LIMITING SPECTRAL DISTRIBUTION OF PATTERNED RANDOM MATRICES

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September 12, 2013

Plan

- Symmetric /skew symmetric patterned random matrices.
- Limiting spectral distribution (LSD).
- Extensions (joint convergence, free limit, half independence).
- Some questions.

Example: Wigner matrix

Symmetric matrix with i.i.d. entries.

$$W_n = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,(n-1)} & x_{1,n} \\ x_{1,2} & x_{2,2} & x_{2,3} & \cdots & x_{2,(n-1)} & x_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1,(n-1)} & x_{2,(n-1)} & x_{3,(n-1)} & \cdots & \cdots & \cdots \\ x_{1,n} & x_{2,n} & x_{3,n} & \cdots & \cdots & \cdots \end{bmatrix}. \quad (1)$$

Example: Toeplitz matrix

$$T_n = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\ x_1 & x_0 & x_1 & \cdots & x_{n-3} & x_{n-2} \\ x_2 & x_1 & x_0 & \cdots & x_{n-4} & x_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n-2} & x_{n-3} & x_{n-4} & \cdots & x_0 & x_1 \\ x_{n-1} & x_{n-2} & x_{n-3} & \cdots & x_1 & x_0 \end{bmatrix}.$$

$(i,j)th$ entry is $x_{|i-j|}$.

Toeplitz operator... Bai (1999).

Example: Hankel matrix

$$H_n = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\ x_2 & x_3 & x_4 & \cdots & x_n & x_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n-2} & x_{n-1} & x_n & \cdots & x_{2n-4} & x_{2n-3} \\ x_{n-1} & x_n & x_{n+1} & \cdots & x_{2n-3} & x_{2n-2} \end{bmatrix}.$$

$(i, j)th$ entry is x_{i+j-2}.

Hankel operator... Bai (1999).

Example: Reverse Circulant and Symmetric Circulant

$$R_n = \begin{bmatrix} x_2 & x_3 & x_4 & \dots & x_0 & x_1 \\ x_3 & x_4 & x_5 & \dots & x_1 & x_2 \\ x_4 & x_5 & x_6 & \dots & x_2 & x_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_0 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & x_0 \end{bmatrix}.$$

$(i,j)th$ entry is $x_{(i+j) \mod n}$.

This is the only k-(left shift) Circulant ($k = n - 1$) matrix which is symmetric (for arbitrary input sequences).

Symmetric circulant SC_n: this is the usual circulant, that is the 1-Circulant, with symmetry imposed on the matrix/sequence.

Example: Triangular matrices

Triangular versions of the matrices. For example, the Triangular Wigner matrix:

$$W_n^u = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,(n-1)} & x_{1,n} \\ x_{1,2} & x_{2,2} & x_{2,3} & \cdots & x_{2,(n-1)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1,(n-1)} & x_{2,(n-1)} & 0 & \ldots & 0 & 0 \\ x_{1,n} & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}. \quad (2)$$

Useful in operator theory (Dykema..).

Shall not discuss Band matrices and r-diagonal matrices.

Other symmetries

Hadamard products of (symmetric) matrices.

Skew symmetric matrices.

Examples: covariance, autocovariance and XX^T

Sample covariance matrix (S matrix). $n^{-1}X_{p\times n}X_{n\times p}^T$, where X is an IID matrix. So S is unadjusted. Here p, n both tend to ∞. [More generally, the columns of X are i.i.d. p-dimensional with dispersion matrix Σ_p].

$n^{-1}X_{p\times n}X_{n\times p}^T$ matrices, where X is not necessarily IID but has some specific pattern: signal processing and wireless communication.

Sample autocovariance matrix. For any time series x_t, $1 \leq t \leq n$,

$$\Gamma_n(X) = ((\gamma_{|i-j|})) \text{ where } \gamma(k) = n^{-1}\sum_{j=1}^{n-k} x_j x_{k+j}.$$

Crucial in time series analysis. It has the Toeplitz structure.

Empirical and limiting spectral distributions: ESD, LSD

ESD:

$$F_n(x) = n^{-1} \sum_{i=1}^{n} I\{\lambda_i \leq x\}.$$

Random probability distribution with mass $1/n$ *at each* λ_i.

How does it behave as $n \to \infty$?

Limiting spectral distribution (LSD): the weak limit of ESD either almost surely or in probability.

