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The Garden of Eden theorem for cellular automata over amenable groups

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Configuration space

- A set, G group, $A^G = \prod_{g \in G} A = \{x : G \to A\}$

- $G \curvearrowright A^G$: $gx(h) = x(g^{-1}h)$

  EX ACTION $\left[1_G x = x \ ; \ (g_1 g_2)x = g_1(g_2 x) \right]$

- Prodiscrete topology = product topology (A discrete top)

  = smallest top s.t. $\pi_g : A^G \to A$, $x \mapsto x(g)$ is continuous $\forall g \in G$

  = SUBBASE elementary cylinders

$$C(g,a) = \{ x : x(g) = a \} ; \ g \in G, \ a \in A$$

  = NEIGHBORHOOD BASE

$$V(x, \Omega) = \{ y : y|_\Omega = x|_\Omega \} ; \ \Omega \subseteq G \text{ finite}$$

- if $\Omega \subset G$ finite, $p \in A^\Omega$ is called pattern. e.g. $x|_\Omega$

  EX
  - A^G is Hausdorff & totally disconnected
  - $G \curvearrowright A^G$ is continuous
  - $|A| < \infty \Rightarrow A^G$ is compact.
  - $|G| \leq |\mathbb{N}| \Rightarrow A^G$ metrizable

- Subshifts $X \subseteq A^G$ G-invariant & closed

  Let $\mathcal{P} \subseteq \bigcup_{\Omega \subset G \text{ finite}} A^\Omega$, $X_\mathcal{P} = \{ x : gx|_\Omega \notin \mathcal{P} \ \forall \Omega \subseteq G \text{ finite}, \forall g \in G \}$

  EX X subshift $\Leftrightarrow \exists \mathcal{P}$ s.t. $X = X_\mathcal{P}$. (defining set of forbidden patterns for X)

  EX $\Omega \subseteq G$ finite, $\mathcal{A} \subseteq A^\Omega$, $X(\mathcal{A}) = \{ x : gx|_\Omega \in \mathcal{A} \ \forall g \in G \}$

  X finite type if (def) $X = X(\mathcal{A})$. (Ω = memory)

  If A is finite X finite type $\Leftrightarrow \exists \mathcal{P}$ finite s.t. $X = X_\mathcal{P}$.

Cellular automata

$\tau : A^G \to B^G$ cellular automaton if $\exists\ \underbrace{M \underset{\text{fin}}{\subset} G}_{\text{memory}},\ \underbrace{\mu : A^M \to B}_{\text{local defining map}}$

$$\tau[x](g) = \mu\left((g^{-1}x)|_M\right) \quad \forall g \in G,\ x \in A^G$$

OBS $\tau[x](1_G) = \mu(x|_M)$.

EX Id_{A^G} ; $\tau(x)(g) = b_0$ (constant) ; $\tau(x)(g) = x(gg_0)$ g_0 fixed.

EXAMPLE (majority) $G = \mathbb{Z}$, $A = \{0, \bullet\}$, $M = \{-1, 0, 1\}$, μ = majority

$$\begin{array}{ll} x & \cdots\ \boxed{0\,|\,0\,|\,\bullet}\,|\,\bullet\ \cdots \\ & \quad\ \ \mu \downarrow \\ \tau(x) & \cdots\ \boxed{0}\,|\,\bullet\ \cdots \end{array}$$

EXAMPLE (Hedlund's marker) $G = \mathbb{Z}$, $A = \{0,1\}$, $M = \{-1, 0, 1, 2\}$

$$\mu(a_{-1}, a_0, a_1, a_2) = \begin{cases} 1 - a_0 & \text{if } (a_{-1}, a_1, a_2) = (0,1,0) \\ a_0 & \text{otherwise} \end{cases}$$

$$\begin{array}{ll} x & 0101\,\boxed{1001}\,001 \\ & \qquad \mu \downarrow \\ \tau(x) & 1001\,\boxed{0}\,110 \end{array}$$

Show that τ is a non-trivial involution : $\tau^2 = \mathrm{Id}_{A^G}$.

EXAMPLE (Conway's game of life) $G = \mathbb{Z}^2$, $A = \{0,1\}$, $M = \boxed{3\times 3}$

μ :

$$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0&0\\\hline 1&1&1\\\hline\end{array} \xrightarrow{\mu} \begin{array}{|c|c|c|}\hline &&\\\hline &1&\\\hline &&\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1&0\\\hline 1&1&*\\\hline\end{array} \xrightarrow{\mu} \begin{array}{|c|c|c|}\hline &&\\\hline &1&\\\hline &&\\\hline\end{array}$$

BIRTH SURVIVAL

otherwise a dead cell will remain dead

$$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1&0\\\hline 0&0&*\\\hline\end{array} \xrightarrow{\mu} \begin{array}{|c|c|c|}\hline &&\\\hline &&\\\hline &&\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline *&*&*\\\hline *&1&1\\\hline 1&1&1\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline &&\\\hline &&\\\hline &&\\\hline\end{array}$$

DEATH DEATH

EXAMPLE G, $S \underset{\text{finite}}{\subset} G$, $A = \mathbb{R}$

$$\Delta_S(x)(g) = x(g) - \frac{1}{|S|}\sum_{s \in S} x(gs) \qquad \text{DISCRETE LAPLACIAN}$$

linear cellular automaton (A is a vector space$/\mathbb{K}$, μ is $\mathbb{K}$-linear $\iff$ τ is $\mathbb{K}$-linear
A^G natural structure of vector space$/\mathbb{K}$)

EXAMPLES Other categories (affine algebraic sets, topological manifolds, etc)

Theorem $\tau : A^G \to A^G$ is continuous & G-equivariant
$g\tau(x) = \tau(gx)$

Proof $\tau(V(x, \Omega M)) \subseteq V(\tau(x), \Omega)$.

Theorem [Curtis Hedlund] $|A| < \infty$, τ is cellular automaton $\iff$ τ is continuous & G-equiv

Corollary $|A| < \infty$, $\tau \circ \sigma$ is a cellular automaton.

EXERCISE [Prop 1.4.9] $\tau : A^G \to B^G$, $\sigma : B^G \to C^G$
$\mu : A^M \to B$ $\quad \nu : B^N \to C$

then $NM = \{nm : n \in N, m \in M\} \subseteq G$ memory set for $\sigma \circ \tau$.
Find $\kappa : A^{NM} \to C$.

EXERCISE $\tau : A^G \to B^G$ cellular automaton $\mu_1 : A^{M_1} \to B$
$\mu_2 : A^{M_2} \to B$ $\Rightarrow$ $\mu : A^M \to B$, $M = M_1 \cap M_2$
$\mu(p) = \mu_i(p_i)$ $\quad p_i|_M = p$ (recall $\tau(x)(1_G) = \mu(x|_M)$).

$CA(G;A)$ MONOID $= \{\tau : A^G \to A^G\}$

3 bis

EXAMPLE G infinite (e.g. $G=\mathbb{Z}$), $A = G$ $\quad \tau : A^G \to A^G$

$$\tau(x)(g) = x(g \cdot x(g))$$

EXERCISE
- τ is continuous
- τ is G-equivariant
- τ is not a cellular automaton!

<u>Induction & restriction</u> G, $\underset{\text{subgroup}}{H \leq G}$, A, B

$M \subseteq H$, $\mu : A^M \to B$ give $\tau^G : A^G \to B^G$

& $\tau_H : A^H \to B^H$.

