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REVIEW OF ELECTRODYNAMICS

Imrana Ashraf Zahid
Quaid-i-Azam University
Islamabad
Pakistan

Strada Costiera, 11 - 34151 - Trieste - Italy • Tel. +39 0402240111 • Fax. +39 040224163 • sci_info@ictp.it • www.ictp.it
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- Magneto- static : Revisited
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Nomenclature

- E = Electric field
- D = Electric displacement
- B = Magnetic flux density
- H = Auxiliary field
- ρ= Charge density
- j = Current density
- μ_0 (permeability of free space) = $4\pi\times10^{-7}$T-m/A
- ε_0 (permittivity of free space) = 8.854×10^{-12}N-m^2/ C^2
- c (speed of light) = 2.99792458×10^{8} m/s

Introduction

- **Electrostatics**

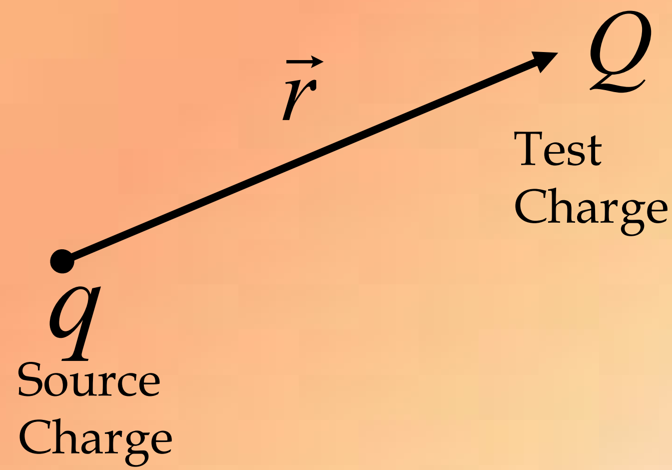
  - **Electrostatic field : Stationary charges produce electric fields that are constant in time. The theory of static charges is called electrostatics.**

Stationary charges → Constant Electric field;

Electrostatic :Revisited

Coulombs Law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{r^2} \hat{r}$$



$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N - m^2}$$

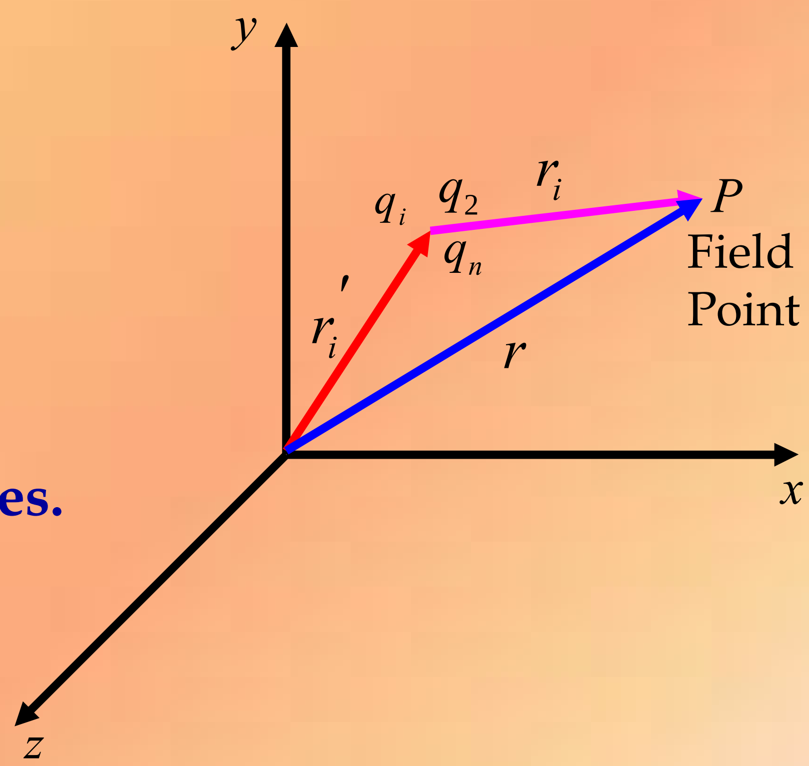
Permittivity of free space

The Electric Field

$$\vec{F} = Q\vec{E}$$

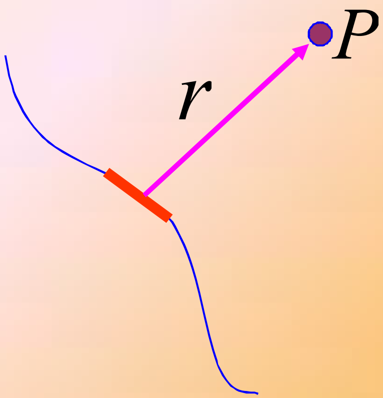
$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^{\,2}}\hat{r}_i$$

$\vec{E}$- **the electric field of the source charges.**



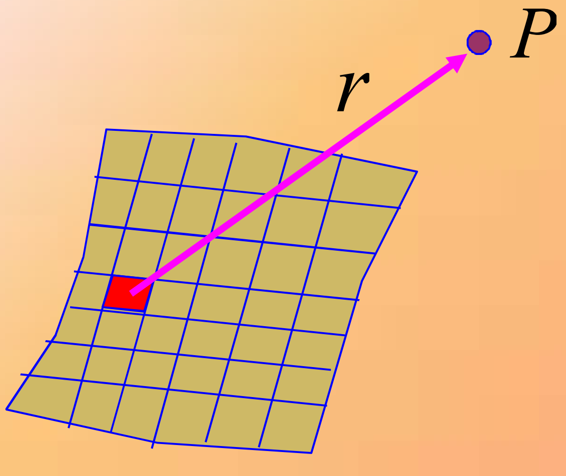
Physically E(P) Is force per unit charge exerted on a test charge placed at P.

The Electric Field: cont'd



$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Line} \frac{\hat{r}}{r^2} \lambda dl$$

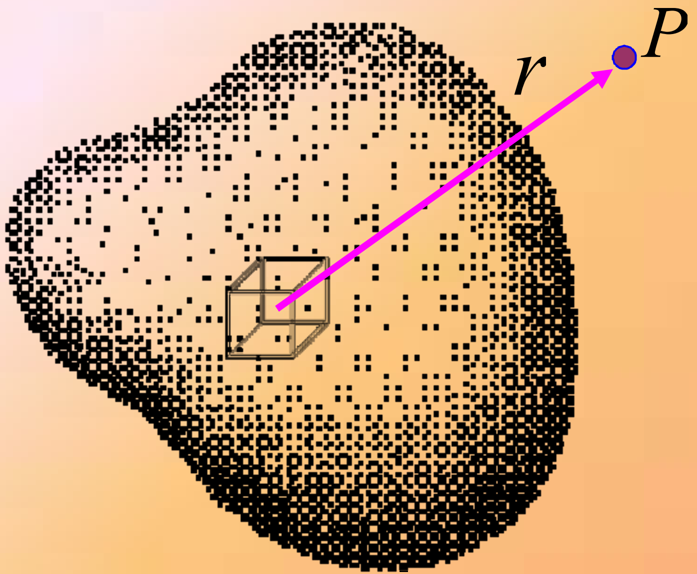
λ **is the line charge density**



$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Surface} \frac{\hat{r}}{r^2} \sigma da$$

σ **is the surface charge density**

The Electric Field: cont'd



$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Volume} \frac{\hat{r}}{r^2} \rho d\tau$$

ρ is the volume charge density

Electric Potential

The work done in moving a test charge Q in an electric field from point P_1 to P_2 with a constant speed.

$$W = Force \bullet distance$$

$$W = -\int_{p_1}^{P_2} Q\vec{E} \bullet d\vec{l}$$

negative sign - work done is against the field.

For any distribution of fixed charges.

$$\oint \vec{E} \bullet d\vec{l} = 0$$

The electrostatic field is conservative

Electric Potential: cont'd

Stokes's Theorem gives

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{E} = -\vec{\nabla} V$$

where *V* is Scalar Potential

The work done in moving a charge Q from infinity to a point P_2 where potential is *V*

$$W = QV$$

V = Work per unit charge

= Volts = joules/Coulomb

Electric Potential : cont'd

Field due to a single point charge q at origin

$$V = \int_r^\infty \frac{q\,dr}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r}$$

$$F \propto \frac{1}{r^2}$$

$$E \propto \frac{1}{r^2}$$

$$V \propto \frac{1}{r}$$

Gauss’s Law

$$\oint \vec{E} \bullet d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

Differential form of Gauss’s Law

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$$

Poisson’s Equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Laplace's Equation

$$\nabla^2 V = 0$$

Electrostatic Fields in Matter

Matter: **Solids, liquids, gases, metal, wood and glasses - behave differently in electric field.**

Two Large Classes of Matter

(i) Conductors

(ii) Dielectric

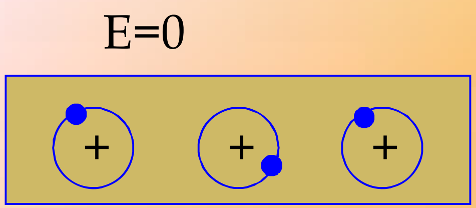
Conductors: **Unlimited supply of free charges.**

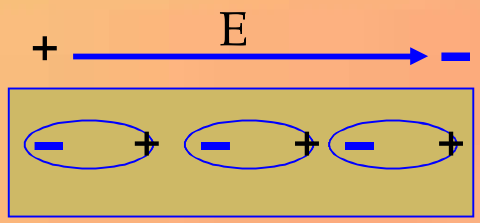
Dielectrics:

- **Charges are attached to specific atoms or molecules- No free charges.**
- **Only possible motion - minute displacement of positive and negative charges in opposite direction.**
- **Large fields- pull the atom apart completely (ionizing it).**

Polarization

A dielectric with charge displacements or induced dipole moment is said to be polarized.





Induced Dipole Moment $$p = \alpha E$$

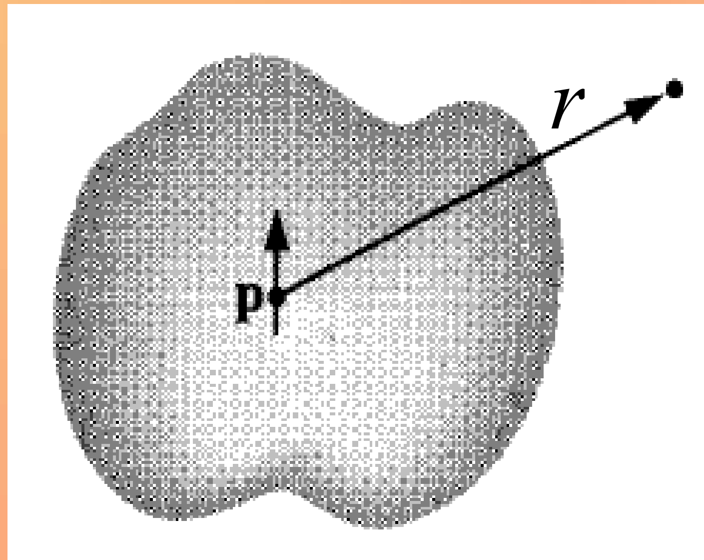
The constant of proportionality α is called the atomic polarizability

P ≡ dipole moment per unit volume

The Field of a Polarized Object

Potential of single dipole p is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\bullet\vec{\mathrm{p}}}{\mathrm{r}^2}$$



$$V = \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\vec{\mathrm{P}}\bullet\hat{r}}{\mathrm{r}^2}d\tau$$

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface}\frac{1}{r}\vec{P}\bullet da - \int_{volume}\frac{1}{r}\left(\vec{\nabla}\bullet\vec{P}\right)d\tau\right]$$

Potential due to dipoles in the dielectric

The Field of a Polarized Object: cont'd

$$\sigma_b = \vec{P} \bullet \hat{n}$$

Bound charges at surface

$$\rho_b = -\vec{\nabla} \bullet \vec{P}$$

Bound charges in volume

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface} \frac{1}{r}\sigma_b \, da + \int_{volume} \frac{1}{r}\rho_b \, d\tau\right]$$

The total field is field due to bound charges plus due to free charges

Gauss's law in Dielectric

- Effect of polarization is to produce accumulations of bound charges.
- The total charge density

$$\rho = \rho_f + \rho_b$$

From Gauss's law

$$\varepsilon_0 \vec{\nabla} \bullet \vec{E} = \rho = \rho_b + \rho_f$$

$$\vec{\nabla} \bullet \vec{D} = \rho_f$$

$$\int \vec{D} \bullet d\vec{a} = Q_{fenc}$$

Q_{fenc}**-Free charges enclosed**

Displacement vector

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

Magnetostatics : Revisited

- Magnetostatics

  - Steady current produce magnetic fields that are constant in time. The theory of constant current is called magnetostatics.