Assumption on entries

Any one:

1. For simplicity assume that it is uniformly bounded.
2. Entries are i.i.d. with finite second or some $4 + \delta$ moment (truncation).
3. Entries are independent with mean zero and variance one with uniformly bounded moment of all orders (joint convergence)

Input, link and patterned matrices

Input sequence. The sequence of variables used to construct the matrix: independent with mean zero and variance one.

Patterned matrix:

$$P_n = ((x_{L(i,j)})), \quad 1 \leq i, j \leq n.$$

Link function: L. $L(i,j) = L(j,i)$ (implies symmetry).

Interested in LSD of $n^{-1/2}P_n$. More generally in joint convergence (limit for traces of polynomials).

When do they exist? Can we have a unified approach? What are the nature of the limits?

Many researchers have established the LSD and its properties for specific matrices.

Link functions of the matrices

Wigner: $L(i,j) = (\min(i,j), \max(i,j))$.

Toeplitz: $L(i,j) = |i-j|$.

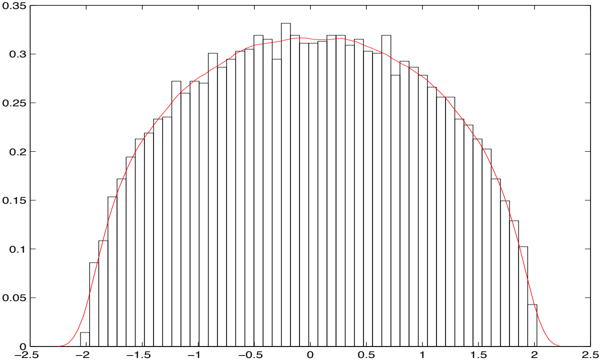
Hankel: $L(i,j) = i+j$.

Reverse Circulant: $L(i,j) = (i+j) \mod n$, $1 \leq i,j \leq n$.

Symmetric Circulant: $L(i,j) = n/2 - |n/2 - |i-j||$, $1 \leq i,j \leq n$.

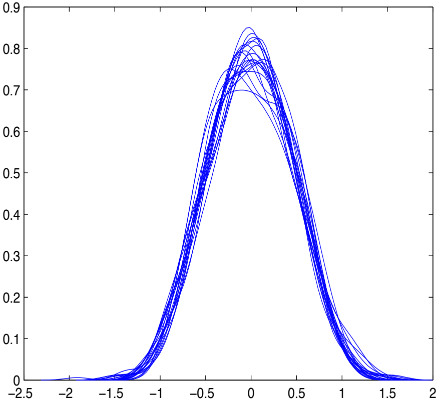
S: $L(i,j) = (i,j)$, $L^T(i,j) = (j,i)$.

Wigner



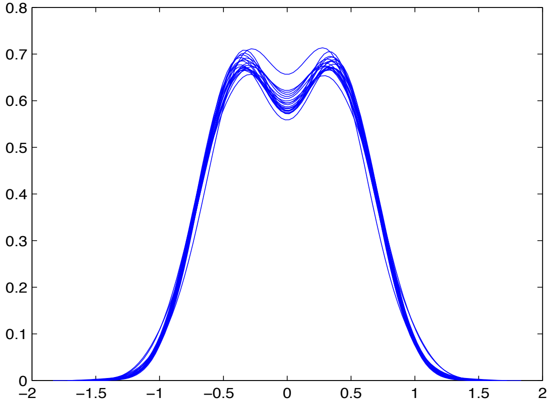
15 (scaled) Wigner matrices of order 400 with Bernoulli entries.

Toeplitz



15 realizations of $n^{-1/2}T_n$ with $n = 400$ and input sequence as i.i.d. normal$(0, 1)$.

Hankel



15 scaled Hankel matrices of order 400 with Bernoulli entries.

Digression to CLT: moment method and pair-partition

$\{x_i\}$ are independent uniformly bounded with mean zero and variance one. Let

$$Y_n = n^{-1/2}(x_1 + x_2 + \ldots + x_n).$$

Hence

$$\mathrm{E}[Y_n]^h = \frac{1}{n^{h/2}} \sum_{1\le i_1, i_2, \ldots, i_h \le n} \mathrm{E}[x_{i_1} x_{i_2} \cdots x_{i_h}].$$

Taking expectation and using elementary order calculations,

$$\begin{aligned} \mathrm{E}[Y_n]^{2k+1} &= o(1) \\ \mathrm{E}[Y_n]^{2k} &= \frac{(2k)!}{2^k k!} + o(1). \end{aligned}$$

$\frac{(2k)!}{2^k k!}$ = Total number of *pair-partitions* of $\{1, 2, \ldots 2k\}$.