- $CA(G,H;A) = \{\tau : A^G \to A^G,\ M \subseteq H\} \longleftrightarrow CA(H;A)$
- $G/H = \{gH : g \in G\}$ <u>left cosets</u> of H in G

$$G = \bigsqcup_{c \in G/H} c \implies A^G = \prod_{c \in G/H} A^c,\quad B^G = \prod_{c \in G/H} B^c$$

$$x \in A^G \Leftrightarrow (x_c)_{c \in G/H} \text{ where } x_c = x|_c .$$

<u>EXERCISE</u>

- $\tau^G = \prod_{c \in G/H} \tau_c$, $\tau_c : A^c \to A^c$ $\tau_c(x_c) = \tau(x)|_c$
- if $c \in G/H$, $g \in c$, $\phi_g : H \to c$, $h \mapsto gh$ BIJECTIVE $\rightsquigarrow$

$\phi_g^* : A^c \to A^H$, $x \mapsto x \circ \phi_g$ BIJECTIVE ; show :

$$\begin{tikzcd} A^c \arrow[r, "\tau_c"] \arrow[d, "\phi_g^*"'] & A^c \arrow[d, "\phi_g^*"] \\ A^H \arrow[r, "\tau_H"'] & A^H \end{tikzcd}$$

commuting diagram!

<u>EXERCISE</u>

τ^G injective $\Leftrightarrow$ τ_H <u>injective</u> (bijective)

τ^G surjective $\Leftrightarrow$ τ_H <u>surjective</u>

$\tau^G(A^G)$ closed in B^G $\Leftrightarrow$ $\tau_H(A^H)$ <u>closed</u> in B^H w.r. prodiscrete topologies (cf Thm 8.8.1)

τ^G pre-injective $\Leftrightarrow$ τ_H <u>pre-injective</u> (Prop. 5.2.2)

τ^G invertible $\Leftrightarrow$ τ_H <u>invertible</u> (Prop 1.10.4)

$$(\tau^{-1})_H = (\tau^G)^{-1}$$

4 bis

Invertible CA: $\tau : A^G \to B^G$ is <u>invertible</u> if
- τ bijective
- $\tau^{-1} : B^G \to A^G$ is a CA.

<u>EX</u> $|A| < \infty$ τ bijective CA $\Rightarrow$ τ invertible. (topological proof)
FIND ALGEBRAIC PROOF [LM]

<u>EXAMPLE</u> $\mathbb{K}$ field, $A = \mathbb{K}[[t]]$ formal power series.
$G = \mathbb{Z}$. Consider $\tau : A^G \to A^G$ $\tau(x)(n) = x(n) - t\,x(n+1)$

- τ is a CA (memory $M = \{0,1\}$, $\mu(a_0, a_1) = a_0 - t a_1$)
- τ is bijective ($\tau^{-1}(x)(n) = x(n) + t\,x(n+1) + t^2 x(n+2) + \cdots$)
- τ^{-1} is <u>NOT</u> a CA ($F \underset{\text{fin}}{\subset} \mathbb{Z}$, let $M \geq 0$ s.t. $F \subset (-\infty, M]$ let $y, z \in A^G$
  $y(n) = \begin{cases} 0 & n \leq M \\ 1 & n \geq M+1 \end{cases}$ $z \equiv 0$; $y|_F \equiv z|_F$ but
  $\tau(y)(0) = t^{M+1} + t^{M+2} + t^{M+3}$ & $\tau(z)(0) = 0$).

The Garden of Eden Theorem of Moore & Myhill 1963

G, A, $\tau : A^G \to A^G$ cellular automaton

- $x \in A^G \setminus \tau(A^G)$ is a <u>GOE configuration</u> (τ surjective $\iff \nexists$ GOE configurations) (<u>Biblical</u> terminology only appearing at time $t \leq 0$.)

- $\Omega \underset{\text{finite}}{\subset} G$, $p \in A^\Omega$ is a <u>GOE pattern</u> if $\nexists\, x \in A^G$ s.t. $\tau(x)\big|_\Omega = p$
  $\iff y \in A^G$ s.t. $y\big|_\Omega = p$ is GOE.

<u>EX</u> τ not surjective $\underset{(\Leftarrow)}{\Longrightarrow} \exists$ GOE pattern.

- $X_1, X_2 \in A^G$ are <u>almost equal</u> if $\{g \in G : X_1(g) \neq X_2(g)\}$ is finite
  $\tau : A^G \to B^G$ is <u>pre-injective</u> if
  $$\left[\begin{matrix} X_1, X_2 \in A^G \text{ almost equal} \\ \tau(X_1) = \tau(X_2) \end{matrix}\right] \Longrightarrow X_1 = X_2.$$

<u>EX</u> $G = \mathbb{Z}$, $A = \mathbb{Z}/3\mathbb{Z}$ $\tau : A^G \to A^G$
$$\tau(x)(n) = x(n-1) + x(n) + x(n+1)$$
Pre-injective not injective
(if X_1, X_2 are almost equal and $\Omega = \{n \in \mathbb{Z} : x_1(n) \neq x_2(n)\}$, if $n_0 = \max \Omega$, then $\tau(X_1)(n_0+1) \neq \tau(X_2)(n_0+1)$;
$X_1 \equiv 0$ & $X_2 \equiv 1$ satisfy $\tau(X_1) = \tau(X_2) \equiv 0$).

- if A is a vector space, $A[G] = \{x \in A^G : \operatorname{Supp} x \text{ is finite}\} \subseteq A^G$
  Show :
  - $\tau(A[G]) \subseteq A[G]$
  - $\tau : A^G \to A^G$ is pre-injective $\iff \tau\big|_{A[G]} : A[G] \to A[G]$ injective.

- given $\tau : A^G \to A^G$, $\Omega \underset{\text{finite}}{\subset} G$, $P_1, P_2 \in A^\Omega$ are called MUTUALLY ERASABLE if $P_1 \neq P_2$ & $\forall\, x_1, x_2 \in A^G$ s.t. $\begin{cases} X_i\big|_\Omega = P_i \\ X_1\big|_{G \setminus \Omega} = X_2\big|_{G \setminus \Omega} \end{cases}$
  one has $\tau(x_1) = \tau(x_2)$.

EXAMPLE (GofLife) & .

PROPOSITION $\tau : A^G \to A^G$ is pre-injective $\iff \nexists$ ME patterns

Theorem (MM 1963) $G = \mathbb{Z}^2$, A finit $\tau : A^G \to A^G$ cellular automata

- $\exists$ GOE patterns $\Longleftrightarrow$ $\exists$ GOE configurations $\Leftrightarrow$ τ not-surjective
- $\exists$ ME patterns $\underset{\text{Prop 5.5.2}}{\Longleftrightarrow}$ τ is not-pre injective

($\Leftrightarrow$ τ is surjective iff τ is pre-injective)

Corollary [Richardson] Every injective CA $\tau : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ with $|A| < \infty$ is surjective.

