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**Preparatory School to the Winter College on Optics: Fundamentals of Photonics - Theory, Devices and Applications**

*3 - 7 February 2014*

# REVIEW OF ELECTRODYNAMICS

Imrana Ashraf Zahid
*Quaid-i-Azam University*
*Islamabad*
*Pakistan*

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# Layout

- Electrostatic : Revisited
- Magneto- static : Revisited
- Introduction to Maxwell's equations
- Electrodynamics before Maxwell
- Maxwell's correction to Ampere's law
- General form of Maxwell's equations
- Maxwell's equations in vacuum
- Maxwell's equations inside matter
- The Electromagnetic wave
- Energy and Momentum of Electromagnetic Waves

# Nomenclature

- $E$ = Electric field
- $D$ = Electric displacement
- $B$ = Magnetic flux density
- $H$ = Auxiliary field
- $\rho$= Charge density
- $j$ = Current density
- $\mu_0$ (permeability of free space) = $4\pi\times10^{-7}$T-m/A
- $\varepsilon_0$ (permittivity of free space) = $8.854\times10^{-12}$N-$m^2$/ $C^2$
- c (speed of light) = $2.99792458\times10^8$ m/s

# Introduction

- **Electrostatics**

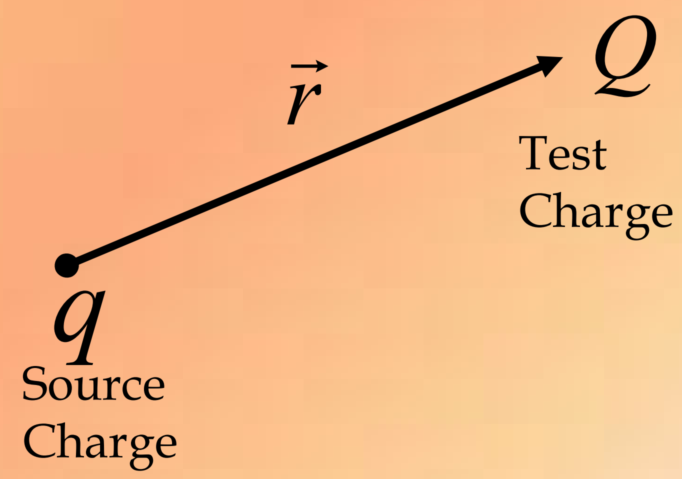
  - **Electrostatic field : Stationary charges produce electric fields that are constant in time. The theory of static charges is called electrostatics.**