Steady currents ⇒ Constant Magnetic field;

Magnetic Forces

Lorentz Force

$$\vec{F} = q\left[\vec{E} + \left(\vec{v} \times \vec{B}\right)\right]$$

- **The magnetic force on a segment of current carrying wire is**

$$F_{mag} = \int \left(\vec{I} \times \vec{B}\right) dl$$

$$F_{mag} = \int I\left(d\vec{l} \times \vec{B}\right)$$

Equation of Continuity

The current crossing a surface s can be written as

$$I = \int_s \vec{J} \bullet d\vec{a} = \int_v \left(\vec{\nabla} \bullet \vec{J}\right) d\tau$$

$$\int_v \left(\vec{\nabla} \bullet \vec{J}\right) d\tau = -\frac{d}{dt} \int \rho\, d\tau = -\int \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

Charge is conserved whatever flows out must come at the expense of that remaining inside - outward flow decreases the charge left in v

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

This is called equation of continuity

Equation of Continuity 1

In Magnetostatic steady currents flow in the wire and its magnitude I must be the same along the line- otherwise charge would be pilling up some where and current can not be maintained indefinitely.

$$\frac{\partial \rho}{\partial t} = 0$$

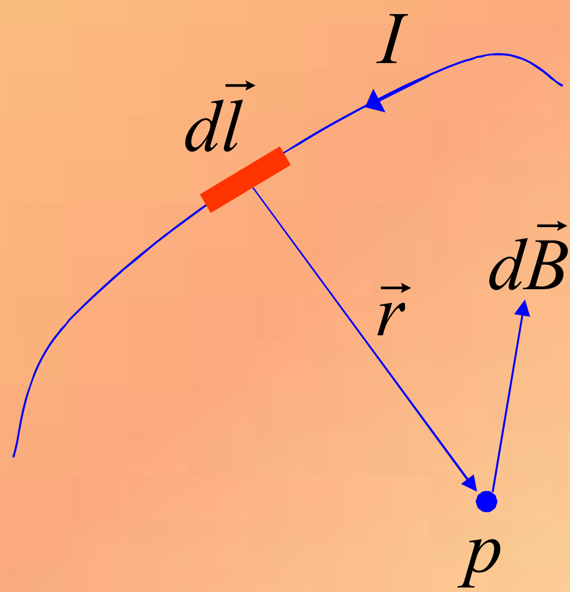
In Magnetostatic and equation of continuity

$$\vec{\nabla} \bullet \vec{J} = 0$$

Steady Currents: The flow of charges that has been going on forever - never increasing - never decreasing.

Magnetostatic and Current Distributions

Biot and Savart Law



$$\vec{B}(p) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \vec{r}}{|\vec{r}|^3} dl$$

dl **is an element of length.**

$\vec{r}$ **vector from source to point p.**

μ_0 **Permeability of free space.**

Unit of B = N/Am = Tesla (T)

Biot and Savart Law for Surface and Volume Currents

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \vec{r}}{|\vec{r}|^3} da$$

For Surface Currents

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \vec{r}}{|\vec{r}|^3} d\tau$$

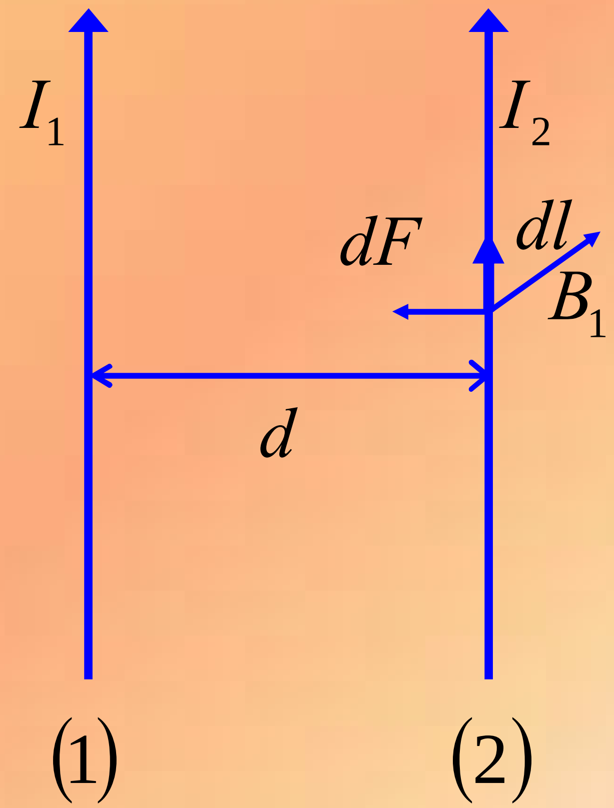
For Volume Currents

Force between two parallel wires

The magnetic field at (2) due to current I_1 is

$$B_1 = \frac{\mu_0 I_1}{2\pi d}$$

Points inside



Magnetic force law

$$dF = \int I_2 \left(d\vec{l}_2 \times \vec{B}_1 \right)$$

$$dF = \int I_2 \left(d\vec{l}_2 \times \frac{\mu_0 I_1}{2\pi d} \hat{k} \right)$$

Force between two parallel wires

$$dF = \frac{\mu_0 I_1 I_2}{2\pi d} dl_2$$

The total force is infinite but force per unit length is

$$\frac{dF}{dl_2} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

If currents are anti-parallel the force is repulsive.

Straight line currents

The integral of B around a circular path of radius s, centered at the wire is

$$\oint \vec{B} \bullet d\vec{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \mu_0 I$$

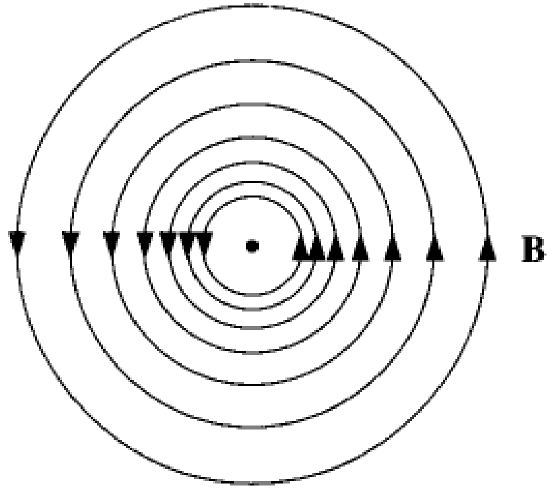
For bundle of straight wires. Wire that passes through loop contributes only.

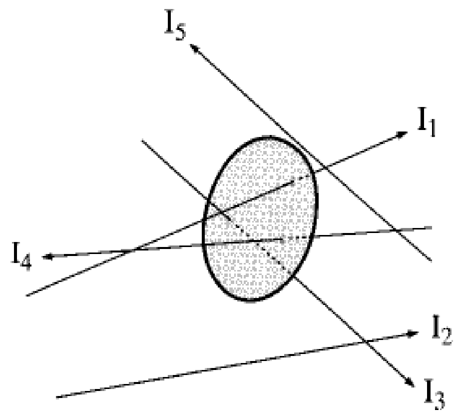
$$\oint \vec{B} \bullet d\vec{l} = \mu_0 I_{enc}$$

Applying Stokes' theorem

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

The current is out of the page

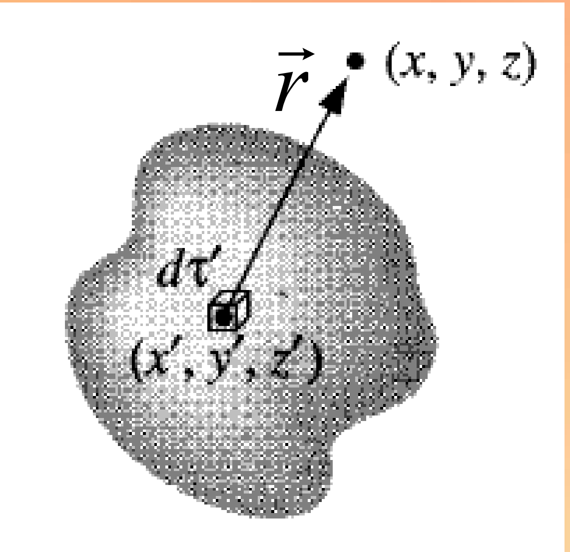




Divergence and Curl of B

Biot-Savart law for the general case of a volume current reads

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(r')\times\vec{r}}{r^3}\,d\tau'$$



B is a function of (x, y, z),

J is a function of (x', y', z'),

$$\vec{r} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

$$\vec{\nabla}\bullet\vec{B} = 0 \quad \textbf{and} \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

Ampere's Law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$ **Ampere's law**

Integral form of Ampere's law

Using Stokes' theorem

$$\int \left(\vec{\nabla} \times \vec{B} \right) \bullet d\vec{a} = \oint \vec{B} \bullet d\vec{l} = \mu_0 \int \vec{J} \bullet d\vec{a}$$

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 I_{enc}$$

Vector Potential

The basic differential law of Magnetostatics

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

B is curl of some vector field called vector potential $A(P)$

$$\vec{B}(P) = \vec{\nabla} \times \vec{A}(P)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_0 J$$

Coulomb's gauge

$$\vec{\nabla} \bullet \vec{A} = 0$$

$$\nabla^2 A = -\mu_0 J$$

Magnetostatic Field in Matter

- Magnetic fields- due to electrical charges in motion.
- Examine a magnet on atomic scale we would find tiny currents.
- Two reasons for atomic currents.
  - Electrons orbiting around nuclei.
  - Electrons spinning on their axes.
- Current loops form magnetic dipoles - they cancel each other due to random orientation of the atoms.
- Under an applied Magnetic field- a net alignment of - magnetic dipole occurs- and medium becomes magnetically polarized or magnetized

Magnetization

If m is the average magnetic dipole moment per unit atom and N is the number of atoms per unit volume, the magnetization is define as

$$\overrightarrow{M} = N\,\overrightarrow{m}$$

$$\overrightarrow{m} = I\,\overrightarrow{a} = Am^2$$

or

$$m = M d\tau$$

$$M = \frac{Am^2}{m^3} = \frac{A}{m}$$

Magnetic Materials

Paramagnetic Materials

The materials having magnetization parallel to B are called paramagnets.

Diamagnetic Materials

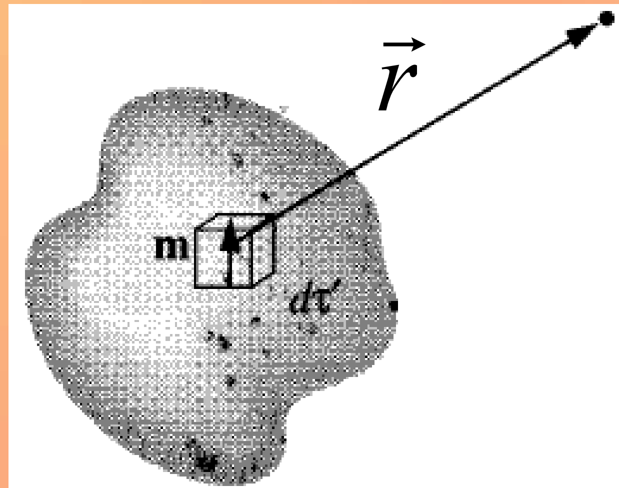
The elementary moment are not permanent but are induced according to Faraday's law of induction. In these materials magnetization is opposite to B.

Ferromagnetic Materials

Have large magnetization due to electron spin. Elementary moments are aligned in form of groups called domain

The Field of Magnetized Object

Using the vector potential of current loop



$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{M}\times\hat{n}}{r}da + \frac{\mu_0}{4\pi}\int\frac{\vec{\nabla}\times\vec{M}}{r}d\tau$$

$$\vec{K_b} = \vec{M}\times\hat{n}$$ **Bound Surface Current**

$$\vec{J_b} = \vec{\nabla}\times\vec{M}$$ **Bound Volume Current**

Ampere's Law in Magnetized Material

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$$

$$\vec{J}=\vec{J}_b+\vec{J}_f$$

$$\frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)=\vec{J}_b+\vec{J}_f=\vec{J}_f+\left(\vec{\nabla}\times\vec{M}\right)$$

$$\vec{\nabla}\times\vec{H}=\vec{J}_f$$

where

$$\vec{H}=\frac{\vec{B}}{\mu_0}-\vec{M}$$

Integral form

$$\oint\vec{H}\bullet d\vec{l}=I_{fenc}$$

Faraday's Law of Induction

- **Faraday's Law - a changing -magnetic flux through circuit induces an electromotive force around the circuit.**

$$\varepsilon = \oint \vec{E} \bullet \vec{dl} = -\frac{d\phi}{dt} = -\frac{d}{dt}\int \vec{B} \bullet \vec{da}$$

ϵ – Induced emf

E – Induced electric field intensity

Differential form of Faraday's law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Faraday's Law of Induction

Induced Electric field intensity in terms of vector potential

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$$

For steady currents

$$\vec{E} = -\vec{\nabla} V$$

V – Scalar potential

Induced emf in a system moving in a changing magnetic field

$$\varepsilon = \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right)$$

MAXWELL'S EQUATIONS

Imrana Ashraf Zahid
Quaid-i-Azam University Islamabad
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Introduction to Maxwell's Equation

- In electrodynamics Maxwell's equations are a set of four equations, that describes the behavior of both the electric and magnetic fields as well as their interaction with matter
- Maxwell's four equations express
  - How electric charges produce electric field (Gauss's law)
  - The absence of magnetic monopoles
  - How currents and changing electric fields produces magnetic fields (Ampere's law)
  - How changing magnetic fields produces electric fields (Faraday's law of induction)

Historical Background

- 1864 Maxwell in his paper “A Dynamical Theory of the Electromagnetic Field” collected all four equations

- 1884 Oliver Heaviside and Willard Gibbs gave the modern mathematical formulation using vector calculus.

- The change to vector notation produced a symmetric mathematical representation, that reinforced the perception of physical symmetries between the various fields.