Def G is surjunctive if $\forall$ finite A, every injective CA
[Gottschalk]
$\tau : A^G \to A^G$ is surjective

$f : A \to A$, $\lvert A \rvert < \infty$ injective $\Rightarrow$ surjective
$f : V \to V$, $\dim_{\mathbb{K}} V < \infty$ linear injective $\Rightarrow$ surjective
$f : A \to A$, affine alg set / $\mathbb{K}$ uncountable alg. closed injective $\xRightarrow{\text{Ax-Grothendieck}}$ surjective e.g. $\mathbb{C}^n$

Thus $\mathbb{Z}^2$ is surjective.

Open problem $\exists$ G which is not surjunctive?

EXERCISE
- class of surjunctive groups closed under SUBGROUPS
- G residually finite $\Rightarrow$ G surjunctive (e.g. free groups)

Thm [3.7.1] Γ group. Then the set $\mathcal{S}(\Gamma)$ of normal subgroups $N \subset \Gamma$ such that Γ/N is surjunctive is closed in $\mathcal{N}(\Gamma)$ (the space of all normal subgroups of Γ equipped with the Gromov-Hausdorff topology).

Amenable groups

D G group is <u>amenable</u> (von Neumann, 1929) if it admits a left-invariant, finitely-additive, probability measure on the set of all its subsets; i.e. $\exists\ \mu : \mathcal{P}(G) \longrightarrow [0,1]$

- $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$
  $[\mu(A \sqcup B) = \mu(A) + \mu(B)]$
- $\mu(gA) = \mu(A)$
- $\mu(G) = 1$

EX G finite $\Rightarrow$ G amenable : $\mu(A) = \dfrac{|A|}{|G|}$.

EX $\mathbb{F}_2 = \langle a, b \rangle$ the free group on two generators is non-amenable

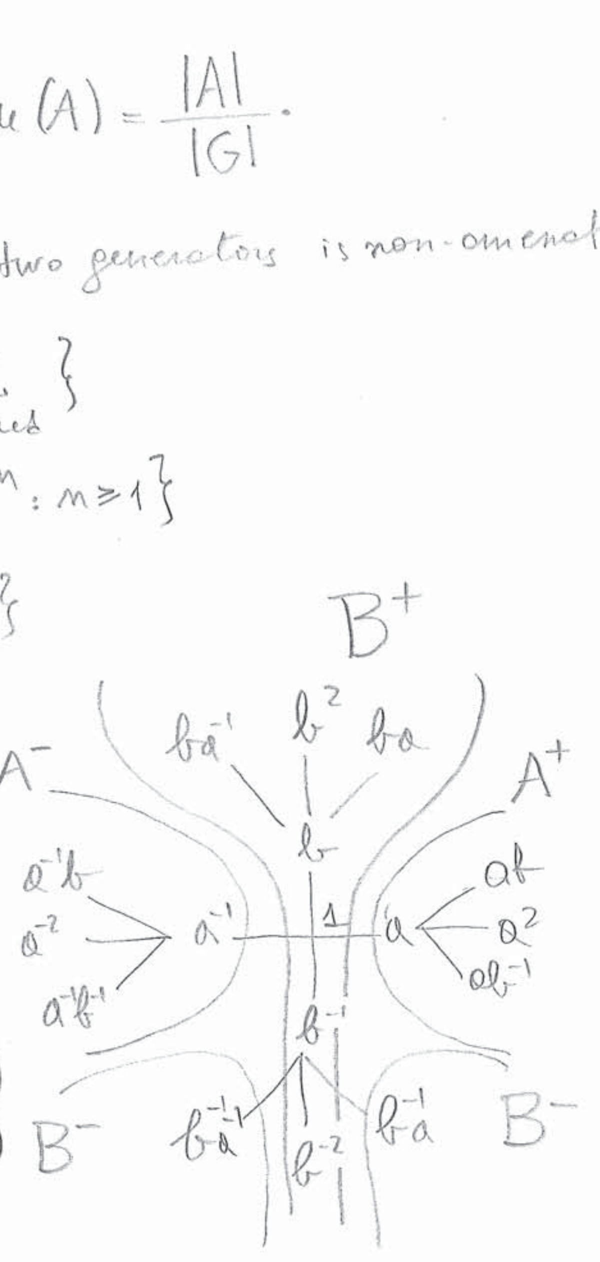
$$A^+ = \{ \underbrace{aw}_{\text{reduced}} \},\quad A^- = \{ \underbrace{a^{-1}w}_{\text{reduced}} \}$$

$$B^+ = \{ \underbrace{bw}_{\text{reduced}} \} \cup \{1\} \cup \{ b^{-n} : n \geq 1 \}$$

$$B^- = \{ \underbrace{b^{-1}w}_{\text{reduced}} \} \setminus \{ b^{-n} : n \geq 0 \}$$

then

$$(*) \begin{cases} \mathbb{F}_2 = A^+ \sqcup A^- \sqcup B^+ \sqcup B^- \\ \quad = A^+ \sqcup aA^- \quad \left(\text{e.g. } ba = a(\underbrace{a^{-1}ba}_{\text{reduced}})\right) \\ \quad = B^+ \sqcup bB^- \end{cases}$$



yielding

$$\begin{aligned} 1 = \mu(\mathbb{F}_2) &= \mu(A^+) + \mu(A^-) + \mu(B^+) + \mu(B^-) \\ &= \mu(A^+) + \mu(aA^-) + \mu(B^+) + \mu(bB^-) \\ &= \mu(\mathbb{F}_2) + \mu(\mathbb{F}_2) \\ &= 2. \ \lightning \end{aligned}$$

The situation in (*) can be generalized as follows:

G is <u>paradoxical</u> if $G = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_m \sqcup B_1 \sqcup \cdots \sqcup B_n$
$= g_1 A_1 \sqcup g_2 A_2 \sqcup \cdots \sqcup g_m A_m$
$= h_1 B_1 \sqcup h_2 B_2 \sqcup \cdots \sqcup h_n B_n$

(**) (m,n)-paradoxical decomposition

<u>EX</u> G <u>paradoxical</u> $\Rightarrow$ G <u>non-amenable</u>.

Consider now the real Banach space $\ell^\infty(G) = \{x : G \to \mathbb{R} \text{ bounded}\}$
$\|x\|_\infty = \sup_{g \in G} |x(g)|$; we set $\underbrace{x \leq y}_{\text{ordering}}$ if $x(g) \leq y(g)\ \forall g \in G$

<u>Def</u> A MEAN on G is a linear map $m : \ell^\infty(G) \to \mathbb{R}$ s.t.
- $m(\mathbb{1}) = 1$
- $m(x) \geq 0\ \forall x \geq 0$

$[g\mu](A) = \mu(g^{-1}A)$
$[\mu g](A) = \mu(Ag^{-1})$
$(gm)(x) = m(g^{-1}x)$
$(mg)(x) = m(xg^{-1})$

<u>EX</u> if $\underset{\text{countable}}{S \subset G}$ and $f : S \to \mathbb{R}$ satisfies $f(s) > 0$, $\sum_{s \in S} f(s) = 1$
then $m_f(x) = \sum_{s \in S} f(s) x(s)$ is a mean.

<u>Prop</u> $m \in (\ell^\infty(E))^*$ and $\|m\| = 1$ $\left(\|m\| = \sup_{\substack{x \in \ell^\infty(G) \\ \|x\|_\infty \leq 1}} |m(x)| \right)$

A mean on G is <u>left-invariant</u> if $m(gx) = m(x)\ \forall g \in G, x \in \ell^\infty(G)$

<u>Theorem</u> $\mathcal{M}(G) = \{\text{means on } G\}$, $\mathcal{PM}(G) = \{\text{probability measures on } G\}$
$\Phi : \mathcal{M}(G) \to \mathcal{PM}(G)$ $\quad \hat{m}(A) = m(\chi_A)$
$m \mapsto \hat{m}$
is bijective.