**Stationary charges ⟹ Constant Electric field;**

# Electrostatic :Revisited

**Coulombs Law**

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$$



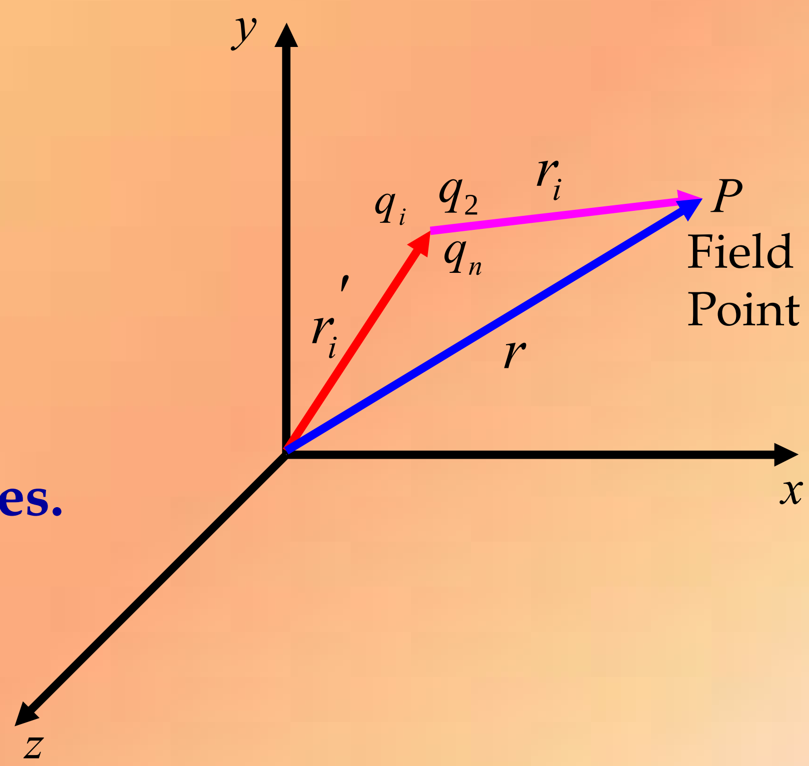

$$\varepsilon_0 = 8.85\times10^{-12}\frac{C^2}{N-m^2}$$

**Permittivity of free space**

# The Electric Field

$$\vec{F} = Q\vec{E}$$

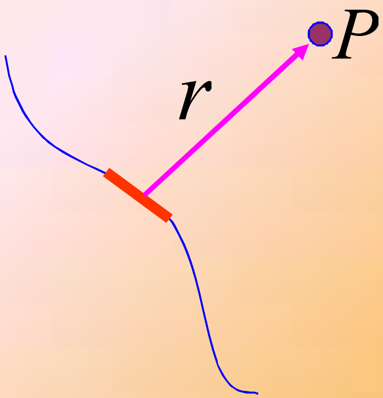
$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^{\,2}}\hat{r}_i$$




$\vec{E}$- **the electric field of the source charges.**

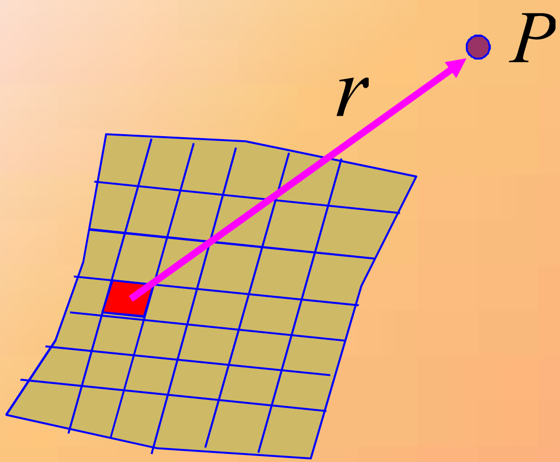
**Physically E(P) Is force per unit charge exerted on a test charge placed at P.**

# The Electric Field: cont'd




$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Line} \frac{\hat{r}}{r^2} \lambda dl$$

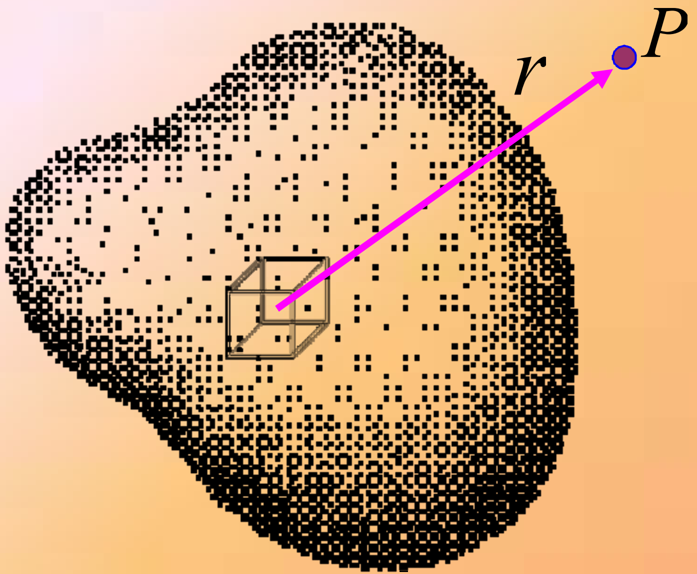
$\lambda$ **is the line charge density**




$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Surface} \frac{\hat{r}}{r^2} \sigma da$$

$\sigma$ **is the surface charge density**

# The Electric Field: cont'd




$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Volume} \frac{\hat{r}}{r^2} \rho \, d\tau$$

**$\rho$ is the volume charge density**

# Electric Potential

The work done in moving a test charge Q in an electric field from point $P_1$ to $P_2$ with a constant speed.

$$W = Force \bullet distance$$

$$W = -\int_{p_1}^{P_2} Q\vec{E} \bullet d\vec{l}$$

negative sign - work done is against the field.