Electrodynamics Before Maxwell

Gauss's Law

$$(i)\vec{\nabla}\bullet\vec{E} = \frac{\rho}{\varepsilon_o}$$

No name

$$(ii)\vec{\nabla}\bullet\vec{B} = 0$$

Faraday's Law

$$(iii)\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Ampere's Law

$$(iv)\vec{\nabla}\times\vec{B} = \mu_o\vec{J}$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

Electrodynamics Before Maxwell (Cont'd)

Apply divergence to (iii)

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\bullet\left(-\frac{\partial\vec{B}}{\partial t}\right)=-\frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{B}\right)$$

The left hand side is zero, because divergence of a curl is zero.

The right hand side is zero because $\vec{\nabla}\bullet\vec{B}=0$.

Apply divergence to (iv)

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{B}\right)=\mu_o\left(\vec{\nabla}\bullet\vec{J}\right)$$

Electrodynamics Before Maxwell (Cont'd)

- **The left hand side is zero, because divergence of a curl is zero.**
- **The right hand side is zero for steady currents i.e.,**

$$\vec{\nabla} \bullet \vec{J} = 0$$

- **In electrodynamics from conservation of charge**

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = 0$$

ρ **is constant at any point in space which is wrong.**

Maxwell's Correction to Ampere's Law

Consider Gauss's Law

$$\vec{\nabla}\bullet\varepsilon_o\vec{E} = \rho$$

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\varepsilon_o\vec{E}\right) = \frac{\partial\rho}{\partial t}$$

$$\Rightarrow \frac{\partial\rho}{\partial t} = \vec{\nabla}\bullet\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

$$\frac{\partial\vec{D}}{\partial t} = \varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

Displacement current

This result along with Ampere's law and the conservation of charge equation suggest that there are actually two sources of magnetic field.

The current density and displacement current.

Maxwell's Correction to Ampere's Law (Cont'd)

Amperes law with Maxwell's correction

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

General Form of Maxwell's Equations

Differential Form

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_o}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Integral Form

$$\oint_S \vec{E} \bullet d\vec{a} = \frac{1}{\varepsilon_o} \int_V \rho dV$$

$$\oint_S \vec{B} \bullet d\vec{a} = 0$$

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \bullet d\vec{a}$$

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_o I_{enc} + \mu_o \varepsilon_o \frac{d}{dt} \oint_S \vec{E} \bullet d\vec{a}$$

Maxwell's Equations in vacuum

- The vacuum is a linear, homogeneous, isotropic and dispersion less medium
- Since there is no current or electric charge is present in the vacuum, hence Maxwell's equations reads as
- These equations have a simple solution in terms of traveling sinusoidal waves, with the electric and magnetic fields direction orthogonal to each other and the direction of travel

$$\vec{\nabla} \bullet \vec{E} = 0$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations Inside Matter

Maxwell's equations are modified for polarized and magnetized materials. For linear materials the polarization *P* and magnetization *M* is given by

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

$$\vec{M} = \chi_m \vec{H}$$

And the *D* and *B* fields are related to *E* and *H* by

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P} = (1+\chi_e)\varepsilon_o \vec{E} = \varepsilon \vec{E}$$

$$\vec{B} = \mu_o\left(\vec{H} + \vec{M}\right) = (1+\chi_m)\mu_o \vec{H} = \mu \vec{H}$$

Where χ_e is the electric susceptibility of material,

χ_m is the magnetic susceptibility of material.

Maxwell's Equations Inside Matter (Cont'd)

- For polarized materials we have bound charges in addition to free charges

$$\sigma_b = \vec{P} \bullet \hat{n}$$

$$\rho_b = -\vec{\nabla} \bullet \vec{P}$$

- For magnetized materials we have bound currents

$$\vec{K_b} = \vec{M} \times \hat{n}$$

$$\vec{J_b} = \vec{\nabla} \times \vec{M}$$

Maxwell's Equations Inside Matter (Cont'd)

- **In electrodynamics any change in the electric polarization involves a flow of bound charges resulting in polarization current J_P**

$$\vec{J}_P = \frac{\partial \vec{P}}{\partial t}$$

Polarization current density is due to linear motion of charge when the Electric polarization changes

Total charge density

$$\rho_t = \rho_f + \rho_b$$

Total current density

$$J_t = J_f + J_b + J_p$$

Maxwell's Equations Inside Matter (Cont'd)

- **Maxwell's equations inside matter are written as**

$$\vec{\nabla}\bullet\vec{E}=\frac{\rho_t}{\varepsilon_o}$$

$$\vec{\nabla}\bullet\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B}=\mu_o\vec{J}_f+\mu_o\vec{J}_p+\mu_o\vec{J}_b+\mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\frac{\vec{B}}{\mu_o}=\vec{J}_f+\frac{\partial\vec{P}}{\partial t}+\vec{\nabla}\times\vec{M}+\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_o}-\vec{M}\right)=\vec{J}_f+\frac{\partial}{\partial t}\left(\varepsilon_o\vec{E}+\vec{P}\right)$$

$$\vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial}{\partial t}\vec{D}$$

Maxwell's Equations Inside Matter (Cont'd)

- **In non-dispersive, isotropic media ε and μ are time-independent scalars, and Maxwell's equations reduces to**

$$\vec{\nabla} \bullet \varepsilon \vec{E} = \rho$$

$$\vec{\nabla} \bullet \mu \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations Inside Matter (Cont'd)

- In uniform (homogeneous) medium ε and μ are independent of position- Maxwell's equations reads as

$$\vec{\nabla}\bullet\vec{D} = \rho_f \qquad \oint_S \vec{D}\bullet d\vec{a} = Q_{f\ enc}$$

$$\vec{\nabla}\bullet\vec{H} = 0 \qquad \oint_S \vec{H}\bullet d\vec{a} = 0$$

$$\vec{\nabla}\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \qquad \oint_C \vec{E}\bullet d\vec{l} = -\mu\frac{d}{dt}\int_S \vec{H}\bullet d\vec{a}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \varepsilon\frac{\partial\vec{E}}{\partial t} \qquad \oint_C \vec{H}\bullet d\vec{l} = I_{f\ enc} + \frac{d}{dt}\int_S \vec{D}\bullet d\vec{a}$$

Generally, ε and μ can be rank-2 tensor (3X3 matrices) describing bi-refringent anisotropic materials.

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Maxwell's equations in integral form:

Gauss' Law:

$$\int_v \vec{\nabla}\cdot\vec{E}(\vec{r},t)\,d\tau' = \frac{1}{\varepsilon_0}\int_v \rho^E_{Tot}(\vec{r},t)\,d\tau' = \frac{1}{\varepsilon_0}\int_v \left(\rho^E_{free}(\vec{r},t) + \rho^E_{bound}(\vec{r},t)\right)d\tau'$$

$$= \oint_S \vec{E}(\vec{r},t)\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q^{enclosed}_{ToT}(t) = \frac{1}{\varepsilon_0}\left(Q^{enclosed}_{free}(t) + Q^{enclosed}_{bound}(t)\right)$$

$$\oint_S \vec{D}(\vec{r},t)\,d\vec{a} = Q^{enclosed}_{free}(t)$$

$$\oint_S \vec{P}(\vec{r},t)\cdot d\vec{a} \equiv -Q^{enclosed}_{bound}(t)$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Auxiliary Relation: $\vec{D}(\vec{r},t) = \varepsilon_o \vec{E}(\vec{r},t) + \vec{P}(\vec{r},t)$

$$\rho_{Bound}(\vec{r},t) \equiv -\vec{\nabla} \cdot \vec{P}(\vec{r},t) \qquad \sigma_{Bound}(\vec{r},t) \equiv \vec{P}(\vec{r},t) \cdot \hat{n}\Big|_{intf}$$

No Magnetic Monopoles: $\int_v \vec{\nabla} \cdot \vec{B}(\vec{r},t)\, d\tau' = \oint_S \vec{B}(\vec{r},t) \cdot d\vec{a} = 0$

Faraday's Law:

$$\int_S \vec{\nabla} \times \vec{E}(\vec{r},t) \cdot d\vec{a} = \oint_C \vec{E}(\vec{r},t) \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}(\vec{r},t)}{\partial t} \cdot d\vec{a} = -\frac{d}{dt}\left[\int_S \vec{B}(\vec{r},t) \cdot d\vec{a}\right]$$

$$\varepsilon mf \;\; \varepsilon(t) \equiv \oint_C \vec{E}(\vec{r},t) \cdot d\vec{\ell} = -\frac{d}{dt}\left[\int_S \vec{B}(\vec{r},t) \cdot d\vec{a}\right] = -\frac{d\Phi_M^{enclosed}(t)}{dt}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Ampere's Law:

$$\int_S \vec{\nabla}\times\vec{B}(\vec{r},t)\cdot d\vec{a} = \oint_C \vec{B}(\vec{r},t)\cdot d\vec{\ell} = \mu_0 \int_S \left(\vec{J}_{ToT}(\vec{r},t)+\vec{J}_D(\vec{r},t)\right)\cdot d\vec{a}$$

$$= \oint_C \vec{B}(\vec{r},t)\cdot d\vec{\ell} = \mu_0\left(I_{ToT}^{encl}(t)+I_D^{encl}(t)\right) = \mu_0\left(I_{free}^{encl}(t)+I_{bound}^{m\,encl}(t)+I_{P_{bound}}^{encl}(t)+I_D^{encl}(t)\right)$$

Auxiliary Relation:

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\vec{B}(\vec{r},t) - \vec{M}(\vec{r},t)$$

$$\vec{J}_{bound}^{m}(\vec{r},t) \equiv \vec{\nabla}\times\vec{M}(\vec{r},t)$$

$$\vec{K}_{bound}^{m}(\vec{r},t) \equiv \vec{M}(\vec{r},t)\times\hat{n}\Big|_{intf}$$

$$\vec{J}_{P_{bound}}(\vec{r},t) \equiv \frac{\partial\vec{P}(\vec{r},t)}{\partial t}$$

$$\rho_m^{Bound}(\vec{r},t) \equiv -\vec{\nabla}\cdot\vec{M}(\vec{r},t)$$

$$\sigma_m^{Bound}(\vec{r},t) \equiv \vec{M}(\vec{r},t)\cdot\hat{n}\Big|_{intf}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

$$\int_S \vec{\nabla}\times\vec{H}(\vec{r},t)\cdot d\vec{a} = \oint_C \vec{H}(\vec{r},t)\cdot d\vec{\ell} = I_{free}^{enclosed}(t) + \int_S \frac{\partial\vec{D}(\vec{r},t)}{\partial t}\cdot d\vec{a} = I_{free}^{enclosed}(t) + \frac{d}{dt}\left[\int_S \vec{D}(\vec{r},t)\cdot d\vec{a}\right]$$

1) Apply the integral form of Gauss' Law at a dielectric interface/boundary using infinitesimally thin Gaussian pillbox extending slightly into dielectric material on either side of interface:

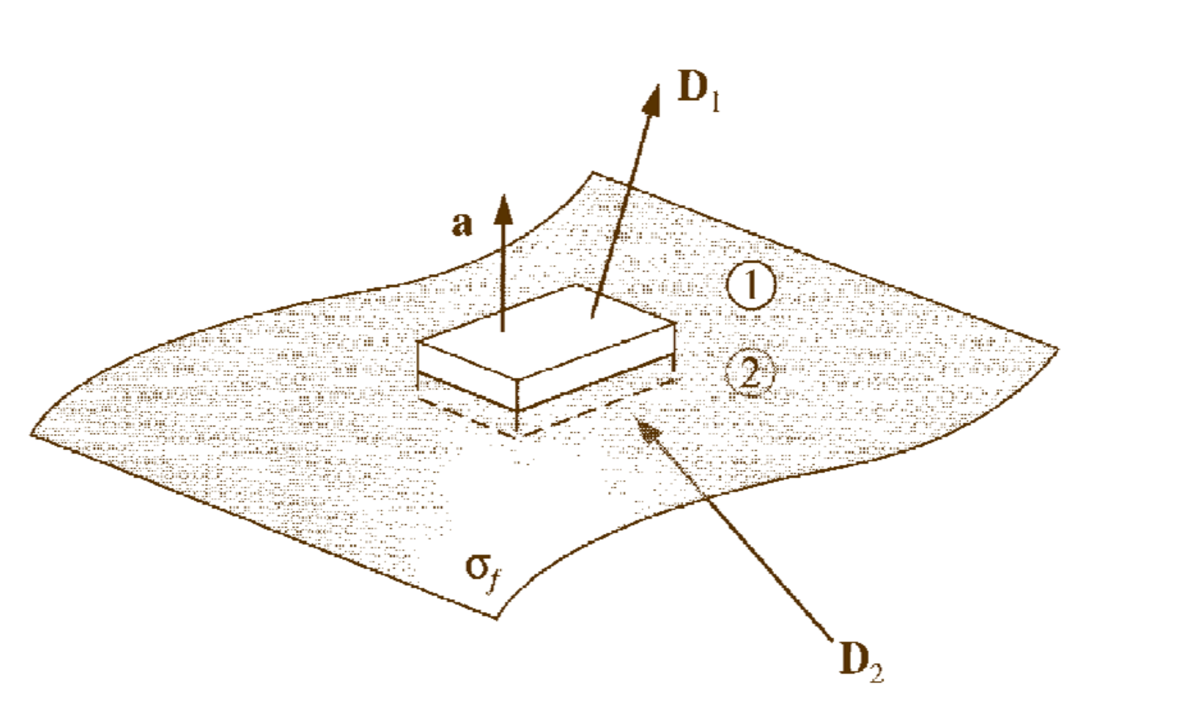
$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{ToT}^{enclosed} = \frac{1}{\varepsilon_0} Q_{free}^{enclosed} + \frac{1}{\varepsilon_0} Q_{bound}^{enclosed} = \frac{1}{\varepsilon_0}\oint_S \sigma_{free}\, da + \frac{1}{\varepsilon_0}\oint_S \sigma_{bound}\, da$$