<u>Proof</u> IDEA $\mathcal{E}(G) := \{x : G \to \mathbb{R}\ \ x(G) \text{ finite}\}$ is <u>dense</u> in $\ell^\infty(G)$
$x = \sum_i a_i \chi_{A_i}$ then $\check{\mu}(x) := \sum_i a_i \mu(A_i)$, $\check{\mu} \in \mathcal{M}(G)$. □

<u>OSS</u> m is left-invariant $\iff$ $\hat{m}$ is left-invariant
$\boxed{\widehat{gm} = g\hat{m}}$

$$\tau(G) \in \{4, 5, 6, \dots, +\infty\}$$

Suppose G paradoxical. Then $\tau(G) = \min\{n+m : (n,m)\text{-paradoxical decomposition}\}$

<u>TARSKI NUMBER</u>. We set $\tau(G) = \infty$ if G is <u>not</u> paradoxical

<u>EX</u> . $H \leq G \Rightarrow \tau(G) \leq \tau(H)$ (cfr. theorem on stability properties of amenability)

. $G \twoheadrightarrow H \Rightarrow \tau(G) \leq \tau(H)$

⊙ $\tau(G) = 4 \underset{(*)}{\Longleftrightarrow} G \supseteq \mathbb{F}_2$

⊙ $\tau(G) \geq 6 \Leftarrow G$ is torsion

(*) <u>Ping-pong lemma (F. Klein)</u>.

Suppose $G = \langle X \rangle$ acts on a set E and $\exists (A_x)_{x\in X}$ $|X|\geq 2$

family of <u>pairwise disjoint</u> subsets of E s.t.

$$x^k\Big(\bigcup_{y\in X\setminus\{x\}} A_y\Big) \subset A_x \quad \forall x \in X,\ \forall k \in \mathbb{Z}\setminus\{0\}$$

Then $G \simeq \mathbb{F}_X$ (free group based at X).

<u>ES</u> $SL_2(\mathbb{Z}) \supseteq \Big\{ \underset{a}{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}, \underset{b}{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}} \Big\}$.

acts on $\mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\} = \{\text{lines of slope } 1/t : t \in \mathbb{R}\cup\{\infty\}\}$

$$gt = \frac{g_{11}t + g_{12}}{g_{21}t + g_{22}} \qquad a^k t = t + 2k$$
$$b^k t = \frac{t}{2kt+1}$$

$$Y =]-1,1[\ , \quad Z = \mathbb{P}_1(\mathbb{R}) \setminus [-1,1]$$

$$a^k Y =]2k-1, 2k+1[\subset Z$$
$$b^k Z =]1/(2k+1), 1/(2k-1)[\subset Y$$

$\Rightarrow \langle a, b \rangle \cong \mathbb{F}_2$.

<u>Theorem</u> $\mathcal{M}(G)$ is a <u>convex</u> <u>compact</u> subset of $(\ell^\infty(G))^*$ w.r.t. the weak-$*$ topology.

<u>Proof</u> convexity check!
closed : limit properties
Banach-Alaoglu Theorem $(\ell^\infty(G))^*_1$ is <u>compact</u>. $\square$

$\underset{\text{left-invariant}}{\mu \in \mathcal{PM}(G)} \Rightarrow \underset{\text{right-invariant}}{\mu^*(A) := \mu(A^{-1})} \quad \mu^* \in \mathcal{PM}(G)$

$\underset{\text{left-invariant}}{m \in \mathcal{M}(G)} \Longrightarrow \underset{\text{right-invariant}}{m^* \in \mathcal{M}(G)}, \quad m^*(X) = m(X^*), \quad X^*(g) = X(g^{-1})$

<u>Prop</u> If $\exists$ left-invariant mean $m : \ell^\infty(G) \to \mathbb{R}$ on G then $\exists$ bi-invariant mean.

<u>Proof:</u> $m \in \mathcal{M}(G)$ left-invariant. Consider $\tilde{x} : G \to \mathbb{R}$

$\tilde{x}(g) = m(xg) \quad \forall g \in G$. Then

$|\tilde{x}(g)| \leq \|xg\|_\infty = \|x\|_\infty \Longrightarrow \tilde{x} \in \ell^\infty(G)$

then $M : \ell^\infty(G) \to \mathbb{R}$ s.t. $M(x) = m(\tilde{x})$ is a mean.
EXERCISE M is bi-invariant.

<u>Cor</u> G amenable $\Longleftrightarrow$ $\exists$ left (resp. bi-invariant) mean on G
$\Longleftrightarrow$ bi-invariant prob. finitely additive prob measure.

Stability properties

Thm AG, the class of all amenable groups, is closed under

- subgroups (so that if $G \supseteq \mathbb{F}_2$, G is non-amenable)
- quotients
- $1 \to \underset{\text{amenable}}{N} \to G \to \underset{\text{amenable}}{G/N} \to 1 \implies G$ amenable
  (Prop. 4.5.5) (so that $G_1 \times G_2$ is amenable, provided G_i are)
- $G = \varinjlim G_i$ is amenable if G_i is amenable $\forall i \in I$.
  (e.g. $G_1 \leq G_2 \leq \cdots$ $G = \bigcup G_i$)

Thm Every abelian group is amenable.

Proof $X = (\ell^\infty(G))^*$ with the weak-$*$ topology is topological vector space.

$K = \mathcal{M}(G)$ is convex $\neq \emptyset$.

$\mathcal{F} = \{ f_g : K \to K \}$, where $f_g(m) = gm$

$\xrightarrow[\text{theorem}]{\text{Markov-K}} \exists\, m_0 \in K$ $\mathcal{F}$-fixed $\iff m_0$ is $\underset{(G \text{ is abelian})}{\text{bi-invariant.}}$ □

Solvable & Nilpotent groups are amenable

Theorem Every solvable (and therefore every nilpotent) group is amenable

Proof By induction on the solvability degree + EXTENSION thm. □

10 bis Subgroups: $\tilde{\mu}(A) = \mu(\bigsqcup_{s\in S} A_s)$

quotients: $\hat{\mu}(A) = \mu(g^{-1}(A))$

extensions: $x\in \ell^\infty(G)$, $\tilde{x}: G/H \to \mathbb{R}$, $\tilde{x}(gH) = m_H((g^{-1}x)|_H)$ well defined

$\& \ \tilde{x}\in \ell^\infty(G/H)$

$m(x) = m_{G/H}(\tilde{x})$.

Thm [Markov-Kakutani] $\underset{\phi^{\#}}{\overset{\text{convex - compact}}{K}} \subseteq \underset{\text{Hausdorff topological vector space}}{X}$

$\mathcal{F} \subseteq \{f: K \to K \text{ continuous, affine}\}$

Suppose $f_1\circ f_2 = f_2 \circ f_1$ $\forall f_1, f_2 \in \mathcal{F}$. Then $\exists$ common fixed point $k_0\in K$:

$f(k_0) = k_0$ $\forall f\in \mathcal{F}$.

