $t_k > 0$.
Proposition 9: (a) If $t \in C(a)$ has exactly two positive components, say $t_1, t_2 > 0$, then

$$B(a, t) = \text{conv}\{z_j : j > 2\}.$$

(b) If $t \in C(a)$ has exactly three positive components, say $t_1, t_2, t_3 > 0$, then

$$B(a, t) = \text{conv}\{z_j : j > 3\}.$$

When $N \geq 4$ we may deal with a such that some $t \in C(a)$ has four positive components. We then obtain the **first** examples where the necessary conditions are **not** sufficient.

Proposition 10: Let $N = 4$ and suppose that z_1, z_2, z_3, z_4 are all extreme in $W(M)$ and are numbered in counterclockwise order. The diagonals $[z_1, z_3]$ and $[z_2, z_4]$ meet at q and divide $W(M)$ into four quadrants. Consider $a \in W(M)$; the possibilities for $B(a)$ are as follows.

(a) See figure 2: a lies in the interior of one of the quadrants. For convenience, assume that $a \in \operatorname{conv}\{z_1, z_2, q\}$; let $x = t(1, 2, 3)$, $y = t(1, 2, 4)$. Then $B(a)$ is the curve traced out by the function $b(r)$ defined for $0 < r < 1$ by

$$b(r) = \sum_{k=1}^{4} \frac{(x_k - y_k)^2}{(1-r)x_k + ry_k} z_k \Big/ \sum_{k=1}^{4} \frac{(x_k - y_k)^2}{(1-r)x_k + ry_k}.$$

Note that $x_4 = 0$ and $y_3 = 0$ so that

$$\lim_{r \to 0} b(r) = z_4, \quad \lim_{r \to 1} b(r) = z_3,$$

and we obtain a continuous curve parametrized on $[0, 1]$ when we interpret $b(0)$ as z_4 and $b(1)$ as z_3. Except for these endpoints, the curve lies in the interior of the opposite quadrant $\operatorname{conv}\{z_3, z_4, q\}$.

(b) If a lies in the interior of one of the sides of $W(M)$ then $B(a)$ is the opposite side (eg if a is inside $[z_1, z_2]$ then $B(a) = [z_3, z_4]$).
If $a = z_k$ then $B(a)$ is the opposite triangle $\text{conv}\{z_j : j \neq k\}$.
(c) See Figure 3: a lies interior to the diagonals but is not q; say a is interior to $[z_1, q]$. Then $B(a)$ is the T–shaped object
$[z_2, z_4] \cup [q, z_3]$.
(d) If $a = q$ then $B(a)$ is the union of the two diagonals.

Figure 2: B(a) is a curve, not the opposite quadrant

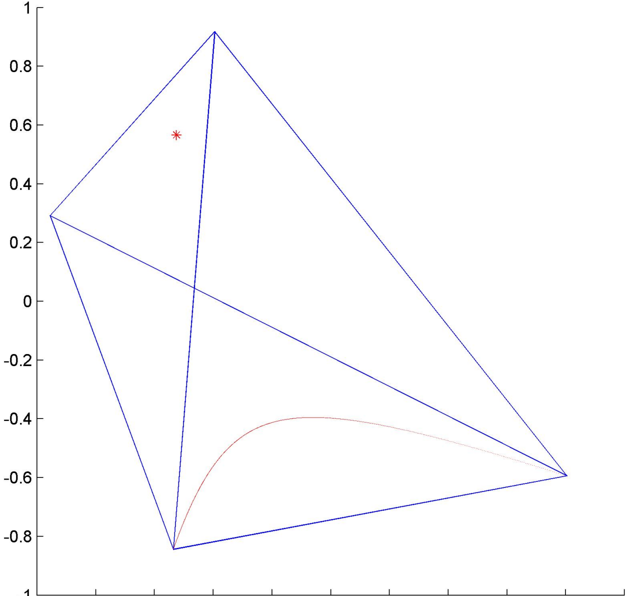
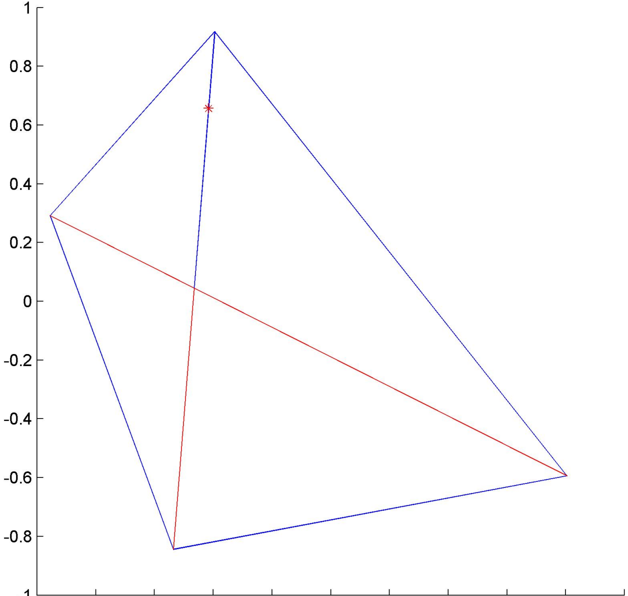


Figure 3: B(a) becomes a T-shaped region



We now have the tools to continue the theme of Proposition 9, treating the case when exactly **four** of the components of $t \in C(a)$ are positive.