For any distribution of fixed charges.

$$\oint \vec{E} \bullet d\vec{l} = 0$$

**The electrostatic field is conservative**

# Electric Potential: cont'd

Stokes's Theorem gives

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{E} = -\vec{\nabla} V$$

where *V* is Scalar Potential

The work done in moving a charge Q from infinity to a point $P_2$ where potential is *V*

$$W = QV$$

**V = Work per unit charge**

**= Volts = joules/Coulomb**

# Electric Potential : cont'd

**Field due to a single point charge q at origin**

$$V = \int_r^{\infty} \frac{q\,dr}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r}$$

$$F \propto \frac{1}{r^2}$$

$$E \propto \frac{1}{r^2}$$

$$V \propto \frac{1}{r}$$

**Gauss's Law**

$$\oint \vec{E} \bullet d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

**Differential form of Gauss's Law**

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$$

**Poisson's Equation**

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

**Laplace's Equation**

$$\nabla^2 V = 0$$

# Electrostatic Fields in Matter

**Matter:** **Solids, liquids, gases, metal, wood and glasses - behave differently in electric field.**

**Two Large Classes of Matter**

**(i) Conductors**

**(ii) Dielectric**

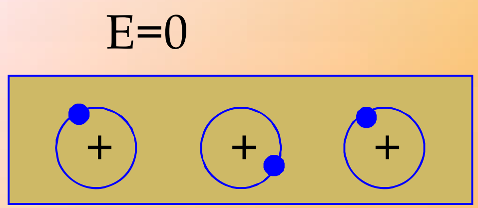
**Conductors:** **Unlimited supply of free charges.**

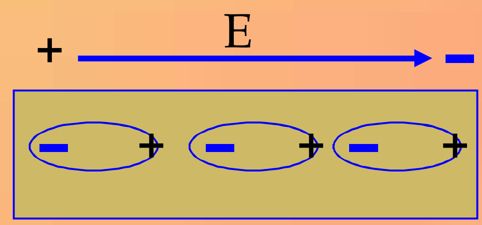
**Dielectrics:**

- **Charges are attached to specific atoms or molecules- No free charges.**
- **Only possible motion - minute displacement of positive and negative charges in opposite direction.**
- **Large fields- pull the atom apart completely (ionizing it).**

# Polarization

A dielectric with charge displacements or induced dipole moment is said to be polarized.




**Induced Dipole Moment**

$$p = \alpha E$$

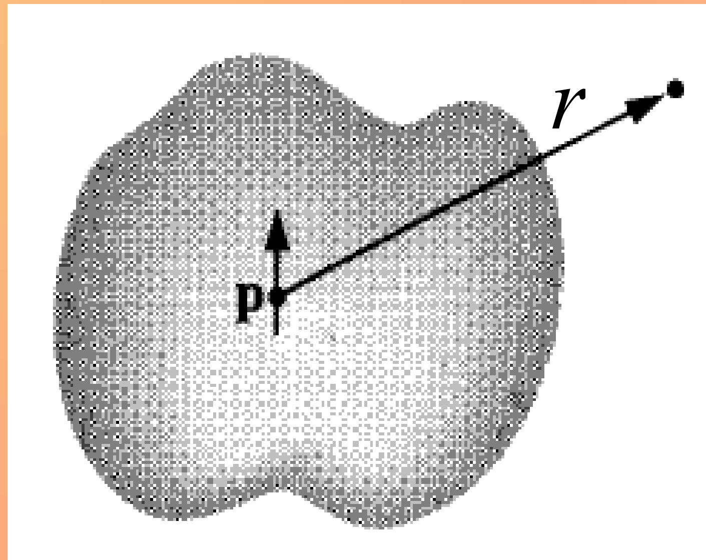
**The constant of proportionality α is called the atomic polarizability**