Commutators

$[h,k] := h^{-1}k^{-1}hk$

$H, K \leq G$, $[H,K] = \langle [h,k] : h\in H, k\in K\rangle$

$D(G) = [G,G]$ derived subgroup

$(D^i(G))_{i\geq 0}$ derived series: $D^{(0)}(G) = G$, $D^{(i)} = [D^{(i-1)}(G), D^{(i-1)}(G)]$ $i\geq 1$

G is solvable if $\exists i_0 \geq 0$ s.t. $D^{(i_0)}(G) = \{1\}$. The smallest i_0 is the solvability degree

$(\gamma_i(G))_{i\geq 0}$ lower central series: $\gamma_0(G) = G$, $\gamma_i(G) = [G, \gamma_{i-1}(G)]$ $i\geq 1$

G is nilpotent if $\exists i_0 \geq 0$ s.t. $\gamma_{i_0}(G) = \{1\}$. The smallest i_0 is the nilpotency degree E.G. Abelian of degree 1

Heisenberg of degree 2

OBS $D^i(G) \subseteq \gamma_i(G)$ $\forall i\geq 0$ (INDUCTION)

Corollary G nilpotent $\Rightarrow$ G solvable

Følner conditions

Prop TFAE for a group G.

(a) $\forall K \subseteq_{\text{finite}} G$, $\forall \varepsilon > 0$ $\exists F \subseteq_{\substack{\text{finite}\\ \neq \emptyset}} G$ s.t. $\frac{|F \setminus kF|}{|F|} < \varepsilon$ $\forall k \in K$;

(b) $\exists (F_j)_{j \in J}$ s.t. $\lim_j \frac{|F_j \setminus gF_j|}{|F_j|} = 0$ $\forall g \in G$

Proof ($a \Rightarrow b$):

$$J = \{(K,\varepsilon) : K \subset_{\text{finite}} G,\ \varepsilon > 0\}$$ with partial ordering

$$(K,\varepsilon) \leq (K',\varepsilon') \text{ if } K \subseteq K' \ \&\ \varepsilon' \leq \varepsilon.$$

is a directed set (indeed J is a lattice) $\sup(J_1, J_2)$ join, $\inf(J_1, J_2)$ meet

Let $(F_j)_{j \in J}$ given by (a).

Fix $g \in G$ and $\varepsilon_0 > 0$. Set $j_0 = (\{g\}, \varepsilon_0)$. If $j = (K,\varepsilon) \geq j_0$ then $g \in K$ and $\varepsilon < \varepsilon_0$; it follows $\frac{|F_j \setminus gF_j|}{|F_j|} < \varepsilon_0$ $\forall j \geq j_0$

($b \Rightarrow a$) Fix $K \subseteq_{\text{finite}} G$ and $\varepsilon > 0$. Then $\forall k \in K$ $\exists j_k \in J$ s.t. $\frac{|F_j \setminus kF_j|}{|F_j|} < \varepsilon$ $\forall j \geq j_k$. Since J is directed, $\exists \bar{j}$ such that $\bar{j} \geq j_k$ $\forall k \in K$. Setting $F := F_{\bar{j}}$ we have $\frac{|F \setminus kF|}{|F|} < \varepsilon$ $\forall k \in K$. $\square$

- G satisfies the <u>Følner condition</u> if it satisfies (a) and/or (b).
- $(F_i)_{i \in I}$ is called a (left) <u>Følner net</u>. (if $|I| = |\mathbb{N}| \to$ <u>Følner sequence</u>)

<u>EX</u>. If $|G| < \infty$ then if $F_n = G$ then $(F_n)_{n \in \mathbb{N}}$ is Følner

<u>EX</u> $G = \mathbb{Z}$ (more generally $G = \mathbb{Z}^d, d \geq 1$) then $F_n = \{-n, -n+1, \dots, -1, 0, \dots, n\}$

<u>EX</u> $G = \mathbb{R}$? (Use the above Prop! And some $\mathbb{Z}$-modules)

<u>EX</u> Suppose G is countable amenable then $\exists\, (F_n)_{n\in\mathbb{N}}$ s.t. $F_n \subseteq F_{n+1}$, $\bigcup_{n=1}^{\infty} F_n = G$

The Theorems of Tarski & Følner (the alternative; combinatorial characterizations)

Thm TFAE

(a) G is amenable;
(b) G admits no paradoxical decomposition;
(c) G satisfies the Følner condition.

We shall prove:

Thm TFAE

(a) G is not amenable
(b) G does not satisfy the Følner condition
(c) $\exists K \subset G$ finite s.t. $|KF| \geq 2|F|$ $\forall F \subset G$ finite (Gromov's doubling condition)

(d) $\exists$ 2-to-1 surjective map $\varphi: G \to G$ and $K \subset G$ finite s.t. $g\varphi(g)^{-1} \in K$ $\forall g \in G$

(e) G admits a paradoxical decomposition.

Proof (a) $\Rightarrow$ (b) Suppose G satisfies the Følner condition and let $(F_j)_{j\in J}$ be a Følner net. Consider $m_j \in \mathcal{M}(G)$

$$m_j(x) = \frac{1}{|F_j|}\sum_{h\in F_j} x(h) \qquad \forall x \in \ell^\infty(G)$$

then $$\left|(gm_j - m_j)(x)\right| \leq 2\,\frac{|F_j \setminus gF_j|}{|F_j|}\,\|x\|_\infty \qquad \forall x \in \ell^\infty(G)$$

so that $\lim (gm_j - m_j) = 0$ weak-$*$ topology. If m is in the weak-$*$-adherence of $(m_j)_{j\in J}$ ($\mathcal{M}(G)$ is compact), then m is left-invariant.
Thus G is amenable.

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Suppose G satisfies Følner condition and let $(F_j)_{j\in J}$ be a Følner net. Consider $\mu_j \in \mathcal{M}(G)$

$$\mu_j(A) = \frac{|A\cap F_j|}{|F_j|} \qquad \forall\, A\subseteq G$$

then $$\left|\mu_j(A) - \mu_j(gA)\right| \leq 2\, \frac{|F_j \setminus g^{-1}F_j|}{|F_j|} \quad (*)$$

if μ = w* limit of the μ_j's then
$\mathcal{M}(G)$ is <u>compact</u> in
$\mu(A) - \mu(gA) = 0$

$$A\cap F_j = (A\cap F_j \cap g^{-1}F_j) \sqcup \sqcup (A\cap F_j \setminus g^{-1}F_j)$$

(b) $\Rightarrow$ (c) Suppose G doesn't satisfy Følner.