Proposition 11: Suppose that $N > 4$ and that $t \in C(a)$ has exactly four positive components; for convenience, assume that $t_1, t_2, t_3, t_4 > 0$ and that a lies in the upper quadrant relative to $Q = \mathrm{conv}\{z_1, z_2, z_3, z_4\}$, ie a is interior to $\mathrm{conv}\{z_1, z_2, q\}$ (see Figure 2, with the understanding that it is now intended to show only the relation of a to z_1, z_2, z_3, z_4, and Proposition 10). Let β be the curve traced out by $b(\cdot)$ of Proposition 10(a) (and shown in Figure 2). Then

$$B(a, t) = \mathrm{conv}\{\beta, z_5, z_6, \ldots, z_N\}.$$

Proposition 11 allows us to understand, in large part, the phenomenon illustrated in Figure 1. Let $N = 5$ and suppose that each eigenvalue z_k is an extreme point of $W(M) = \text{conv}\{z_1, \ldots, z_5\}$ (eg whenever M is unitary). For convenience, label the z_k in counterclockwise order. Suppose that a lies strictly inside the central pentagon, ie $\Omega_2(M)$ as defined in (3). For each k let β_k denote the curve obtained as in Proposition 11 by regarding a as an element of the quadrilateral $Q_k = \text{conv}\{z_j : j \neq k\}$. Note that β_k connects z_{k+2} and z_{k+3} (numbering modulo 5) and lies in the quadrant of Q_k opposite to the one containing a. We claim that (as illustrated in Figure 1) $B(a)$ includes the whole “starfish” region bounded by $\beta_1, \beta_2, \ldots, \beta_5$.

To see this note that the starfish is the union of the wedges $W_k = \text{conv}\{\beta_k, z_k\}$, so it suffices to show that each $W_k \subseteq B(a)$. Since $a \in Q_k$ there is $t \in C(a)$ such that $t_k = 0$. Then Proposition 11 tells us that $B(a, c) = W_k$.

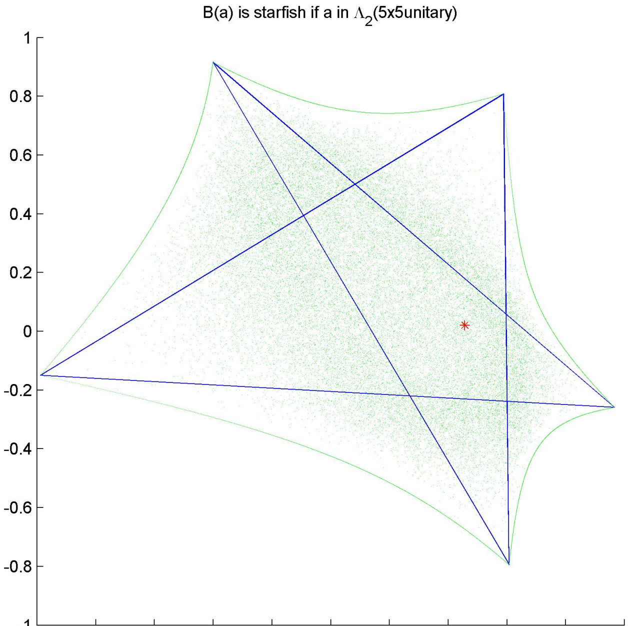
Figure 1 was obtained by first computing $C(a)$ via Lemma 8 as

$$\mathrm{conv}\{t(k, k+2, k+3) : k = 1, 2, \dots, 5\}$$

(note that for a in the inner pentagon, the only eigenvalue triangles containing a correspond to the triples z_k, z_{k+2}, z_{k+3}). To generate each of the thousands of b's in $B(a)$, plotted as green points in Figure 1, our MATLAB program first chose a "random" point $t \in C(a)$ (ie a random convex combination of the five $c(k, k+2, k+3)$), put $u = \sqrt{t}$, then computed $b = (Nw, w)$ where w was chosen "randomly" in

$$\mathbb{C}^5 \ominus \mathrm{span}\{u, u \circ \mathrm{Re}(z), u \circ \mathrm{Im}(z)\}$$

(and normalized so that $\|w\| = 1$). The curves β_k were added using the formula of Proposition 10(a). Such simulations strongly suggest the following "starfish conjecture", since no green dots fall outside the starfish: in such a situation (and in particular when $N = 5$ and M is unitary), $B(a)$ not only contains the starfish but is equal to it.