$(ab....a..b..c..c...)$ (k letters, introduced alphabetically)

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Moment Method

The h-th moment of the ESD has the following nice form:

$$\frac{1}{n}\sum_{i=1}^{n}\lambda_i^h = \frac{1}{n}\operatorname{tr}(A^h) = \beta_h(A) \text{ (say)}.$$

If

(M1) $E[\beta_h(A_n)] \longrightarrow \beta_h$ (convergence of the average ESD).

(M2) $V[\beta_h(A_n)] \longrightarrow 0$ (relatively easier than (M1)).

(C) $\{\beta_h\}$ satisfies *Carleman's condition*:

$$\sum_{h=1}^{\infty}\beta_{2h}^{-1/2h} = \infty.$$

Then LSD of $\{A_n\}$ exists (in probability).

We will focus mainly on condition (M1).

Property B: subsequential limits, Gaussian domination, Carleman condition

Property B: Δ, the maximum number of times any variable appears in any row remains finite as dimension increases.

$\Delta = 1$: Wigner and Reverse Circulant.
$\Delta = 2$: Toeplitz, Hankel and Symmetric Circulant.

Result 1

Suppose Property B holds. Then the ESD of $\{n^{-1/2}P_n\}$ is tight a.s.. Any subsequential limit G satisfies,

(i) $\beta_{2k}(G) \leq \frac{(2k)!\Delta^k}{k!2^k}$ (implies Carleman's condition).

(ii) $\beta_{2k+1}(G) = 0$ (implies symmetry of G)

(iii) LSD exists for $\{n^{-1/2}P_n\}$ iff for every h,

$$\lim \mathrm{E}[\beta_h(n^{-1/2}P_n)] = \beta_h \text{ (say)}. \tag{3}$$

Circuit

h-th moment of $n^{-1/2}P_n$ is:

$$\frac{1}{n}\operatorname{Tr}\left(\frac{P_n}{\sqrt{n}}\right)^h = \frac{1}{n^{1+h/2}} \sum_{1 \le i_1, i_2, \ldots, i_h \le n} x_{L(i_1,i_2)} x_{L(i_2,i_3)} \cdots x_{L(i_{h-1},i_h)} x_{L(i_h,i_1)}. \tag{4}$$

Circuit $\pi : \{0, 1, 2, \cdots, h\} \to \{1, 2, \cdots, n\}$ with $\pi(0) = \pi(h)$.

$$\mathrm{E}[\beta_h(n^{-1/2}P_n)] = \frac{1}{n^{1+h/2}} \sum_{\pi:\ \pi \text{ circuit}} \mathrm{E}\,\mathbb{X}_\pi$$

where

$$\mathbb{X}_\pi = x_{L(\pi(0),\pi(1))} x_{L(\pi(1),\pi(2))} \cdots x_{L(\pi(h-2),\pi(h-1))} x_{L(\pi(h-1),\pi(h))}.$$

Matched and pair-matched circuits

Matched circuit: For any i, there is at least one $j \neq i$ such that

$$L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j)).$$

Pair-matched circuit: such values occur only in pairs.

π *non-matched* implies $\mathrm{E}(\mathbb{X}_\pi) = 0$. Hence

$$\mathrm{E}[\beta_h(n^{-1/2} P_n)] = \frac{1}{n^{1+h/2}} \sum_{\pi:\ \pi \text{ matched circuit}} \mathrm{E}\,\mathbb{X}_\pi .$$

Only matched circuits matter.

Equivalence of circuits: words

π_1 and π_2 are *equivalent* if they match at the same positions.

Equivalence class $\leftrightarrow$ partition of $\{1, 2, \cdots, h\} \leftrightarrow$ *word* w of length h of letters where the first occurrence of each letter is in alphabetical order.

Example: $h = 5$. $\{\{1, 3, 5\}, \{2, 4\}\} \leftrightarrow$ *ababa*.

$\Pi(w)$ = Equivalence class corresponding to w.

Then

$$\mathrm{E}[\beta_h(n^{-1/2}P_n)] = \frac{1}{n^{1+h/2}} \sum_{w} \sum_{\pi \in \Pi(w)} \mathrm{E}\, \mathbb{X}_\pi.$$

Limit moments: pair-matched words

- The first sum is a finite sum.
- $\mathrm{E}\,\mathbb{X}_\pi = 1$ for pair-matched words.