Gives:

$$E_{\underset{above}{2}}^{\perp} - E_{\underset{below}{1}}^{\perp} = \frac{1}{\varepsilon_0}\sigma_{ToT} = \frac{1}{\varepsilon_0}\left(\sigma_{free} + \sigma_{bound}\right)$$

(at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter



Maxwell's Equations and Boundary Conditions at Interfaces in Matter

The positive direction is from medium 2 (below) to medium 1 (above)

$$\oint_S \vec{D}\cdot d\vec{a} = Q_{free}^{enclosed} = \oint_S \sigma_{free} da \quad \Rightarrow \quad \boxed{\underset{above}{D_2^{\perp}} - \underset{below}{D_1^{\perp}} = \sigma_{free}}$$ (at interface)

Likewise: $$\oint_S \vec{\mathrm{P}}\cdot d\vec{a} = Q_{bound}^{enclosed} = -\oint_S \sigma_{bound} da \Rightarrow \boxed{\underset{above}{P_2^{\perp}} - \underset{below}{P_1^{\perp}} = \sigma_{bound}}$$ (at interface)

$$\vec{E} \equiv -\vec{\nabla} V$$

Since: $$\boxed{\left(\frac{\partial V_2^{above}}{\partial n} - \frac{\partial V_1^{below}}{\partial n}\right)\Bigg|_{interface} = -\frac{1}{\varepsilon_0}\sigma_{ToT} = -\frac{1}{\varepsilon_0}\left(\sigma_{free} + \sigma_{bound}\right)}$$ (at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

$$\vec{D} = \varepsilon\vec{E} = -\varepsilon\vec{\nabla}V$$

Since: $$\left.\left(\varepsilon_2 \frac{\partial V_2^{above}}{\partial n} - \varepsilon_1 \frac{\partial V_1^{below}}{\partial n}\right)\right|_{interface} = -\sigma_{free}$$ (at interface)

Similarly, for $\int_v \vec{\nabla}\bullet\vec{B}\,d\tau' = \oint_S \vec{B}\bullet d\vec{a} = 0$ (no magnetic monopoles), then at an interface:

$$\vec{B}_2^{above}\bullet\vec{a} - \vec{B}_1^{above}\bullet\vec{a} = 0 \Rightarrow \boxed{B^{\perp}_{\underset{above}{2}} - B^{\perp}_{\underset{below}{1}} = 0} \text{ or: } \boxed{B^{\perp}_{\underset{above}{2}} = B^{\perp}_{\underset{below}{1}}}$$ (at interface)

Since: $$\vec{H} = \left(\frac{1}{\mu_o}\right)\vec{B} - \vec{M} \qquad \underline{\text{Then}}: \qquad \vec{B} = \mu_o\left(\vec{H} + \vec{M}\right)$$

$$\oint_S \vec{B}\bullet d\vec{a} = \mu_o \oint_S \left(\vec{H} + \vec{M}\right)\bullet d\vec{a} = 0 \qquad \underline{\text{or}}: \qquad \oint_S \vec{H}\bullet d\vec{a} = -\oint_S \vec{M}\bullet d\vec{a}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Then: $\vec{H}_2^{above} \bullet \vec{a} - \vec{H}_1^{below} \bullet \vec{a} = -\left(\vec{M}_2^{above} \bullet \vec{a} - \vec{M}_1^{below} \bullet \vec{a} \right)$ (at interface)

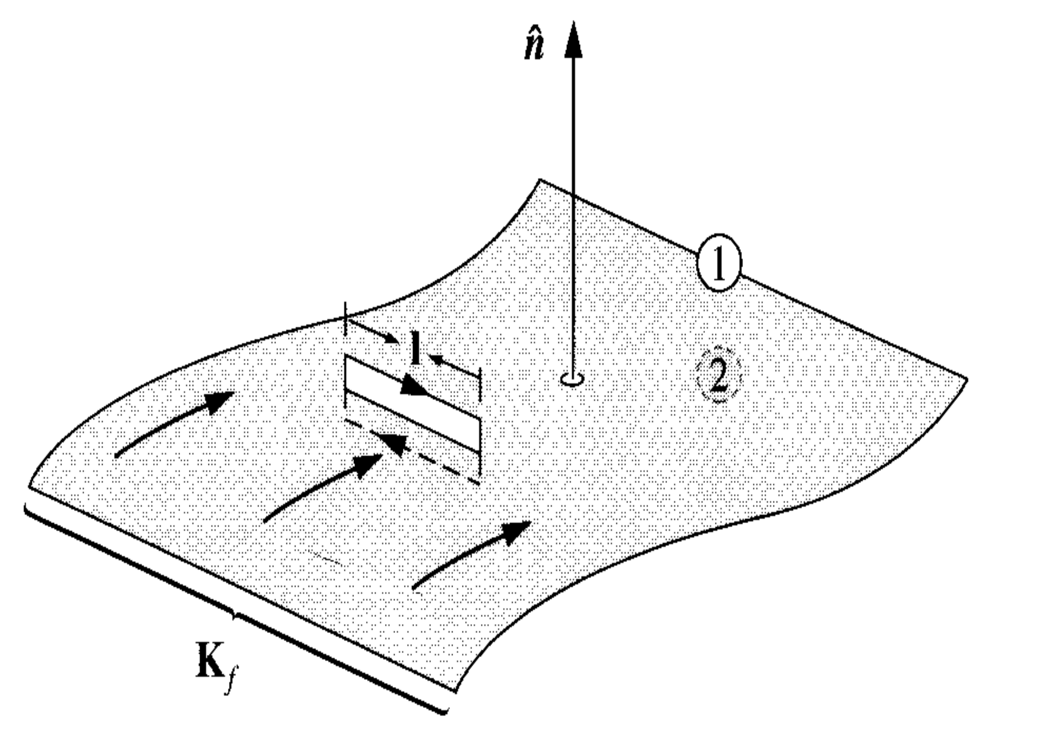
Or: $$\boxed{\left(\underset{above}{H_2^{\perp}} - \underset{below}{H_1^{\perp}} \right) = -\left(\underset{above}{M_2^{\perp}} - \underset{below}{M_1^{\perp}} \right) = -\sigma_{magnetic}^{bound}}$$ (at interface)

Effective bound magnetic charge at interface

3) For Faraday's Law: EMF, $\varepsilon = \oint_C \vec{E} \bullet d\vec{\ell} = -\frac{d}{dt}\left(\oint_S \vec{B} \bullet d\vec{a} \right) = -\frac{d\Phi_m}{dt}$

At interface between two different media, taking a closed contour C of width l extending slightly into the material on either side of interface.

Maxwell's Equations and Boundary Conditions at Interfaces in Matter



Maxwell's Equations and Boundary Conditions at Interfaces in Matter

$$\vec{E}_2^{above}\bullet\vec{\ell} - \vec{E}_1^{below}\bullet\vec{\ell} = -\frac{d}{dt}\oint_S \vec{B}\bullet d\vec{a} = 0$$

(in limit area of contour loop → 0, magnetic flux enclosed → 0)

$$\boxed{\underset{above}{E_2^{\parallel}} - \underset{below}{E_1^{\parallel}} = 0}$$ (at interface) or: $$\boxed{\underset{above}{E_2^{\parallel}} = \underset{below}{E_1^{\parallel}}}$$ (at interface)

Since: $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$ And: $\varepsilon_o \vec{E} = \vec{D} - \vec{P}$

Thus: $$\left(\vec{E}_2^{above}\bullet\vec{\ell} - \vec{E}_1^{below}\bullet\vec{\ell}\right) = \left(\vec{D}_2^{above}\bullet\vec{\ell} - \vec{D}_1^{below}\bullet\vec{\ell}\right) - \left(\vec{P}_2^{above}\bullet\vec{\ell} - \vec{P}_1^{below}\bullet\vec{\ell}\right) = 0$$

In limit area of contour loop → 0 magnetic flux enclosed → 0

$$\Rightarrow \boxed{\left(\underset{above}{\vec{D}_2^{\parallel}} - \underset{below}{\vec{D}_1^{\parallel}}\right) = \left(\underset{above}{\vec{P}_2^{\parallel}} - \underset{below}{\vec{P}_1^{\parallel}}\right)}$$ (at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

4) Finally, for Ampere's Law: $\oint_C \vec{B} \bullet d\vec{\ell} = \mu_o \left(I_{TOT}^{encl} + I_D^{encl} \right)$

$$\vec{B}_2^{above} \bullet \vec{\ell} - \vec{B}_1^{below} \bullet \vec{\ell} = \mu_o I_{TOT}^{encl} + \mu_o I_D^{encl}$$

$$I_D^{encl} = \int_S \vec{J}_D \bullet d\vec{a} = \varepsilon_o \int_S \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a}$$

$$I_{TOT}^{encl} = I_{free}^{encl} + I_{bound}^{encl} + I_{P_{bound}}^{encl}$$

$$I_{P_{bound}}^{encl} = \int_S \vec{J}_{P_{bound}} \bullet d\vec{a} = \int_S \frac{\partial \vec{P}}{\partial t} \bullet d\vec{a}$$

$$I_{bound}^{encl} = \int_S \vec{J}_m^{bound} \bullet d\vec{a} = \int_S \vec{\nabla} \times \vec{M} \bullet d\vec{a}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Where I_{TOT}^{encl} = TOTAL current (free + bound + polarization) passing through enclosing Amperian loop contour C

No volume current density $\vec{J}_{TOT}, \vec{J}_{free}, \vec{J}_{bound}^{m}$ or $\vec{J}_p$ contributes to I_{TOT}^{encl} in the limit area of contour loop $\rightarrow 0$, however a surface current $\vec{K}_{TOT}, \vec{K}_{free}, \vec{K}_{bound}^{m} = \vec{M} \times \hat{n}$ can contribute!

In the limit that the enclosing Amperian loop contour C shrinks to zero height above/below interface- the enclosed area of loop contour → 0,

Then: $$I_D^{encl} = \varepsilon_o \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \varepsilon_o \frac{d}{dt}\left[\int_S \vec{E} \cdot d\vec{a}\right] = \varepsilon \frac{d\Phi_E}{dt} \rightarrow 0$$

$(\Phi_E \equiv \int_S \vec{E} \cdot d\vec{a}$ = enclosed flux of electric field lines$)$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Similarly:

$$I_{P_{bound}}^{encl} = \int_S \frac{\partial \vec{P}}{\partial t} \cdot d\vec{a} = \frac{d}{dt}\left[\int_S \vec{P} \cdot d\vec{a}\right] = \frac{d\Phi_P}{dt} \rightarrow 0$$

$$(\Phi_P \equiv \int_S \vec{P} \cdot d\vec{a} = \text{enclosed flux of electric polarization field lines})$$

If $\hat{n}$ is unit normal/perpendicular to interface, note that $(\hat{n} \times \vec{\ell})$ is normal/perpendicular to plane of the Amperian loop contour.

$$I_{TOT}^{encl} = \vec{K}_{TOT} \cdot (\hat{n} \times \vec{\ell}) = (\vec{K}_{TOT} \times \hat{n}) \cdot \vec{\ell}$$

$$I_{free}^{encl} = \vec{K}_{free} \cdot (\hat{n} \times \vec{\ell}) = (\vec{K}_{free} \times \hat{n}) \cdot \vec{\ell}$$

$$I_{bound}^{encl} = \vec{K}_{bound} \cdot (\hat{n} \times \vec{\ell}) = (\vec{K}_{bound}^{m} \times \hat{n}) \cdot \vec{\ell}$$

Using: $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A})$

$= \vec{C} \cdot (\vec{A} \times \vec{B})$

$\{= (\vec{A} \times \vec{B}) \cdot \vec{C}\}$

$$I_{TOT} = I_{free} + I_{bound} \qquad \vec{K}_{TOT} = \vec{K}_{free} + \vec{K}_{bound}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

In the limit that the enclosing Amperian loop contour C (of width l) shrinks to zero height above/below interface, causing area of enclosed loop contour → 0, then:

$$\vec{B}_2^{above}\bullet\vec{\ell}-\vec{B}_1^{below}\bullet\vec{\ell}=\mu_o I_{TOT}^{encl}+\overbrace{\cancel{\mu_o I_D^{encl}}}^{=0}=\mu_o I_{TOT}^{encl}=\left(\vec{K}_{TOT}\times\hat{n}\right)\bullet\vec{\ell}$$

$$\boxed{\underset{above}{B_2^{\parallel}}-\underset{below}{B_1^{\parallel}}=\mu_o\vec{K}_{TOT}\times\hat{n}=\mu_o\left(\vec{K}_{free}+\vec{K}_{bound}^{m}\right)\times\hat{n}}\quad \text{(at interface)}$$