Then $\exists\, \underset{\text{finite}}{K_0 \subset G}$ and $\varepsilon_0 > 0$ s.t. $\dfrac{|F \setminus k_0 F|}{|F|} \geq \varepsilon_0$

$\forall\, \underset{\text{finite}}{F \subset G}$ $\longrightarrow$ $\exists\, k_0(F) \in K$

Set $K_1 = K_0 \cup \{1_G\}$. Then $K_1 F \supset F$ and $K_1 F \setminus F = K_0 F \setminus F$. It follows

$$|K_1 F| - |F| = |K_1 F \setminus F| = |K_0 F \setminus F| \geq$$
$$\geq |k_0 F \setminus F| = |F \setminus k_0 F| \quad (\text{since } |F| = |k_0 F|)$$
$$\geq \varepsilon_0 |F|$$

so that $|K_1 F| \geq (1+\varepsilon_0)|F|$. Let $n_0 \in \mathbb{N}$ be such that $(1+\varepsilon_0)^{n_0} \geq 2$ and set $K = K_1^{n_0}$, then

$$|KF| = |K^{n_0} F| = |K(K^{n_0-1}F)| \geq (1+\varepsilon_0)|K^{n_0-1}F|$$
$$\geq \dots \geq (1+\varepsilon_0)^{n_0}|F| \geq 2|F|. \square$$

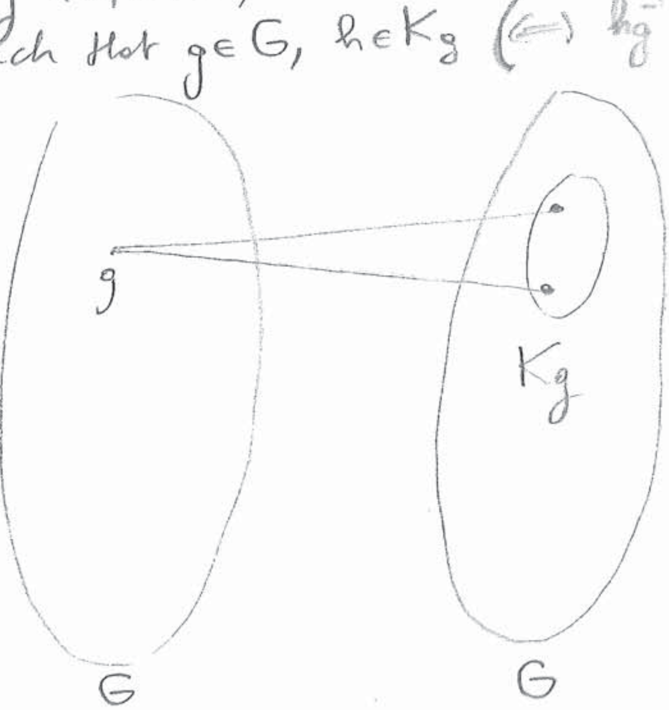
(c) $\Rightarrow$ (d). Suppose $\exists\, \underset{\text{finite}}{K \subset G}$ s.t. $|KF| \geq 2|F|$ for all $\underset{\text{finite}}{F \subset G}$.

Consider the bipartite graph $\mathcal{G}: (G, G; E)$ where $E \subset G \times G$ consists of all pairs (g,h) such that $g \in G$, $h \in Kg$ ($\Leftrightarrow hg^{-1} \in K$)

$\mathcal{G}$ satisfies the <u>Hall 2-harem</u> condition

$\mathcal{N}_R(F) = KF \Rightarrow$
$|\mathcal{N}_R(F)| = |KF| \geq 2|F|$

$\mathcal{N}_L(F) = K^{-1}F \Rightarrow$
$|\mathcal{N}_L(F)| = |K^{-1}F| \geq$
$|k^{-1}F| = |F| \geq \frac{1}{2}|F|$



14 bis

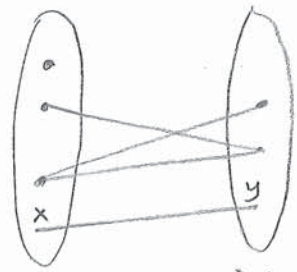
Bipartite graphs

$\mathcal{G} = (X, Y; E)$, where X, Y are arbitrary sets and $E \subseteq X \times Y$
$\downarrow$ left $\downarrow$ right

adjacent vertices

$N_R(x) \subset Y$

$N_L(y) \subset X$



X boys $\qquad$ Y girls

adjacent edges

$N_R(A) = \bigcup_{x \in A} N_R(x)$

$N_L(B) = \bigcup_{y \in B} N_L(y)$

Matchings
- A matching in $\mathcal{G} = (X, Y; E)$ is a subset $M \subseteq E$ of pairwise nonadjacent edges
- M is left (right)-perfect if $\forall x \in X$ ($\forall y \in Y$) $\exists y \in Y$ ($x \in X$) s.t. $(x, y) \in M$.

M is perfect if it is left and right perfect

Def (Hall conditions) $\mathcal{G}$ satisfies the left (right) Hall condition

$$|N_R(A)| \geq |A| \quad \forall A \subset X \text{ fin} \qquad (|N_L(B)| \geq |B| \quad \forall B \subset Y \text{ finite})$$

G satisfies the Hall marriage condition if it satisfies both left & right Hall conditions.

P/M Theorem (Hall) G admits a perfect matching $\iff$ satisfies H-m-c.

Def Hall 2theorem cond

$$|N_R(A)| \geq 2|A| \quad \forall A \subset X \text{ fin} \quad \& \quad |N_L(B)| \geq |B| \quad \forall B \subset Y \text{ fin}$$

Thm (Hall*) G admits a perfect $(2,1)$-matching $\iff$ satisfies $(2\text{-}1)$ Horem condition.

By the Hall ~~harem~~ Theorem $\exists$ perfect $(2,1)$-matching $M \subset E$. In other words $\boxed{\exists\ \varphi : G \to G \text{ surjective 2-1 map}\ h\varphi(h)^{-1} \in K\ \ \forall h \in G.}$ s.t. $(\varphi(h), h) \in E$ equiv.

(d) $\Rightarrow$ (e). Suppose $\exists\ \varphi : G \to G \ \underset{\text{surjective}}{\text{2-to-1}}$ s.t. $\boxed{g\varphi(g)^{-1} \in K}$ $\forall g \in G$. By the (AC) $\exists\ \psi_1, \psi_2 : G \to G$ s.t. $\forall g \in G$

$\varphi^{-1}(g) = \{\psi_1(g), \psi_2(g)\}$.

Then $\vartheta_1(g) = \psi_1(g)g^{-1}$ & $\vartheta_2(g) = \psi_2(g)g^{-1} \in K$ $\forall g \in G$ by $\square$

$\boxed{\text{Setting } A'_k = \{g \in G : \vartheta_1(g) = k\},\ B'_k = \{g \in G : \vartheta_2(g) = k\}}$

we have

$$G = \left(\bigsqcup_{k \in K} kA'_k\right) \bigsqcup \left(\bigsqcup_{k \in K} kB'_k\right) \equiv \bigsqcup_{k \in K} A_k \uplus \left(\bigsqcup_{k \in K} B_k\right)$$

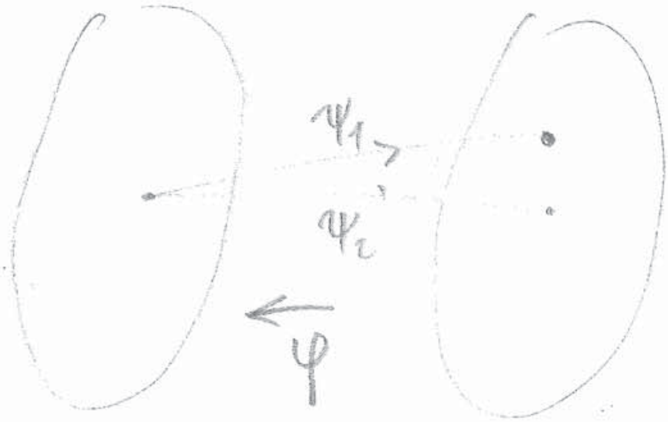
$$= \bigsqcup_{k \in K} A'_k \equiv \bigsqcup_{k \in K} kA_k \qquad (A_k = kA'_k)$$

$$= \bigsqcup_{k \in K} B'_k \equiv \bigsqcup_{k \in K} kB_k \qquad (B_k = kB'_k)$$

thus G is paradoxical.