Let (for w of length h),

$$p(w) = \lim_n \frac{1}{n^{1+h/2}} \#\Pi(w) \quad \textit{whenever the limit exists.}$$

- Only pair-matched words survive if we assume Property B:

$$p(w) = 0 \text{ if } w \text{ is not pair-matched.}$$

Hence $\beta_{2k+1} = 0$ for all k.

IF $p(w)$ exists for each pair-matched word, then

$$\beta_{2k} = \sum_{w \text{ pair-matched}} p(w)$$

(w is of length $2k$).

The five matrices

Result 2

LSD exists for Wigner, Toeplitz, Hankel, Symmetric Circulant and Reverse Circulant.

Wigner matrix: Wigner (1955, 1958).....

Toeplitz (and Hankel): Bryc, Dembo and Jiang (2006); Hammond and Miller (2005).

Symmetric circulant: Bose and Mitra (2002); Bose and Sen (2008).

Reverse circulant: Bose and Mitra (2002); Bose and Sen (2008); Bose, Hazra and Saha (2011).

Symmetric Circulant, Gaussian limit

- Recall that Guassian is an “upper bound”.

- For Symmetric Circulant, $p(w) = 1$ for all (pair-matched) w.
Hence the LSD is standard Gaussian.

- Not so for other link functions.

- Guassian limit is an “exception”.

Highly palindromic Toeplitz (symmetric) circulant. Miller, Jackson and Pham (2012).

Symmetric words and Reverse Circulant

Symmetric word: each letter appears exactly once in an odd and exactly once in an even position.

abcabc is symmetric while *abab* is not.

There are $k!$ symmetric words of length $2k$.

For Reverse Circulant, $p(w) = 1$ for all symmetric words (and 0 otherwise). Hence the LSD is

$$f_R(x) = |x|e^{-x^2}, \ -\infty < x < \infty.$$

This is the distribution of the symmetrized square root of chi-square two distribution.

Catalan words, Wigner matrix and Semi Circle law

Catalan: a symmetric word where sequentially deleting all double letters leads to the empty word.

abcabc is symmetric but not Catalan. *abccba* is Catalan.

Catalan words

$\Updownarrow$

non-crossing pair-partitions

$\Updownarrow$

SRW paths, origin to origin, on or above the axis.

There are $\frac{(2k)!}{k!(k+1)!}$ Catalan words of length $2k$.

For the Wigner matrix $p(w) = 1$ if w is Catalan (and 0 otherwise). Hence the LSD is the semicircle law

$$f_w(x) = \frac{1}{2\pi}\sqrt{4 - x^2}, \ -2 \leq x \leq 2.$$

Toeplitz and Hankel

For both, $p(w) = 1$ if w is Catalan (Bose and Sen, 2008).

For symmetric words $p_T(w) = p_H(w)$ (Bose and Sen,2008).

LSD are unbounded (Bryc, Dembo and Jiang, 2006, Hammond and Miller, 2005).

Hankel LSD is not unimodal (Bryc, Dembo and Jiang, 2006).

Toeplitz LSD has a density which is bounded (Sen and Virag, 2011). Also, almost sure behavior of Max eval.

Word contribution

Table : Pair-matched words and moments, X symmetric.

MATRIX	w Cat.	w sym. not cat.	Other w	β_{2k} or LSD
SC	1	1	1	$\frac{(2k)!}{2^k k!}$, $N(0,1)$
R	1	1	0	$k!$, $\pm\sqrt{\chi_2^2}$
T	1	$p_T(w) \neq 0, 1$	$p_T(w) \neq 0, 1$	$\frac{(2k)!}{k!(k+1)!} \leq \beta_{2k} \leq \frac{(2k)!}{k! 2^k}$
H	1	$p_H(w) = p_T(w)$	0	$\frac{(2k)!}{k!(k+1)!} \leq \beta_{2k} \leq k!$
W	1	0	0	$\frac{(2k)!}{k!(k+1)!}$, semicircle

Triangular matrices

Result 3 (Basu, Bose, Ganguly and Hazra, 2012)

LSD exists for triangular versions of Wigner, Toeplitz, Hankel, Symmetric Circulant and Reverse Circulant. The LSD is symmetric.

Moments of LSD not known except for the triangular Wigner.