Since: $\vec{H}=\frac{1}{\mu_o}\vec{B}-\vec{M}$ and: $\frac{1}{\mu_o}\vec{B}=\vec{H}+\vec{M}$ then:

$$\boxed{\frac{1}{\mu_o}\left(\vec{B}_2^{above}\bullet\vec{\ell}-\vec{B}_1^{below}\bullet\vec{\ell}\right)=\left(\vec{H}_2^{above}\bullet\vec{\ell}-\vec{H}_1^{below}\bullet\vec{\ell}\right)+\left(\vec{M}_2^{above}\bullet\vec{\ell}-\vec{M}_1^{below}\bullet\vec{\ell}\right)=\left[\left(\vec{K}_{free}\times\hat{n}\right)+\left(\vec{K}_{bound}\times\hat{n}\right)\right]}$$

(at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

We also see that: $\underset{above}{H_2^{\parallel}} - \underset{below}{H_1^{\parallel}} = \vec{K}_{free} \times \hat{n}$ (at interface)

and: $\underset{above}{M_2^{\parallel}} - \underset{below}{M_1^{\parallel}} = \vec{K}_{bound}^{m} \times \hat{n}$ (at interface)

- $\parallel$- components of ***B*** are discontinuous at interface by $\mu_o \vec{K}_{TOT} \times \hat{n}$
- $\parallel$- components of ***H*** are discontinuous at interface by $\vec{K}_{free} \times \hat{n}$
- $\parallel$- components of **M** are discontinuous at interface by $\vec{K}_{bound}^{m} \times \hat{n}$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

If $\vec{B} = \vec{\nabla} \times \vec{A}$

where *A* is the magnetic vector potential - then:

$$\left(\frac{1}{\mu_0}\right)\left[B^{\parallel}_{\underset{above}{2}} - B^{\parallel}_{\underset{below}{1}}\right] = \vec{K}_{TOT} \times \hat{n}$$

(at interface) is equivalent to:

$$\left(\frac{1}{\mu_o}\right)\left(\frac{\partial \vec{A}_2^{above}}{\partial n} - \frac{\partial \vec{A}_1^{below}}{\partial n}\right)\Bigg|_{interface} = -\vec{K}_{TOT}$$

(at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

For linear magnetic media:

$$\vec{B} = \mu\vec{H} \quad \text{or:} \quad \vec{H} = \frac{1}{\mu}\vec{B}$$

$$\left[H^{\parallel}_{\underset{above}{2}} - H^{\parallel}_{\underset{below}{1}} \right] = \vec{K}_{free} \times \hat{n}$$ (at interface) is equivalent to:

$$\left(\frac{1}{\mu_{\underset{above}{2}}} \right) \frac{\partial \vec{A}_2^{above}}{\partial n}\Bigg|_{\text{interface}} - \left(\frac{1}{\mu_{\underset{below}{1}}} \right) \frac{\partial \vec{A}_1^{below}}{\partial n}\Bigg|_{\text{interface}} = -\vec{K}_{free}$$ (at interface)

Potential Formulation of Electrodynamics 1

- In electrostatic

$$\vec{\nabla}\times\vec{E}(\vec{r},t)=0$$

$$\vec{E}(\vec{r},t)=-\vec{\nabla}V(\vec{r},t)$$

In electrodynamics

$$\vec{\nabla}\times\vec{E}(\vec{r},t)\neq 0$$

But

$$\vec{\nabla}\bullet\vec{B}(\vec{r},t)=0$$

$$\vec{B}(\vec{r},t)=\vec{\nabla}\times\vec{A}(\vec{r},t)$$

Putting this in Faraday's Law

$$\vec{\nabla}\times\vec{E}(\vec{r},t)=-\vec{\nabla}\times\left(\frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right)$$

$$\vec{\nabla}\times\left[\vec{E}(\vec{r},t)+\frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right]=0$$

$$\left[\vec{E}(\vec{r},t)+\frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right]\equiv-\vec{\nabla}V(\vec{r},t)$$

$$\vec{E}(\vec{r},t)=-\vec{\nabla}V(\vec{r},t)-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}$$

Potential Formulation of Electrodynamics 2

$$\vec{\nabla}\bullet\vec{B}(\vec{r},t)=\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{A}(\vec{r},t)\right)$$

and from

$$\vec{\nabla}\times\vec{E}(\vec{r},t)=-\vec{\nabla}\times\vec{\nabla}V(\vec{r},t)-\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}(\vec{r},t)$$

$$\vec{\nabla}\times\vec{E}(\vec{r},t)=0-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}$$

Explain Maxwell's ii and iii Equations

Potential Formulation of Electrodynamics 3

Now consider Maxwell's i and iv equations

As

$$\vec{\nabla}\bullet\vec{E}(\vec{r},t)=\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$

Gauss's Law

$$\vec{\nabla}\bullet\left[-\vec{\nabla}V(\vec{r},t)-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right]=\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$

$$\nabla^2 V(\vec{r},t)+\frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{A}(\vec{r},t)\right)=-\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$

This replaces Poisson's Equation in electrodynamics

Potential Formulation of Electrodynamics 4

Now consider Ampere's Law:

$$\vec{\nabla} \times \vec{B}(\vec{r},t) = \mu_o \vec{J}_{tot}(\vec{r},t) + \mu_o \varepsilon_o \frac{\partial \vec{E}(\vec{r},t)}{\partial t}$$

with: $$\vec{E}(\vec{r},t) = -\vec{\nabla} V(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$$

and: $$\vec{B}(\vec{r},t) = \vec{\nabla} \times \vec{A}(\vec{r},t)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}(\vec{r},t)\right) = \mu_o \vec{J}_{tot}(\vec{r},t) - \mu_o \varepsilon_o \vec{\nabla}\left(\frac{\partial V(\vec{r},t)}{\partial t}\right) - \mu_o \varepsilon_o \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2}$$

Potential Formulation of Electrodynamics 5

Using vector identity:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)\equiv\vec{\nabla}\left(\vec{\nabla}\bullet\vec{A}\right)-\nabla^2\vec{A}$$

Re-arranging:

$$\left(\nabla^2\vec{A}(\vec{r},t)-\mu_o\varepsilon_o\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2}\right)-\vec{\nabla}\left(\vec{\nabla}\bullet\vec{A}(\vec{r},t)+\mu_o\varepsilon_o\frac{\partial V(\vec{r},t)}{\partial t}\right)=-\mu_o\vec{J}_{tot}(\vec{r},t)$$

These equation carry all information in Maxwell's Equations

Potential Formulation of Electrodynamics 6

$$\nabla^2 V(\vec{r},t) + \frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{A}(\vec{r},t)\right) = -\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$

$$\left(\nabla^2\vec{A}(\vec{r},t) - \mu_o\varepsilon_o\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\bullet\vec{A}(\vec{r},t) + \mu_o\varepsilon_o\frac{\partial V(\vec{r},t)}{\partial t}\right) = -\mu_o\vec{J}_{tot}(\vec{r},t)$$

Four Maxwell's equations reduced to two equations using potential formulation.
Potentials V and A are not uniquely defined by above equations.

Gauge Transformations

Suppose we have e.g. two sets of potentials

$$\left\{V(\vec{r},t),\vec{A}(\vec{r},t)\right\} \text{ and } \left\{V'(\vec{r},t),\vec{A}'(\vec{r},t)\right\}$$

That correspond to the same physical fields

$$\vec{E}(\vec{r},t) \quad \text{and} \quad \vec{B}(\vec{r},t)$$

These two sets of potentials must be related to each other by:

$$\vec{A}'(\vec{r},t)=\vec{A}(\vec{r},t)+\vec{\alpha}(\vec{r},t)$$

and

$$V'(\vec{r},t)=V(\vec{r},t)+\beta(\vec{r},t)$$

Because:

$$\vec{B}(\vec{r},t)=\vec{\nabla}\times\vec{A}(\vec{r},t)=\vec{\nabla}\times\vec{A}'(\vec{r},t)=\vec{\nabla}\times\left(\vec{A}(\vec{r},t)+\vec{\alpha}(\vec{r},t)\right)$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{\alpha}(\vec{r},t)\equiv 0$$

Gauge Transformations

But if: $\left(\vec{\nabla}\times\vec{\alpha}(\vec{r},t)\right)=0$, then since $\left(\vec{\nabla}\times\vec{\nabla}f(\vec{r},t)\right)\equiv 0$

we can always write $\vec{\alpha}(\vec{r},t)\equiv\vec{\nabla}\lambda(\vec{r},t)$

And if:

$$\vec{E}(\vec{r},t)=-\vec{\nabla}V(\vec{r},t)-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}$$

$$=-\vec{\nabla}V'(\vec{r},t)-\frac{\partial\vec{A}'(\vec{r},t)}{\partial t}=-\vec{\nabla}V(\vec{r},t)-\vec{\nabla}\beta(\vec{r},t)-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}-\frac{\partial\vec{\alpha}(\vec{r},t)}{\partial t}$$

Then we see that $\vec{\nabla}\beta(\vec{r},t)+\frac{\partial\vec{\alpha}(\vec{r},t)}{\partial t}=0$

Gauge Transformations

But:

$$\vec{\alpha}(\vec{r},t) \equiv \vec{\nabla}\lambda(\vec{r},t) \Rightarrow \vec{\nabla}\beta(\vec{r},t) + \frac{\partial\left(\vec{\nabla}\lambda(\vec{r},t)\right)}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla}\left(\beta(\vec{r},t) + \frac{\partial\lambda(\vec{r},t)}{\partial t}\right) = 0$$

which must hold for arbitrary all space-time points $(\vec{r},t)$

$$\Rightarrow \beta(\vec{r},t) + \frac{\partial\lambda(\vec{r},t)}{\partial t} = 0$$

Gauge Transformations

Note that

$$\vec{\nabla}\left(\beta(\vec{r},t)+\frac{\partial\lambda(\vec{r},t)}{\partial t}\right)=0$$

can also be satisfied if

$$\left(\beta(\vec{r},t)+\frac{\partial\lambda(\vec{r},t)}{\partial t}\right)=\kappa(t)$$

i.e. the scalar function κ (t)depends only on time, t.

Thus we see that:

$$\beta(\vec{r},t)=-\frac{\partial\lambda(\vec{r},t)}{\partial t}+\kappa(t)$$

But we can always "absorb" κ (t)

$$\lambda'(\vec{r},t)=\lambda(\vec{r},t)+\int_{t'=0}^{t'=t}\kappa(t)\,dt'$$

Gauge Transformations

Note also that since the scalar function κ (t) depends only on time t, this will not affect the gradient of $\vec{\nabla}\lambda(\vec{r},t)$ in any way, and hence $\vec{\alpha}(\vec{r},t)=\vec{\nabla}\lambda(\vec{r},t)$ is completely unaffected by this!

Thus:

$$\vec{A}'(\vec{r},t)=\vec{A}(\vec{r},t)+\alpha(\vec{r},t)=\vec{A}(\vec{r},t)+\vec{\nabla}\lambda(\vec{r},t)$$

or

$$\vec{\nabla}\lambda(\vec{r},t)=\vec{A}'(\vec{r},t)-\vec{A}(\vec{r},t)\equiv\Delta\vec{A}(\vec{r},t)$$

And:

$$V'(\vec{r},t)=V(\vec{r},t)-\frac{\partial\lambda(\vec{r},t)}{\partial t}$$

Such changes in V and A are called Gauge Transformations

or

$$-\frac{\partial\lambda(\vec{r},t)}{\partial t}=V'(\vec{r},t)-V(\vec{r},t)\equiv\Delta V(\vec{r},t)$$

Coulomb's and Lorentz Gauges

Coulomb Gauge $\vec{\nabla}\bullet\vec{A}(\vec{r},t)=0$

Using this we get $\nabla^2 V(\vec{r},t)=-\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$

It is Poisson's equation, setting $V(\vec{r}=\infty,t)=0$

we get $V(\vec{r},t)=\frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho_{tot}(\vec{r}',t)}{\boldsymbol{r}}d\tau'$

$\vec{\boldsymbol{r}}\equiv\vec{r}-\vec{r}'$

$\boldsymbol{r}=|\vec{\boldsymbol{r}}|=\sqrt{r^2-r'^2}$

Scalar potential is easy to calculate in Coulomb's gauge but vector potential is difficult to calculate

Coulomb's Gauge

The differential equations for V and A in Coulombs gauge reads

$$\nabla^2 V(\vec{r},t) = -\frac{1}{\varepsilon_o}\rho_{TOT}(\vec{r},t)$$

$$\nabla^2 \vec{A}(\vec{r},t) - \mu_o\varepsilon_o \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = -\mu_o \vec{J}_{tot}(\vec{r},t) + \mu_o\varepsilon_o \vec{\nabla}\left(\frac{\partial V(\vec{r},t)}{\partial t}\right)$$

Lorentz Gauge

The Lorentz gauge:

$$\vec{\nabla}\bullet\vec{A}(\vec{r},t)=-\mu_o\varepsilon_o\frac{\partial V(\vec{r},t)}{\partial t}$$

This is design to eliminate the middle term in eqn. for A

$$\nabla^2\vec{A}(\vec{r},t)-\mu_o\varepsilon_o\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2}=-\mu_o\vec{J}_{tot}(\vec{r},t)$$

And equation for V will become

$$\left(\nabla^2V(\vec{r},t)-\varepsilon_o\mu_o\frac{\partial^2V(\vec{r},t)}{\partial t^2}\right)=-\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$

Lorentz Gauge

The Lorentz gauge treats V and A on equal footing. The same differential operator

$$\square^2 \equiv \nabla^2 - \varepsilon_o \mu_o \frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

called the d'Alembertian

$$\square^2 \vec{A}(\vec{r},t) = -\mu_o \vec{J}_{tot}(\vec{r},t)$$

and

$$\square^2 V(\vec{r},t) = -\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$

ELECTROMAGNETIC WAVES IN VACUUM

Imrana Ashraf Zahid

Quaid-i-Azam University Islamabad Pakistan

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ELECTROMAGNETIC WAVES IN VACUUM

THE WAVE EQUATION

- In regions of free space (i.e. the vacuum) - where no electric charges - no electric currents and no matter of any kind are present - Maxwell's equations (in differential form) are:

1) $\vec{\nabla}\cdot\vec{E}(\vec{r},t)=0$

2) $\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0$

3) $\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}$

4) $\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu_o\varepsilon_o\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}=\dfrac{1}{c^2}\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}$

$\left(c^2=1/\varepsilon_o\mu_o\right)$

Set of coupled first-order partial differential equations

ELECTROMAGNETIC WAVES IN VACUUM . . .