(e) $\Rightarrow$ (a) already observed (proved for $\mathbb{F}_2'$). $\square$

15 bis



(AC)

$\exists\, \psi_1, \psi_2 : G \to G$

$$G = \psi_1(G) \sqcup \psi_2(G)$$
$$= \left(\bigsqcup_{k \in K_1} (k A_k) \right) \sqcup \left(\bigsqcup_{k \in K_2} (k B_k) \right)$$

$\Rightarrow A_k = \{ g \in G$

Interiors, Closures & Boundaries

G group

$E, \Omega \subseteq G$

E-interior $\Omega^{-E} = \{g \in G : gE \subset \Omega\} \equiv \bigcap_{e \in E} \Omega e^{-1}$

E-exterior $\Omega^{+E} = \{g \in G : gE \cap \Omega \neq \phi\} = \bigcup_{e \in E} \Omega e^{-1} = \Omega E^{-1}$

E-boundary $\partial_E \Omega = \Omega^{+E} \setminus \Omega^{-E}$

Proposition Let G be a group and $(F_j)_{j \in J}$ a net of finite nonempty subsets

TFAE: (a) $(F_j)_{j \in J}$ is a ~~left~~ right Følner net

(b) $\lim_j \frac{|\partial_E(F_j)|}{|F_j|} = 0 \quad \forall E \subset_{\text{finite}} G$

Proof (b) $\Rightarrow$ (a). Let $g \in G$ and set $E = \{1_G, g^{-1}\}$.

Then $F_j \setminus F_j g \subseteq F_j^{+E} \setminus (F_j \cap F_j g) = \partial_E(F_j)$

F_j^{+E}: because $1_G \in E$; $F_j \cap F_j g = F_j^{-E}$

Thus $\lim_j \frac{|F_j \setminus F_j g|}{|F_j|} = 0$.

(a) $\Rightarrow$ (b) Let $E \subseteq_{\text{fin}} G$

$$\partial_E(F_j) = \left(\bigcup_{a \in E} F_j a^{-1}\right) \setminus \left(\bigcap_{b \in E} F_j b^{-1}\right)$$

$$= \left(\bigcup_{a \in E} F_j a^{-1}\right) \cap \left(G \setminus \bigcap_{b \in E} F_j b^{-1}\right)$$

$$= \left(\bigcup_{a \in E} F_j a^{-1}\right) \cap \left(\bigcup_{b \in E} (G \setminus F_j b^{-1})\right)$$

$$= \bigcup_{a, b \in E} \left(F_j a^{-1} \setminus F_j b^{-1}\right)$$

It follows

$$|\partial_E(F_j)| \leq \sum_{a,b\in E} |F_j a^{-1} \setminus F_j b^{-1}|$$

$$= \sum_{a,b\in E} |F_j \setminus F_j b^{-1} a|$$

$$\leq |E|^2 \max_{k\in K} |F_j \setminus F_j k|$$

where $K = \{E^{-1}E\} \underset{\text{finite}}{\subset} G$. □

<u>Corollary</u> Let G be a group. TFAE

(a) G is amenable

(b) $\forall E \subset G$, $\forall \varepsilon > 0$, $\exists F \underset{\text{finite}}{\subset} G$ s.t. $\dfrac{|\partial_E(F)|}{|F|} < \varepsilon$.

<u>Entropy</u> Let G be an amenable group $\mathcal{F} = (F_j)_{j\in J}$ a <u>right</u> Følner net, A a finite set.

If $X \subset A^G$, $\Omega \underset{\text{finite}}{\subset} G$ we set $X_\Omega := \{x|_\Omega : x \in X\} \subseteq A^\Omega$

<u>Def</u> $\mathrm{ent}_{\mathcal{F}}(X) = \limsup_j \dfrac{\log|X_{F_j}|}{|F_j|}$

<u>Immediate properties</u>

- $\mathrm{ent}_{\mathcal{F}}(A^G) = \log|A|$
- $X \subset Y \Rightarrow \mathrm{ent}_{\mathcal{F}}(X) \leq \mathrm{ent}_{\mathcal{F}}(Y) \leq \log|A|$

Proposition $\tau : A^G \to A^G$ cellular automaton, $X \subset A^G$.
Then $\quad ent_{\mathcal{F}}(\tau(X)) \leq ent_{\mathcal{F}}(X)$.

Proof $Y := \tau(X)$, $M \subset G$ memory set containing 1_G.
$\forall \Omega$ we have $\tau_\Omega : X_\Omega \longrightarrow Y_{\Omega^{-M}}$ where
SURJECTION
$$u \longmapsto \tau(x)|_{\Omega^{-M}}$$

$x \in X$ is any st. $x|_\Omega = u$ (recall $\tau(V(x, \Omega)) \subset V(\tau(x), \Omega^{-M})$)

$$\Rightarrow \quad |Y_{\Omega^{-M}}| \underset{(*)}{\leq} |X_\Omega|$$

Now
$$\begin{aligned}
\log|Y_\Omega| &\leq \log|Y_{\Omega^{-M}} \times A^{\Omega \setminus \Omega^{-M}}| \\
&= \log|Y_{\Omega^{-M}}| + |\Omega \setminus \Omega^{-M}| \log|A| \\
&\underset{(*)}{\leq} \log|X_\Omega| + |\Omega \setminus \Omega^{-M}| \log|A| \\
&\leq \log|X_\Omega| + |\partial_M \Omega| \log|A|
\end{aligned}$$

$$\Rightarrow \frac{\log|Y_{F_i}|}{|F_i|} \leq \frac{\log|X_{F_i}|}{|F_i|} + \frac{|\partial_M F_i|}{|F_i|} \log|A|$$

$$\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \text{ Følner} \\
ent_{\mathcal{F}}(Y) & ent_{\mathcal{F}}(X) & 0
\end{array} \quad . \quad \square$$

Def $E, E' \subset G$. A subset $T \subset G$ is an (E, E')-tiling if
(i) $g_1 E \cap g_2 E = \emptyset$ if $g_1, g_2 \in T$, $g_1 \neq g_2$
(ii) $G = \bigcup_{g \in T} gE'$.

EXAMPLES ① $\mathbb{Z}$ is a $([0,1[, [0,1[)$-tiling of $\mathbb{R}$; ② $[0,1]$ is a $(2\mathbb{Z}, \mathbb{Z})$-tiling

<u>Lemma</u> $X \subset A^G$ s.t. $\exists\ E, E' \underset{\text{finite}}{\subseteq} G$ and $T \subseteq G$ (E,E')-tiling s.t. $X_{gE} \underset{\circledcirc}{\subsetneq} A^{gE}$ $\forall g \in T$. Then $\mathrm{ent}_{\mathcal{F}}(X) < \log |A|$.