Triangular Wigner

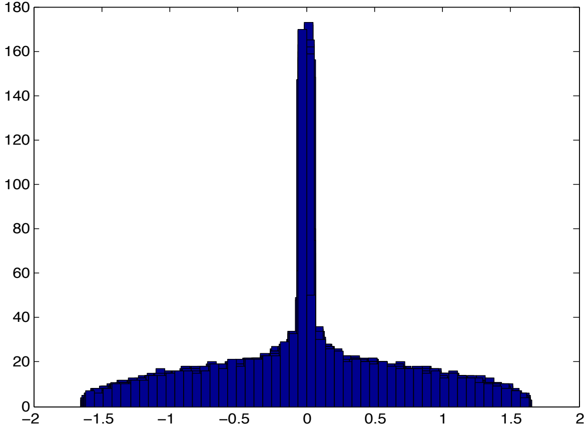
Only Catalan words contribute but not equally.

LSD is supported in $[-\sqrt{e},\ \sqrt{e}]$ with density unbounded at 0.

$$\beta_{2k} = \frac{k^k}{(k+1)!}.$$

Word	$p_u(w)$
aa	1/2
aabb	1/3
abba	1/3
aabbcc	1/4
abbcca	1/4
abbacc	5/24
aabccb	5/24
abccba	5/24

Triangular Wigner



Triangular Wigner. Similar distributions for other matrices.

S matrix and XX^T matrix limits

S matrix: Marčenko-Pastur (1967), Bai.....

XX^T: Bose, Gangopadhyay and Sen (2010).

MATRIX	w Cat.	w sym. not cat.	Other w	β_k or LSD
$p/n \to 0$ $\sqrt{\frac{n}{p}}(S - I_p)$, $\sqrt{\frac{n}{p}}(n^{-1}XX^T - I_p)$, $X = T,\ H,\ R,\ C$	1 (in p)	0	0	$\frac{(2k)!}{k!(k+1)!}$, semicircle $\mathcal{L}_T$
$p/n \to y \neq 0, \infty$ S $n^{-1}XX^T$ $X = T, H, R, C$	1	0	0	$\sum_{t=0}^{k-1} \frac{1}{t+1}\binom{k}{t}\binom{k-1}{t}y^t$ different, but universal

Autocovariance function and autocovariance matrix

$X = \{X_t\}$: *weakly stationary* process;

$\mathrm{E}(X_t) = 0$ and $\mathrm{E}(X_t^2) < \infty$.

Autocovariance function (ACVF) $\gamma(\cdot)$ and the *autocovariance matrix* (ACVM) Σ_n of order n are:

$\gamma(k) = cov(X_0, X_k),\ k = 0, 1, \ldots$ and $\Sigma_n = ((\gamma_X(i-j)))_{1 \le i,j \le n}$.

Σ_n is a Toeplitz matrix. By Szego's theorem, its LSD is given by $f(U)$ where U is a uniform random variable and f (assume $\sum |\gamma(k)| < \infty$) is given by

$$f(t) = \sum_{k=-\infty}^{\infty} \exp(-2\pi i t k)\gamma(k),\ t \in (0,\ 1]. \tag{5}$$

Assumptions

$X = \{X_t\}_{t\in\mathbb{Z}}$ is a linear (MA(∞)) process

$$X_t = \sum_{k=0}^{\infty} \theta_k \varepsilon_{t-k} \tag{6}$$

where $\{\varepsilon_t,\ t \in \mathbb{Z}\}$ is a sequence of random variables which are either

(a) $\{\varepsilon_t\}$ are *i.i.d.* with $\mathrm{E}[\varepsilon_t] = 0$ and $\mathrm{E}[\varepsilon_t^2] = 1$.

OR

(b) $\{\varepsilon_t\}$ are independent, uniformly bounded with $\mathrm{E}[\varepsilon_t] = 0$ and $\mathrm{E}[\varepsilon_t^2] = 1$.

Further, assume

$\sum_{j=0}^{\infty} |\theta_j| < \infty$.

LSD of sample autocovariance matrix

$\gamma(k)$ are estimated by $\hat{\gamma}(k) = n^{-1} \sum_{i=1}^{n-|k|} X_i X_{i+|k|}.$

The corresponding *Sample autocovariance matrix* is given by

$$\Gamma_n = ((\hat{\gamma}(i-j)))_{1 \le i,j \le n}. \tag{7}$$

Result 4 (Basak, Bose and Sen, 2013)

(i) The LSD of Γ_n exists.

(ii) The LSD is unbounded.

(iii) The LSD is never equal to $f(U)$ (inconsistency).

Tapering or banding alleviates this problem.

Skew symmetric

Wigner, Toeplitz and Symmetric circulant–remain unchanged.