- We can de-couple Maxwell's equations -by applying the curl operator to equations 3) and 4):

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$=\vec{\nabla}\left(\overset{=0}{\cancel{\vec{\nabla}\bullet\vec{E}}}\right)-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

$$=-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)$$

$$=\boxed{\nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=\vec{\nabla}\times\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)$$

$$=\vec{\nabla}\left(\overset{=0}{\cancel{\vec{\nabla}\bullet\vec{B}}}\right)-\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right)$$

$$=-\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$=\boxed{\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}}$$

ELECTROMAGNETIC WAVES IN VACUUM . . .

- These are three-dimensional de-coupled wave equations.
- Have exactly the same structure – both are linear, homogeneous, 2nd order differential equations.
- Remember that each of the above equations is explicitly dependent on space and time,

$$i.e.\ \vec{E} = \vec{E}(\vec{r},t) \text{ and } \vec{B} = \vec{B}(\vec{r},t):$$

$$\nabla^2 \vec{E}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = 0$$

$$\nabla^2 \vec{B}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2} = 0$$

ELECTROMAGNETIC WAVES IN VACUUM . . .

- Maxwell's equations implies that empty space – the vacuum (not empty at the microscopic scale) – supports the propagation of (macroscopic) electromagnetic waves - which propagate at the speed of light (in vacuum):

$$c = 1/\sqrt{\varepsilon_o \mu_o} = 3 \times 10^8 \text{ m/s}$$

MONOCHROMATIC EM PLANE WAVES

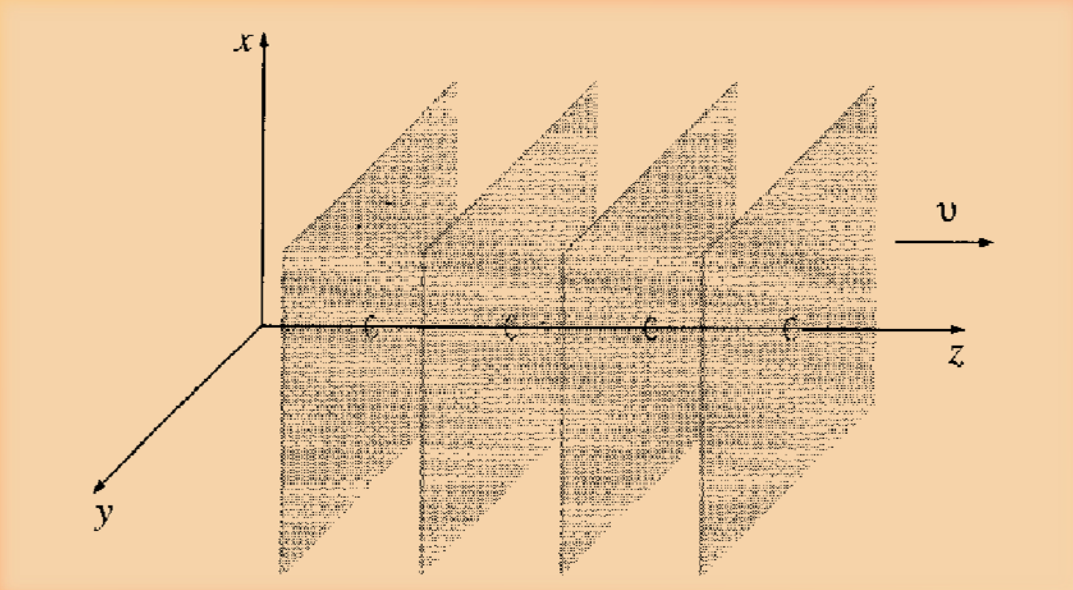
Monochromatic EM plane waves propagating in free space are waves consisting of a single frequency f , wavelength $\lambda = c f$, angular frequency $\omega = 2\pi f$ and wave-number $k = 2\pi /\lambda$ - propagate with speed $c = f\lambda = \omega k$.

In the visible region of the EM spectrum [$\{\sim 380$ *nm (violet)* $\leq \lambda \leq \sim 780$ *nm (red)*$\}$- EM light waves of a given frequency or wavelength are perceived by the human eye as having a specific- single colour.

Single- frequency EM waves are called mono-chromatic.

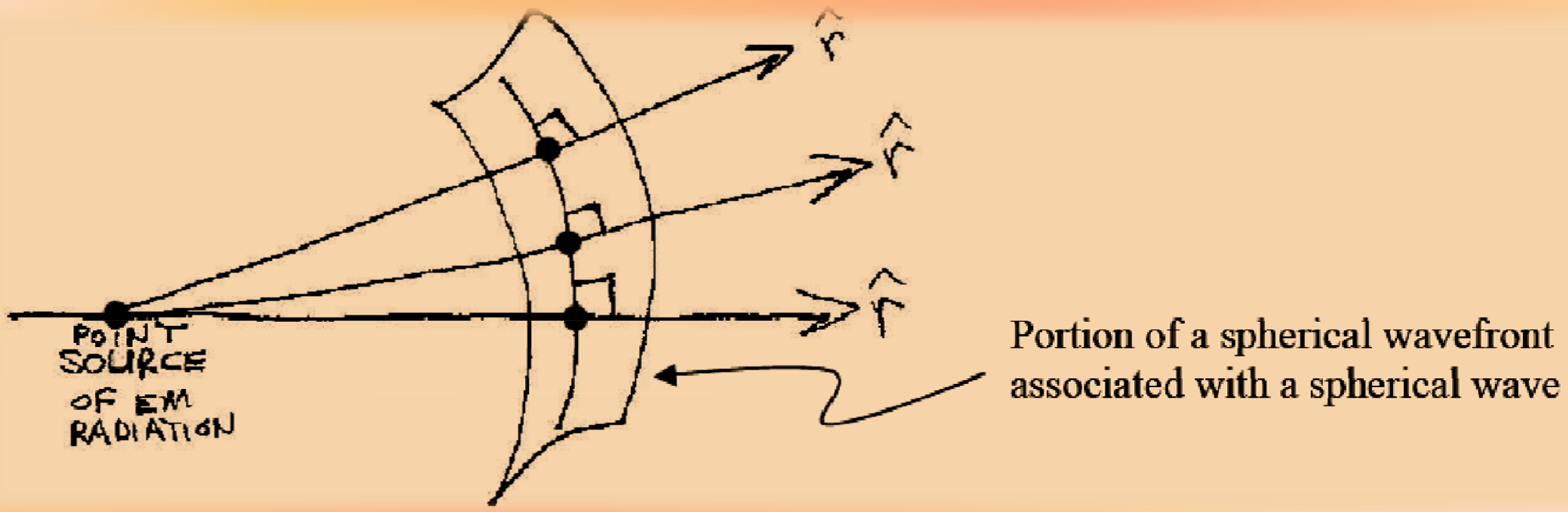
MONOCHROMATIC EM PLANE WAVES

EM waves that propagate e.g. in the $+\hat{z}$ direction but which additionally have no explicit *x*- or *y*-dependence are known as plane waves- for a given time t the wave fronts of the EM wave lie in a plane- ⊥ to the $\hat{z}$ *-axis*,



MONOCHROMATIC EM PLANE WAVES

There also exist spherical EM waves – emitted from a point source – the wave-fronts associated with these EM waves are spherical - and thus do not lie in a plane ⊥ to the direction of propagation of the EM wave



MONOCHROMATIC EM PLANE WAVES

If the point source is infinitely far away from observer- then a spherical wave → plane wave in this limit, (the radius of curvature → ∞); a spherical surface becomes planar as $R_C \rightarrow \infty$.

Criterion for a plane wave: $\lambda \ll R_C$

Monochromatic plane waves associated with $\vec{E}$ and $\vec{B}$

$$\vec{\tilde{B}}(z,t) = \vec{\tilde{B}}_o e^{i(kz-\omega t)}$$

$$\vec{\tilde{E}}(z,t) = \vec{\tilde{E}}_o e^{i(kz-\omega t)}$$

MONOCHROMATIC EM PLANE WAVES

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{i(kz-\omega t)}$$

$$\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_o e^{i(kz-\omega t)}$$

Propagating in $+\hat{z}$ direction

Propagating in $+\hat{z}$ direction

$n.b.$ complex vectors:

e.g. $\tilde{\vec{E}}_o = E_o e^{i\delta}\hat{x}$

$n.b.$ complex vectors:

e.g. $\tilde{\vec{B}}_o = B_o e^{i\delta}\hat{y}$

$n.b.$ The *real, physical* (instantaneous) fields are:

$$\left\{ \begin{array}{l} \vec{E}(\vec{r},t) \equiv \mathrm{Re}\left(\tilde{\vec{E}}(\vec{r},t)\right) \\ \vec{B}(\vec{r},t) \equiv \mathrm{Re}\left(\tilde{\vec{B}}(\vec{r},t)\right) \end{array} \right\}$$

Very important to keep in mind!!

MONOCHROMATIC EM PLANE WAVES

Maxwell's equations for free space impose additional constraints on $\tilde{\vec{E}}_o$ and $\tilde{\vec{B}}_o$

Since: $\vec{\nabla}\bullet\vec{E} = 0$ and: $\vec{\nabla}\bullet\vec{B} = 0$

$= \mathrm{Re}\left(\vec{\nabla}\bullet\tilde{\vec{E}}\right) = 0$ $= \mathrm{Re}\left(\vec{\nabla}\bullet\tilde{\vec{B}}\right) = 0$

These two relations can only be satisfied

$$\forall(\vec{r},t) \text{ if } \vec{\nabla}\bullet\tilde{\vec{E}} = 0 \ \ \forall(\vec{r},t) \text{ and } \vec{\nabla}\bullet\tilde{\vec{B}} = 0 \ \ \forall(\vec{r},t)$$

In Cartesian coordinates: $\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

<u>Thus</u>: $\left(\vec{\nabla}\bullet\tilde{\vec{E}}\right) = 0$ and $\left(\vec{\nabla}\bullet\tilde{\vec{B}}\right) = 0$ become:

MONOCHROMATIC EM PLANE WAVES

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(\vec{\tilde{E}}_o e^{i(kz-\omega t)}\right)=0 \quad \text{and} \quad \left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(\vec{\tilde{B}}_o e^{i(kz-\omega t)}\right)=0$$

Now suppose we do allow:

$$\vec{\tilde{E}}_o=\underbrace{\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)}_{\text{polarization in } \hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{E}_o e^{i\delta}$$

$$\vec{\tilde{B}}_o=\underbrace{\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)}_{\text{polarization in } \hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{B}_o e^{i\delta}$$

Then

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0$$

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0$$

MONOCHROMATIC EM PLANE WAVES

E_{ox}, E_{oy}, E_{oz} = Amplitudes (constants) of the electric field components in x, y, z directions respectively.

B_{ox}, B_{oy}, B_{oz} =Amplitudes (constants) of the magnetic field components in *x, y, z* directions respectively.

$$\frac{\partial}{\partial x}\hat{x}\bullet E_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial y}\hat{y}\bullet E_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial x}\hat{x}\bullet B_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial y}\hat{y}\bullet B_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial z}\left(e^{az}\right)=ae^{az}$$

MONOCHROMATIC EM PLANE WAVES . . .

$$\frac{\partial}{\partial z}\hat{z}\bullet E_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta} = ikE_{oz}e^{i(kz-\omega t)}e^{i\delta} = 0 \quad \Leftarrow \text{ true } \textit{iff } \boxed{E_{oz} \equiv 0} \text{ !!!}$$

$$\frac{\partial}{\partial z}\hat{z}\bullet B_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta} = ik\mathrm{E}_{oz}e^{i(kz-\omega t)}e^{i\delta} = 0 \quad \Leftarrow \text{ true } \textit{iff } \boxed{B_{oz} \equiv 0} \text{ !!!}$$

- Maxwell's equations additionally impose the restriction that an electromagnetic plane wave cannot have any component of **E** or **B** || to (or anti- || to) the propagation direction (in this case here, the z -direction)
- Another way of stating this is that an EM wave cannot have any longitudinal components of **E** and **B** (i.e. components of **E** and **B** lying along the propagation direction).