<u>Proof</u> $\boxed{\begin{array}{l}\text{Technical result: } T_j = T \cap F_j^{-E} = \{g \in T : gE \subseteq F_j\} \\ \text{then } \exists\ \alpha > 0 \text{ and } j_0 \in J \text{ s.t. } |T_j| \geq \alpha |F_j|\end{array}}$

Set also $\quad F_j^* = F_j \setminus \underbrace{\bigsqcup_{g \in T_j} gE}_{\subseteq F_j}$ so $|F_j|^* = |F_j| - |T_j| \cdot |E|$

$$
\begin{aligned}
\log |X_{F_j}| &\leq \log \Big| A^{F_j^*} \times \prod_{g \in T_j} X_{gE} \Big| \\
&= |F_j^*| \log |A| + \sum_{g \in T_j} \log |X_{gE}| \\
&\underset{\circledcirc}{\leq} |F_j^*| \log |A| + \sum_{g \in T_j} \log \left(A^{|gE|} - 1 \right) \\
&\underset{\substack{\text{simple} \\ \text{manipulation}}}{=} |F_j| \log |A| + |T_j| \underbrace{\log \left(1 - |A|^{-|E|} \right)}_{-c}
\end{aligned}
$$

$$
\underset{\substack{\downarrow \\ \mathrm{ent}_{\mathcal{F}}(X)}}{\frac{\log |X_{F_j}|}{|F_j|}} \leq \log |A| - c \frac{|T_j|}{|F_j|} \leq \log |A| - c\alpha < \log |A|. \qquad \square
$$

<u>Theorem</u> TFAE

(a) τ is surjective

(b) $\mathrm{ent}_{\mathcal{F}}(\tau(A^G)) = \log|A|$

(c) τ is pre-injective

<u>Proof</u> (a) $\Rightarrow$ (b): trivial. $\square$

(b) $\Rightarrow$ (a): if τ is <u>not</u> surjective then $\exists$ GOE pattern

thus $\exists \underset{\text{finite}}{E \subset G}$ s.t. $(\tau(A^G))_E \subsetneq A^E$.

Setting $E' = EE^{-1}$ we can find an (E,E')-tiling $T \subseteq G$

Since $(\tau(A^G))_E \subsetneq A^E$ we have $(\tau(A^G))_{gE} \subsetneq A^{gE}$

$\forall g \in G$ in part $\forall g \in T \overset{\text{lemma}}{\Longrightarrow} \mathrm{ent}_{\mathcal{F}}(\tau(A^G)) < \log|A|$. $\square$

(b) $\Rightarrow$ (c): Suppose τ is <u>not</u> pre-injective. Then

$\exists\, x_1, x_2 \in A^G$ s.t. $\tau(x_1) = \tau(x_2)$ such that $\Omega := \{g \in G : x_1(g) \neq x_2(g)\}$

is finite and $\neq \emptyset$. Set $R = M^{-1}M$ $(\ni 1_G)$.

Let $E = \Omega^{+R}$ and $T \subset G$ and (E,E')-tiling

Let $Z = \{ z \in A^G : z|_{hE} \neq (hx_1)|_{hE} \quad \forall h \in T \}$

Observe that $Z_{hE} \subsetneq A^{hE} \overset{\text{lemma}}{\Longrightarrow} \mathrm{ent}_{\mathcal{F}}(Z) < \log|A|$

$\Rightarrow \mathrm{ent}_{\mathcal{F}}(\tau(Z)) < \log|A|$.

But one can show that $\tau(Z) = \tau(A^G)$ since $hx_1|_{hE}$ and $hx_2|_{hE}$ basically have the same image. Thus $\mathrm{ent}_{\mathcal{F}}(\tau(A^G)) < \log|A|$.

(c) $\Rightarrow$ (b): Suppose $\mathrm{ent}(\tau(A^G)) < \log|A|$.

Let $Y := \tau(A^G)$.

$$\log|Y_{F_j^{+M}}| \le \log|Y_{F_j}| + |F_j^{+M} \setminus F_j| \log|A|$$
$$\le \log|Y_{F_j}| + |\partial_M F_j| \log|A|$$

$$\Rightarrow \frac{\log|Y_{F_j^{+M}}|}{|F_j|} \le \underbrace{\frac{\log|Y_{F_j}|}{|F_j|}}_{\downarrow \ \mathrm{ent}_{\mathcal{F}}(Y)} + \underbrace{\frac{|\partial_M F_j|}{|F_j|}}_{\downarrow \ 0} \log|A|$$

$$< \log|A|$$

$$\underset{\exists j_0 \in J}{\Rightarrow} \frac{\log|Y_{F_{j_0}^{+M}}|}{|F_{j_0}|} < \log|A|.$$

For fixed $\sigma_0 \in A$ let $Z = \{z \in A^G : z(g) = \sigma_0 \ \forall g \in G \setminus F_{j_0}\}$.

Note that $Z \approx A^{F_{j_0}} \Rightarrow |Z| = |A|^{|F_{j_0}|}$.

Now $|\tau(Z)| \le |\tau(Z)_{F_{j_0}^{+M}}| \le |Y_{F_{j_0}^{+M}}| < |A|^{|F_{j_0}|} = |Z|$

This means that $\exists\, z_1 \neq z_2 \in Z$ s.t. $\tau(z_1) = \tau(z_2)$. Thus τ is not pre-injective (since z_1 is a.e. to z_2). $\square$

Counterexamples

- Surjective not pre-injective over $G = \mathbb{F}_2$

<u>Majority action</u> $A = \{0,1\}$

$$\tau(x)(g) = \begin{cases} 1 & \text{if } \sum_{s \in \{a, a^{-1}, b, b^{-1}\}} x(gs) > 2 \\ 0 & \text{"} \quad \text{"} \quad < 2 \\ x(g) & \text{otherwise} \end{cases}$$

- Pre-injective not surjective over $G = \mathbb{F}_2$

$A = H \times H$, H nontrivial abelian group (e.g. $\mathbb{Z}/2\mathbb{Z}$)

$P_1(u) = (h_1, 0)$ $\quad P_2(u) = (h_2, 0)$ $\quad \forall u = (h_1, h_2)$

$$\tau(x)(g) = P_1(x(ga)) + P_2(x(gb)) + P_1(x(ga^{-1})) + P_2(x(gb^{-1}))$$

(non-surjective $\tau(A^G) \subseteq (H \times \{0\})^G \subsetneq A^G$.)

<u>BARTHOLDI's THEOREM</u> G non-amenable. $\exists A$ finite $\tau : A^G \to A^G$ surjective not pre-injective.

<u>Proof</u> Since G is non-amenable $\exists\ \varphi : G \to G$ (2-to-1)-map surjective $\exists K \underset{\text{finite}}{\subset} G$ s.t. $\varphi(g)^{-1} g \in K$ $\forall g \in G$.

$A := K \times K$ / Fix a total order on K and fix $k_0 \in K$.

$\mu : A^K \to A$

$$\mu(y) = \begin{cases} (k', h') & \text{if } \exists!\ (k,h) \in K \times K \text{ with } k < h \\ & \text{s.t. } y(k) = (k, k') \ \&\ y(h) = (h, h') \\ (k_0, k_0) & \text{otherwise} \end{cases}$$

then the associated c.a. $\tau : A^G \to A^G$ is surjective not pre-injective

<u>Corollary</u> CHARACTERIZATION of AMENABILITY

<u>PROBLEM</u> Suppose G non-amenable. $\exists ?$ A finite $\tau : A^G \to A^G$ pre-injective not surjective?