Hankel and reverse circulant–changes , no closed form, (unimodal?).

Joint convergence: non-commutative prob. space

Non-commutative probability space: $(\mathcal{A}, \phi)$. $\mathcal{A}$ is a unital algebra, $\phi : \mathcal{A} \to \mathbb{C}$ is a linear functional satisfying $\phi(1) = 1$.

$\mathcal{A}_n$ = $n \times n$ real symm. random matrices, $\phi_n = \frac{1}{n}[\mathrm{Tr}(\cdot)]$.

$\mathcal{A}$ = algebra of polynomials in J non-commutative variables $a_1, \ldots, a_J$.

Result 7 (Bose, Hazra and Saha, 2011)

Consider J (indices) sequences of independent matrices $\{A_{i,n}, 1 \leq i \leq J\}$ where the link function L satisfies Property B. If $p(w)$ exists, then for any k and any monomial,

$$\lim \mathrm{Tr}\left[\frac{1}{n}\frac{A_{i_1,n}}{\sqrt{n}} \ldots \frac{A_{i_k,n}}{\sqrt{n}}\right] = \lim \phi_n(a_{i_1} \ldots a_{i_k}) = \phi(a_{i_1} \ldots a_{i_k}) \text{ (say)}$$

exists almost surely. $(\mathcal{A}_n, \phi_n) \to (\mathcal{A}, \phi)$.

Joint convergence: total, free and half independence

• Symmetric Circulants commute.
Each index pair-matched word contributes 1: totally indept.

• The Wigner limit.
Each indexed Catalan word contributes 1: free independent.

• The Reverse Circulants satisfy $ABC = CBA$ (half commute).
Each indexed symmetric word contributes 1: half independent.

It is not known what type of dependence is exhibited by Toeplitz and Hankel matrices.

Colors and indices

Different patterns: colors. So we have five colors (Wigner, Toeplitz, Hankel, Symmetric Circulant and Reverse Circulant).

Multiple copies: indices.

Result 8 (Basu, Bose, Ganguly and Hazra, 2012)

(i) Joint convergence holds if we have several indices of all the five colors.

(ii) The Wigner matrices are free of any of the other matrices (one color at a time).

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Remarks

(i) Property B implies convergence of the moments $\leftrightarrow$ existence of LSD. Subsequential limits always exist. What further restrictions on the link function implies the convergence of moments?

(ii) Under what conditions on the link does the LSD have bounded or unbounded support? Density? One mode?

(iii) Classes of limits possible under Property B ?

(iv) More on classes of contributing words and link functions?

(v) More about the LSD of T, H ?

(vi) More about the LSD of triangular T, H, ...?

(v) Further properties of joint convergence?

Half independence

Symmetric elements. $\{a_i\}_{i \in J} \subset \mathcal{A}$. For any $\{i_j\} \subset J$, let $a = a_{i_1} a_{i_2} \cdots a_{i_k}$.

Let $E_i(a)$ and $O_i(a)$ = the number of times a_i has occurred in the even/odd positions in a.

a is symmetric (wrt $\{a_i\}_{i \in J}$) if $E_i(a) = O_i(a)$ for all $i \in J$.

Half independent elements. Suppose $\{a_i\}_{i \in J}$ half commute (that is $a_i a_j a_k = a_k a_j a_i$ for all i, j, k).

$\{a_i\}$ are half independent if the following conditions are satisfied.

1. $\{a_i^2\}_{i \in J}$ are independent (joint moment splits).
2. For all non-symmetric a, $\phi(a) = 0$.

Boundedness assumption

$$
\begin{aligned}
k_n &= \#\{L_k(i,j) : 1 \le i,j,k \le n\}, \\
\alpha_n &= \max_k \#\{(i,j) : L_n(i,j) = k,\ 1 \le i,\ j \le n\}.
\end{aligned}
$$

$\alpha_n = 2$ for Wigner and $\alpha_n = O(n)$ for Symmetric Circulant, Reverse Circulant, Toeplitz and Hankel.

$k_n = O(n^2)$ for Wigner and $k_n = O(n)$ for Symmetric Circulant, Reverse Circulant, Toeplitz and Hankel.

Result

(The i.i.d. case): if $\alpha_n k_n = O(n^2)$, $k_n \to \infty$, then it is enough to restrict to bounded inputs among finite variance inputs.

Alternate assumption: (The non-i.i.d. case) Any power of the input sequence is uniformly integrable.

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