B(a) is starfish if a in Λ_2(5x5unitary)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1

Continuity of $B(\cdot)$

A natural assertion of "continuity" for $B(\cdot)$ might be that $d_H(B(a'), B(a)) \to 0$ as $a' \to a$, where $d_H(X, Y)$ is the Hausdorff distance between compact nonempty sets $X, Y \subset \mathbb{C}$. Recall that

$$d_H(X, Y) = \max\{\hat{d}_H(X, Y), \hat{d}_H(Y, X)\},$$

where

$$\hat{d}_H(X, Y) = \max_{x \in X}(\min_{y \in Y} |x - y|).$$

However, we have seen simple examples where this fails: consider again Figure 2 and 3.

Figure 2: B(a) is a curve, not the opposite quadrant

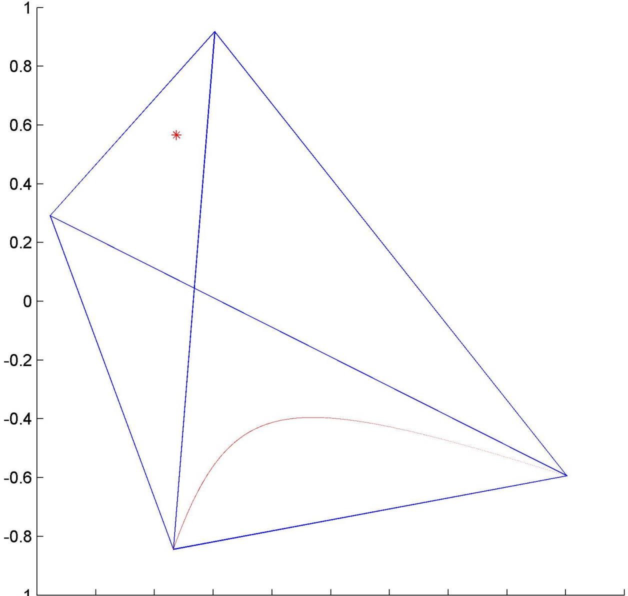
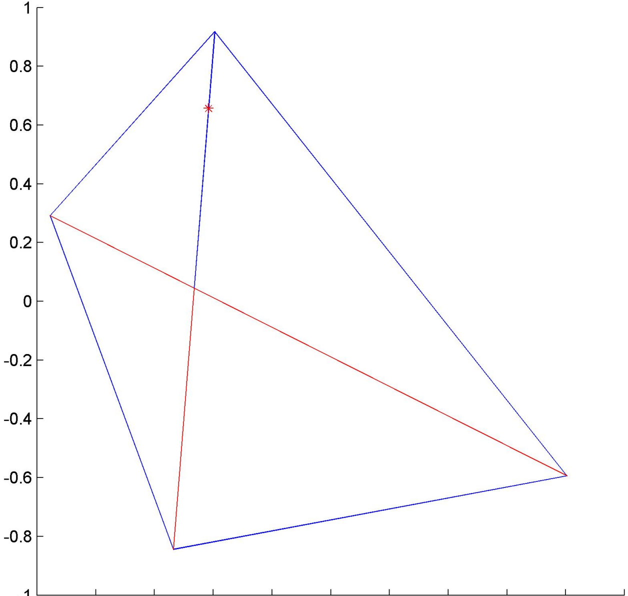


Figure 3: B(a) becomes a T-shaped region



In spite of such "failures" we'll show that $B(\cdot)$ is continuous with respect to Hausdorff distance at most points of $W(M)$ and enjoys a "one–sided" Hausdorff continuity in general.

Proposition 12: If $N \geq 4$, $B(a)$ is a compact nonempty set for any $a \in W(M)$.
The proof is a routine compactness argument and a related argument shows that, in general, $B(\cdot)$ is continuous in a one–sided Hausdorff sense.
Proposition 13: If $a, a_n \in W(M)$ and $a_n \to a$, then

$$\hat{d}_H(B(a_n), B(a)) \to_n 0. \tag{4}$$

In terms of the obvious extension of Hausdorff distance to compact nonempty subsets of Δ_N, we note that $C(\cdot)$ is continuous and in fact satisfies a Lipschitz condition for each fixed M.

Proposition 14: There is a constant $K < \infty$ depending only on M such that for all $a, a' \in W(M)$

$$d_H(C(a), C(a')) \leq K|a - a'|.$$

Next we show that $B(\cdot)$ is d_H–continuous at any point that is "off the grid", and that continuity is uniform if we stay bounded away from the grid.

Proposition 15: If $a \in W(M)$ but a does not lie on any line segment $[z_i, z_j]$, then $a' \to a$ implies that

$$d_H(B(a'), B(a)) \to 0.$$

In fact, on any subset $S(d) \subset W(M)$ that is a positive distance d from the grid

$$G = \bigcup \{[z_i, z_j] : i, j = 1, \ldots, N\},$$

so that

$$S(d) = \{a \in W(M) : \min_{g \in G} |a - g| \geq d\},$$

the map $a \mapsto B(a)$ is uniformly continuous.

Questions:

Many questions are suggested by this work. Here are a few natural examples.

If $a, b, c \in \Lambda_3(M)$ for normal M, does it follow that $\mathrm{diag}(a, b, c)$ is a compression?

More generally, what can we say about normal compressions (ie diagonal compressions) that are 3×3 and larger?

Can we prove the "starfish conjecture"?

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