MONOCHROMATIC EM PLANE WAVES . . .

- Thus, Maxwell's equations additionally tell us that an EM wave is a purely transverse wave (at least for propagation in free space) – the components of **E** and **B** must be ⊥ to propagation direction.
- The plane of polarization of an EM wave is defined (by convention) to be parallel to **E.**

MONOCHROMATIC EM PLANE WAVES . . .

Maxwell's equations impose another restriction on the allowed form of E and B for an EM wave:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = \mathrm{Re}\left(\vec{\nabla}\times\tilde{\vec{E}}\right) = \mathrm{Re}\left(-\frac{\partial\tilde{\vec{B}}}{\partial t}\right)$$

and/or:

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mathrm{Re}\left(\vec{\nabla}\times\tilde{\vec{B}}\right) = \mathrm{Re}\left(\frac{1}{c^2}\frac{\partial\tilde{\vec{E}}}{\partial t}\right)$$

Can *only* be satisfied $\forall\ (\vec{r},t)$ ***iff***:

$$\vec{\nabla}\times\tilde{\vec{E}} = -\frac{\partial\tilde{\vec{B}}}{\partial t}$$

and/or:

$$\vec{\nabla}\times\tilde{\vec{B}} = \frac{1}{c^2}\frac{\partial\tilde{\vec{E}}}{\partial t}$$

MONOCHROMATIC EM PLANE WAVES . . .

$$\vec{\nabla}\times\vec{\tilde{E}}=\left(\overbrace{\frac{\partial\tilde{E}_z}{\partial y}}^{=0}-\frac{\partial\tilde{E}_y}{\partial z}\right)\hat{x}+\left(\frac{\partial\tilde{E}_x}{\partial z}-\overbrace{\frac{\partial\tilde{E}_y}{\partial x}}^{=0}\right)\hat{y}+\left(\overbrace{\frac{\partial\tilde{E}_y}{\partial x}}^{=0}-\overbrace{\frac{\partial\tilde{E}_x}{\partial y}}^{=0}\right)\hat{z}=-\frac{\partial\tilde{B}_x}{\partial t}\hat{x}-\frac{\partial\tilde{B}_y}{\partial t}\hat{y}-\overbrace{\frac{\partial\tilde{B}_z}{\partial t}}^{=0}\hat{z}$$

$$\vec{\nabla}\times\tilde{B}=\left(\overbrace{\frac{\partial\tilde{B}_z}{\partial y}}^{=0}-\frac{\partial\tilde{B}_y}{\partial z}\right)\hat{x}+\left(\frac{\partial\tilde{B}_x}{\partial z}-\overbrace{\frac{\partial\tilde{B}_y}{\partial x}}^{=0}\right)\hat{y}+\left(\overbrace{\frac{\partial\tilde{B}_y}{\partial x}}^{=0}-\overbrace{\frac{\partial\tilde{B}_x}{\partial y}}^{=0}\right)\hat{z}=\frac{1}{c^2}\frac{\partial\tilde{E}_x}{\partial t}\hat{x}+\frac{1}{c^2}\frac{\partial\tilde{E}_y}{\partial t}\hat{y}+\frac{1}{c^2}\overbrace{\frac{\partial\tilde{E}_z}{\partial t}}^{=0}\hat{z}$$

$$\vec{\tilde{E}}=\tilde{E}_x\hat{x}+\tilde{E}_y\hat{y}+\overbrace{\tilde{E}_z}^{=0}\hat{z}=\left(E_{ox}\hat{x}+E_{oy}\hat{y}+\overbrace{E_{oz}}^{=0}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}$$

$$\vec{\tilde{B}}=\tilde{B}_x\hat{x}+\tilde{B}_y\hat{y}+\overbrace{\tilde{B}_z}^{=0}\hat{z}=\left(B_{ox}\hat{x}+B_{oy}\hat{y}+\overbrace{B_{oz}}^{=0}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}$$

MONOCHROMATIC EM PLANE WAVES . . .

$$\vec{\tilde{E}} = \tilde{E}_x \hat{x} + \tilde{E}_y \hat{y} = \left(E_{ox} \hat{x} + E_{oy} \hat{y} \right) e^{i(kz-\omega t)} e^{i\delta}$$

$$\vec{\tilde{B}} = \tilde{B}_x \hat{x} + \tilde{B}_y \hat{y} = \left(B_{ox} \hat{x} + B_{oy} \hat{y} \right) e^{i(kz-\omega t)} e^{i\delta}$$

$$\vec{\nabla} \times \vec{\tilde{E}} = -\frac{\partial \tilde{E}_y}{\partial z} \hat{x} + \frac{\partial \tilde{E}_x}{\partial z} \hat{y} = -\frac{\partial \tilde{B}_x}{\partial t} \hat{x} - \frac{\partial \tilde{B}_y}{\partial t} \hat{y}$$

$$\vec{\nabla} \times \vec{\tilde{B}} = -\frac{\partial \tilde{B}_y}{\partial z} \hat{x} + \frac{\partial \tilde{B}_x}{\partial z} \hat{y} = \frac{1}{c^2} \frac{\partial \tilde{E}_x}{\partial t} \hat{x} + \frac{1}{c^2} \frac{\partial \tilde{E}_y}{\partial t} \hat{y}$$

Can only be satisfied / can only be true ***iff*** the $\hat{x}$ and $\hat{y}$ relations are <u>separately</u> / <u>independently</u> satisfied $\forall \ (\vec{r}, t)$!

MONOCHROMATIC EM PLANE WAVES . . .

$\vec{\nabla}\times\vec{\tilde{E}}:$

$$-\frac{\partial\tilde{E}_y}{\partial z}\hat{x} = -\frac{\partial\tilde{B}_x}{\partial t}\hat{x} \Rightarrow \frac{\partial\tilde{E}_y}{\partial z} = \frac{\partial\tilde{B}_x}{\partial t} \Rightarrow ikE_{oy} = -i\omega B_{ox} \qquad (1)$$

$$+\frac{\partial\tilde{E}_x}{\partial z}\hat{y} = -\frac{\partial\tilde{B}_y}{\partial t}\hat{y} \Rightarrow \frac{\partial\tilde{E}_x}{\partial z} = -\frac{\partial\tilde{B}_y}{\partial t} \Rightarrow ikE_{ox} = +i\omega B_{oy} \qquad (2)$$

$\vec{\nabla}\times\tilde{B}:$

$$-\frac{\partial\tilde{B}_y}{\partial z}\hat{x} = \frac{1}{c^2}\frac{\partial\tilde{E}_x}{\partial t}\hat{x} \Rightarrow -\frac{\partial\tilde{B}_y}{\partial z} = \frac{1}{c^2}\frac{\partial\tilde{E}_x}{\partial t} \Rightarrow -ikB_{oy} = -\frac{1}{c^2}i\omega E_{ox} \qquad (3)$$

$$+\frac{\partial\tilde{B}_x}{\partial z}\hat{y} = \frac{1}{c^2}\frac{\partial\tilde{E}_y}{\partial t}\hat{y} \Rightarrow \frac{\partial\tilde{B}_x}{\partial z} = \frac{1}{c^2}\frac{\partial\tilde{E}_y}{\partial t} \Rightarrow ikB_{ox} = -\frac{1}{c^2}i\omega E_{oy} \qquad (4)$$

From (1): $ik\tilde{E}_{oy} = -i\omega B_{ox} \Rightarrow E_{oy} = -\left(\frac{\omega}{k}\right)B_{ox}$ or: $B_{ox} = -\left(\frac{k}{\omega}\right)E_{oy}$

MONOCHROMATIC EM PLANE WAVES . . .

From (2): $ik\tilde{E}_{ox} = +i\omega B_{oy}$ $\Rightarrow$ $E_{ox} = +\left(\frac{\omega}{k}\right)B_{oy}$ or: $B_{oy} = +\left(\frac{k}{\omega}\right)E_{ox}$

From (3): $-ikB_{oy} = -\frac{1}{c^2}i\omega E_{ox}$ $\Rightarrow$ $B_{oy} = +\frac{1}{c^2}\left(\frac{\omega}{k}\right)E_{ox}$

From (4): $ikB_{ox} = -\frac{1}{c^2}i\omega E_{oy}$ $\Rightarrow$ $B_{ox} = -\frac{1}{c^2}\left(\frac{\omega}{k}\right)E_{oy}$

$$c = f\lambda = (2\pi f)\left(\frac{\lambda}{2\pi}\right) = \left(\frac{\omega}{k}\right) \qquad \frac{1}{c} = \left(k/\omega\right) \qquad \left(k = 2\pi/\lambda\right)$$

MONOCHROMATIC EM PLANE WAVES . . .

$\underline{\vec{\nabla}\times\tilde{\vec{E}}:}$ (1) $B_{ox} = -\frac{1}{c}E_{oy}$

(2) $B_{oy} = +\frac{1}{c}E_{ox}$

$\underline{\vec{\nabla}\times\tilde{\vec{B}}:}$ (3) $B_{oy} = +\frac{1}{c}E_{ox}$

(4) $B_{ox} = -\frac{1}{c}E_{oy}$

Maxwell's Equations also have some redundancy encrypted into them!

Actually we have only two independent relations:

$B_{ox} = -\frac{1}{c}E_{oy}$ and $B_{oy} = +\frac{1}{c}E_{ox}$

But: $\hat{z}\times\hat{y} = -\hat{x}$ $\hat{z}\times\hat{x} = +\hat{y}$

MONOCHROMATIC EM PLANE WAVES . . .

Very Useful Table:

$\hat{x}\times\hat{y}=\hat{z}$	$\hat{y}\times\hat{x}=-\hat{z}$
$\hat{y}\times\hat{z}=\hat{x}$	$\hat{z}\times\hat{y}=-\hat{x}$
$\hat{z}\times\hat{x}=\hat{y}$	$\hat{x}\times\hat{z}=-\hat{y}$

Two relations can be written compactly into one relation:

$$\vec{\tilde{B}}_o=\frac{1}{c}\left(\hat{z}\times\vec{\tilde{E}}_o\right)$$

Physically this relation states that E and B are:

- in phase with each other.
- mutually perpendicular to each other - (**E**⊥**B**)⊥ $\hat{z}$

MONOCHROMATIC EM PLANE WAVES . . .

The **E** and **B** fields associated with this monochromatic plane EM wave are purely transverse { n.b. this is as also required by relativity at the microscopic level – for the extreme relativistic particles – the (massless) real photons travelling at the speed of light c that make up the macroscopic monochromatic plane EM wave.}

The real amplitudes of E and B are related to each other by:

$$B_o = \frac{1}{c} E_o$$

with

$$B_o = \sqrt{B_{ox}^2 + B_{oy}^2}$$

and

$$E_o = \sqrt{E_{ox}^2 + E_{oy}^2}$$

Instantaneous Poynting's Vector for a linearly polarized *EM wave*

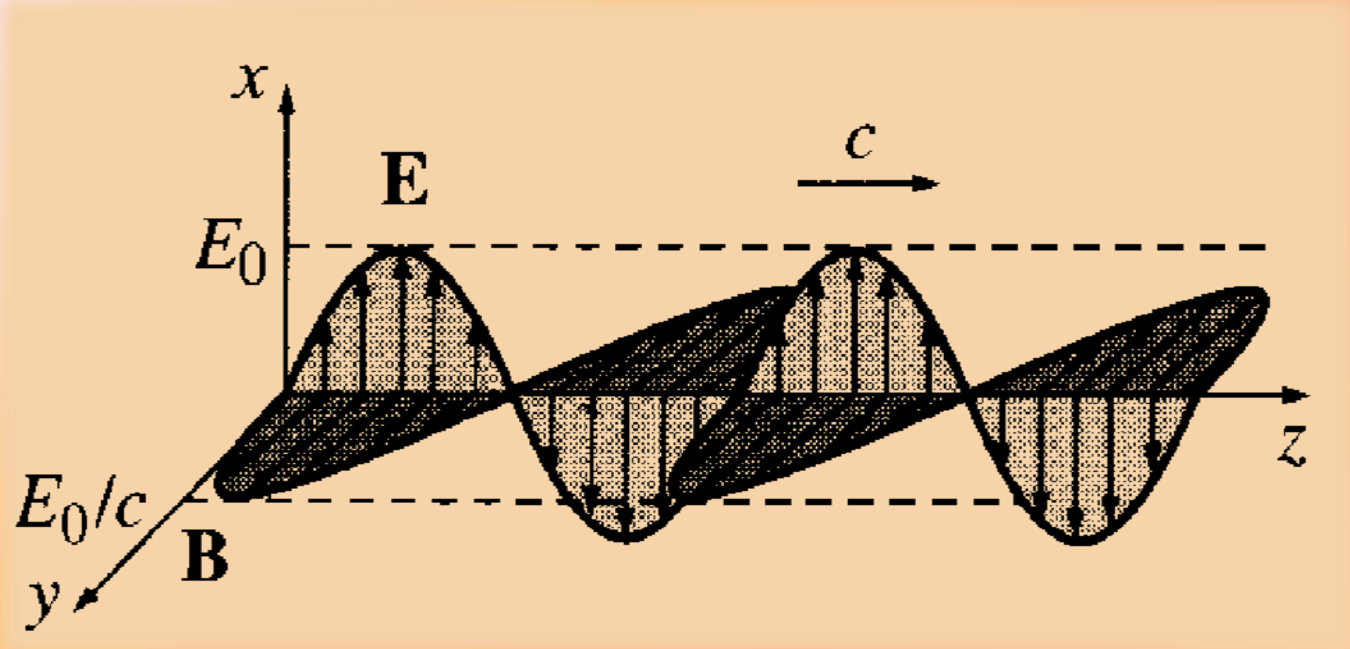
$$\vec{S}(z,t) = \frac{1}{\mu_o}\vec{E}(z,t)\times\vec{B}(z,t) = \frac{1}{\mu_o}\mathrm{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\mathrm{Re}\left\{\tilde{\vec{B}}(z,t)\right\}$$

$$\vec{S}(z,t) = \frac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\underbrace{(\hat{x}\times\hat{y})}_{=\hat{z}}$$

$$\vec{S}(z,t) = \frac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\,\hat{z} \qquad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

⇒EM Power flows in the direction of propagation of the EM wave (here, the +zˆ direction)

Instantaneous Poynting's Vector for a linearly polarized *EM wave*



This is the paradigm for a monochromatic plane wave. The wave as a whole is said to be polarized in the x direction (by convention, we use the direction of E to specify the polarization of an electromagnetic wave).

