Associative algebras and Lie algebras defined by Lyndon words

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Assume that $X = \{x_1, \cdots, x_g\}$ is a finite alphabet and $\mathbf{k}$ is a field. We study the class $\mathfrak{C}(X, W)$ of associative graded $\mathbf{k}$-algebras A generated by X and with a fixed obstructions set W consisting of Lyndon words in the alphabet X. Important examples are the monomial algebras $A = \mathbf{k}\langle X\rangle/(W)$, where W is an antichain of Lyndon words of arbitrary cardinality and the enveloping algebra $U\mathfrak{g}$ of any X-generated Lie $\mathbf{k}$-algebra $\mathfrak{g}$. We prove that all algebras A in $\mathfrak{C}(X, W)$ share the same Poincaré-Birkhoff-Witt type $\mathbf{k}$-basis built out of the so called *Lyndon atoms* N (determined uniquely by W) but, in general, N may be infinite. We prove that A has polynomial growth if and only if the set of Lyndon atoms N is finite. In this case A has a $\mathbf{k}$-basis $\mathfrak{N} = \{l_1^{\alpha_1} l_2^{\alpha_2} \cdots l_d^{\alpha_d} \mid \alpha_i \geq 0, 1 \leq i \leq d\}$, where $N = \{l_1, \cdots, l_d\}$. Surprisingly, in the case when A has polynomial growth its global dimension does not depend on the shape of its defining relations but only on the set of obstructions W. We prove that if A has polynomial growth of degree d then A has global dimension d and is standard finitely presented, with $d-1 \leq |W| \leq d(d-1)/2$. We study when the set of standard bracketings $[W] = \{[w] \mid w \in W\}$ is a Groebner-Shirshov Lie basis. We use our general results to classify the Artin-Schelter regular algebras A generated by two elements, with defining relations $[W]$ and global dimension ≤ 7. We give an extremal class of monomial algebras, the Fibonacci-Lyndon algebras, F_n, with global dimension n and polynomial growth, and show that the algebra F_6 of global dimension 6 cannot be deformed, keeping the multigrading, to an Artin-Schelter regular algebra.