Instantaneous Energy & Linear Momentum & Angular Momentum in *EM Waves*

Instantaneous Energy Density Associated with an *EM Wave:*

$$u_{EM}(\vec{r},t)=\frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t)+\frac{1}{\mu_o}B^2(\vec{r},t)\right)=u_{elect}(\vec{r},t)+u_{mag}(\vec{r},t)$$

where $$u_{elect}(\vec{r},t)=\frac{1}{2}\varepsilon_o E^2(\vec{r},t)$$

and $$u_{mag}(\vec{r},t)=\frac{1}{2\mu_o}B^2(\vec{r},t)=\frac{1}{2}\varepsilon_o E^2(\vec{r},t)$$

Instantaneous Energy & Linear Momentum & Angular Momentum in *EM* Waves

But $B^2 = \frac{1}{c^2} E^2$ - EM waves in vacuum, and $\frac{1}{c^2} = \varepsilon_o \mu_o$

$$u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \frac{\varepsilon_o \not{\mu_o}}{\not{\mu_o}} E^2(\vec{r},t) \right) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \varepsilon_o E^2(\vec{r},t)\right)$$

$$\boxed{u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2 \cos^2\left(\vec{k}\bullet\vec{r} - \omega t + \delta\right)} \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

$u_{elect}(\vec{r},t) = u_{mag}(\vec{r},t)$ - EM waves propagating in the vacuum !!!!

Instantaneous Poynting's Vector Associated with an *EM Wave*

$$\vec{S}(\vec{r},t) = \frac{1}{\mu_o}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t) = \frac{1}{\mu_o}\mathrm{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\mathrm{Re}\left\{\tilde{\vec{B}}(z,t)\right\} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

For a linearly polarized monochromatic plane EM wave propagating in the vacuum,

$$\vec{S}(\vec{r},t) = c\left(\frac{\varepsilon_o \mu_o}{\mu_o}\right)E_o^2\cos^2(kz-\omega t+\delta)\hat{z} = c\varepsilon_o E_o^2\cos^2(kz-\omega t+\delta)\hat{z}$$

But $u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2\cos^2(kz-\omega t+\delta)$

$$\vec{S}(\vec{r},t) = c\,u_{EM}(\vec{r},t)\hat{z}$$

Instantaneous Poynting's Vector Associated with an *EM Wave*

The propagation velocity of energy $\vec{v}_{prop} = c\hat{z}$

Poynting's Vector = Energy Density * Propagation Velocity

$$\vec{S}(\vec{r},t) = u_{EM}(\vec{r},t)\vec{v}_{prop}$$

Instantaneous Linear Momentum Density Associated with an EM Wave:

$$\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

Instantaneous Linear Momentum Density Associated with an *EM Wave*

For linearly polarized monochromatic plane EM waves propagating in the vacuum:

$$\vec{\wp}_{EM} = \frac{1}{c^{\not{2}}} \not{c}\, \varepsilon_o E_o^2 \cos^2(kz - \omega t + \delta)\,\hat{z} = \frac{1}{c} \underbrace{\varepsilon_o E_o^2 \cos^2(kz - \omega t + \delta)}_{=u_{EM}}\,\hat{z}$$

But: $$u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2 \cos^2(kz - \omega t + \delta)$$

$$\boxed{\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) = \frac{1}{c} u_{EM}(\vec{r},t)\,\hat{z}} \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

Instantaneous Angular Momentum Density Associated with an *EM wave*

$$\vec{\ell}_{EM}(\vec{r},t) = \vec{r} \times \vec{\wp}_{EM}(\vec{r},t) \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

But: $$\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) = \frac{1}{c} u_{EM}(\vec{r},t)\hat{z} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

For an EM wave propagating in the +z^ direction:

$$\vec{\ell}_{EM}(\vec{r},t) = \frac{1}{c^2}\vec{r} \times \vec{S}(\vec{r},t) = \frac{1}{c} u_{EM}(\vec{r},t)(\vec{r} \times \hat{z}) \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

Depends on the choice of origin

Instantaneous Power Associated with an *EM wave*

The instantaneous EM power flowing into/out of volume v with bounding surface S enclosing volume v (containing EM fields in the volume v) is:

$$P_{EM}(t) = \frac{\partial U_{EM}(t)}{\partial t} = \int_v \frac{\partial u_{EM}(\vec{r},t)}{\partial t} d\tau = -\oint_S \vec{S}(\vec{r},t) \bullet d\vec{a}$$

The instantaneous EM power crossing (imaginary) surface is:

$$P_{EM}(t) = -\int_S \vec{S}(\vec{r},t) \bullet d\vec{a}_{\perp}$$

The instantaneous total *E*M energy contained in volume v

$$U_{EM}(t) = \int_v u_{EM}(\vec{r},t)\, d\tau \quad \text{(Joules)}$$

Instantaneous Angular Momentum Density Associated with an *EM wave*

The instantaneous total EM linear momentum contained in the volume v is:

$$\vec{p}_{EM}(t) = \int_v \vec{\wp}_{EM}(\vec{r},t)\, d\tau \qquad \left(\frac{\text{kg-m}}{\text{sec}}\right)$$

The instantaneous total EM angular momentum contained in the volume v is:

$$\vec{\mathcal{L}}_{EM}(t) = \int_v \vec{\ell}_{EM}(\vec{r},t)\, d\tau \qquad \left(\frac{\text{kg-m}^2}{\text{sec}}\right)$$

Time-Averaged Quantities Associated with EM Waves

Usually we are not interested in knowing the instantaneous power P(t), energy / energy density, Poynting's vector, linear and angular momentum, *etc.*- because experimental measurements of these quantities are very often averages over many extremely fast cycles of oscillation. For example period of oscillation of light wave

$$\tau_{light} = 1/f_{light} \simeq \frac{1}{10^{15}\,\text{cps}} = 10^{-15}\ \text{sec/cycle} = 1\,\text{femto-sec}\ )$$

We need time averaged expressions for each of these quantities - in order to compare directly with experimental data- for monochromatic plane EM light waves:

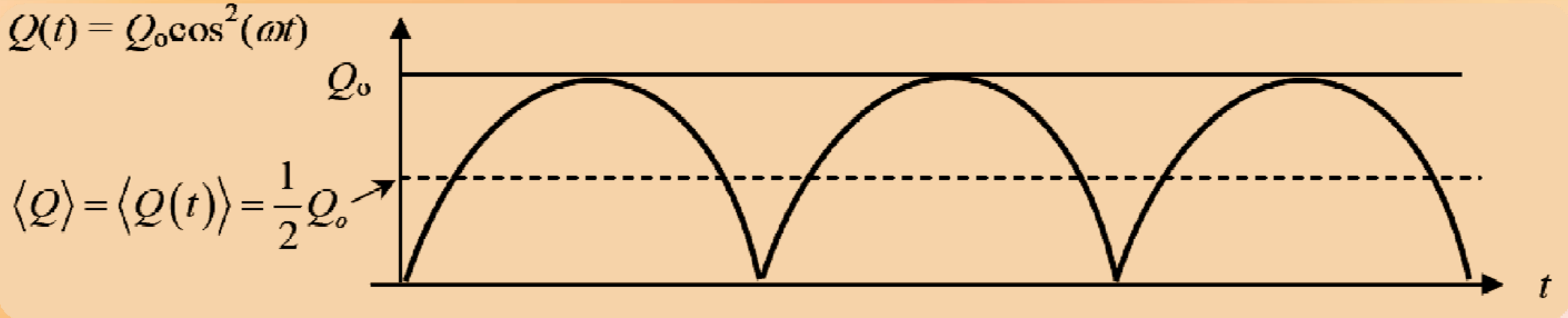
Time-Averaged Quantities Associated with EM Waves

If we have an instantaneous physical quantity of the form:

$$Q(t) = Q_o \cos^2(\omega t)$$

The time-average of *Q(t)* is defined as:

$$\langle Q(t) \rangle \equiv \langle Q \rangle = \frac{1}{\tau} \int_{t=0}^{t=\tau} Q(t)\, dt = \frac{Q_o}{\tau} \int_{t=0}^{t=\tau} \cos^2(\omega t)\, dt$$



Time-Averaged Quantities Associated with EM Waves

The time average of the $\cos^2(\omega t)$ function:

$$\frac{1}{\tau}\int_0^{\tau}\cos^2(\omega t)\,dt = \frac{1}{\tau}\left[\frac{t}{2}+\frac{\sin 2\omega t}{4\omega}\right]_{t=0}^{t=\tau} = \frac{1}{2\tau}\left[(\tau-0)+\left(\frac{\sin 2\omega\tau}{2\omega}-0\right)\right] = \frac{1}{2\tau}\left[\tau+\frac{\sin 2\omega\tau}{2\omega}\right]$$

$$\omega\tau = 2\pi f\tau \qquad f = 1/\tau \qquad \omega\tau = 2\pi(\tau/\tau) = 2\pi \qquad \sin(\omega\tau) = \sin(2\pi) = 0$$

$$\frac{1}{\tau}\int_0^{\tau}\cos^2(\omega t)\,dt = \frac{1}{2\not{t}}\left[\not{t}\right] = \frac{1}{2} \qquad \langle Q(t)\rangle = \langle Q\rangle = \frac{1}{2}Q_o$$

Thus, the time-averaged quantities associated with an EM wave propagating in free space are:

Time-Averaged Quantities Associated with EM Waves

Intensity of an *EM* wave:

$$I(\vec{r}) \equiv \langle S(\vec{r},t) \rangle = \langle |\vec{S}(\vec{r},t)| \rangle = c \langle u_{EM}(\vec{r},t) \rangle = \frac{1}{2} c \varepsilon_o E_o^2 \quad \left(\frac{\text{Watts}}{\text{m}^2} \right)$$

The intensity of an EM wave is also known as the irradiance of the EM wave – it is the radiant power incident per unit area upon a surface.

Time-Averaged Quantities Associated with EM Waves

EM Energy Density: $u_{EM}(\vec{r},t) \Rightarrow \langle u_{EM}(\vec{r},t) \rangle$

Total EM Energy: $U_{EM}(t) \Rightarrow \langle U_{EM}(t) \rangle$

Poynting's Vector: $\vec{S}(\vec{r},t) \Rightarrow \langle \vec{S}_{EM}(\vec{r},t) \rangle$

EM Power: $P_{EM}(t) \Rightarrow \langle P_{EM}(t) \rangle$

Time-Averaged Quantities Associated with EM Waves

Linear Momentum Density: $\vec{\wp}_{EM}(\vec{r},t) \Rightarrow \left\langle \vec{\wp}_{EM}(\vec{r},t) \right\rangle$

Linear Momentum: $\vec{p}_{EM}(t) \Rightarrow \left\langle \vec{p}_{EM}(t) \right\rangle$

Angular Momentum Density: $\vec{\ell}_{EM}(\vec{r},t) \Rightarrow \left\langle \vec{\ell}_{EM}(\vec{r},t) \right\rangle$

Angular Momentum: $\vec{\mathcal{L}}_{EM}(t) \Rightarrow \left\langle \vec{\mathcal{L}}_{EM}(t) \right\rangle$

Time-Averaged Quantities Associated with EM Waves

For a monochromatic EM plane wave propagating in free space / vacuum in ^z direction:

$$\langle u_{EM}(\vec{r},t)\rangle = \frac{1}{2}\varepsilon_o E_o^2 \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}c\varepsilon_o E_o^2 \hat{z} = c\langle u_{EM}(\vec{r},t)\rangle \hat{z} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

$$\langle \vec{\wp}_{EM}(\vec{r},t)\rangle = \frac{1}{2c}\varepsilon_o E_o^2 \hat{z} = \frac{1}{c^2}\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{c}\langle u_{EM}(\vec{r},t)\rangle \hat{z} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

$$\langle \ell_{EM}(\vec{r},t)\rangle = \left(\vec{r}\times\langle \vec{\wp}_{EM}(\vec{r},t)\rangle\right) = \frac{1}{c^2}\left(\vec{r}\times\langle \vec{S}(\vec{r},t)\rangle\right) = \frac{1}{c}\langle u_{EM}(\vec{r},t)\rangle(\hat{r}\times\hat{z}) \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

Story has not finished yet

To be continued…

THANKS FOR TIME BEING