**P ≡ dipole moment per unit volume**

# The Field of a Polarized Object

Potential of single dipole p is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\bullet\vec{\mathrm{p}}}{\mathrm{r}^2}$$




$$V = \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\vec{\mathrm{P}}\bullet\hat{r}}{\mathrm{r}^2}d\tau$$

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface}\frac{1}{r}\vec{P}\bullet da - \int_{volume}\frac{1}{r}\left(\vec{\nabla}\bullet\vec{P}\right)d\tau\right]$$

**Potential due to dipoles in the dielectric**

# The Field of a Polarized Object: cont'd

$$\sigma_b = \vec{P} \bullet \hat{n}$$

**Bound charges at surface**

$$\rho_b = -\vec{\nabla} \bullet \vec{P}$$

**Bound charges in volume**

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface} \frac{1}{r}\sigma_b \, da + \int_{volume} \frac{1}{r}\rho_b \, d\tau\right]$$

**The total field is field due to bound charges plus due to free charges**

# Gauss's law in Dielectric

- Effect of polarization is to produce accumulations of bound charges.
- The total charge density

$$\rho = \rho_f + \rho_b$$

**From Gauss's law**

$$\varepsilon_0 \vec{\nabla} \bullet \vec{E} = \rho = \rho_b + \rho_f$$

$$\vec{\nabla} \bullet \vec{D} = \rho_f$$

$$\int \vec{D} \bullet d\vec{a} = Q_{fenc}$$

$Q_{fenc}$**-Free charges enclosed**

**Displacement vector**

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

# Magnetostatics : Revisited

- Magnetostatics
  - Steady current produce magnetic fields that are constant in time. The theory of constant current is called magnetostatics.

**Steady currents → Constant Magnetic field;**

# Magnetic Forces

**Lorentz Force**

$$\vec{F} = q\left[\vec{E} + \left(\vec{v} \times \vec{B}\right)\right]$$

- **The magnetic force on a segment of current carrying wire is**

$$F_{mag} = \int \left(\vec{I} \times \vec{B}\right) dl$$

$$F_{mag} = \int I\left(d\vec{l} \times \vec{B}\right)$$

# Equation of Continuity

The current crossing a surface s can be written as

$$I = \int_s \vec{J} \bullet d\vec{a} = \int_v \left(\vec{\nabla} \bullet \vec{J}\right) d\tau$$

$$\int_v \left(\vec{\nabla} \bullet \vec{J}\right) d\tau = -\frac{d}{dt} \int \rho \, d\tau = -\int \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

Charge is conserved whatever flows out must come at the expense of that remaining inside - outward flow decreases the charge left in v

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

**This is called equation of continuity**

# Equation of Continuity 1

**In Magnetostatic steady currents flow in the wire and its magnitude I must be the same along the line- otherwise charge would be pilling up some where and current can not be maintained indefinitely.**

$$\frac{\partial \rho}{\partial t} = 0$$

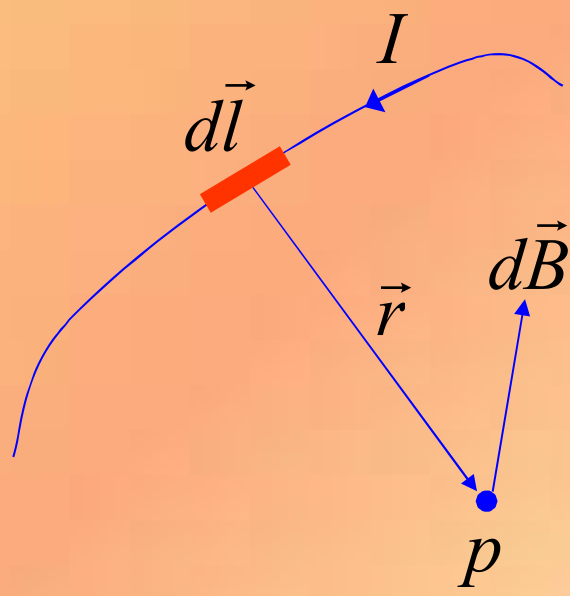
**In Magnetostatic and equation of continuity**

$$\vec{\nabla} \bullet \vec{J} = 0$$

**Steady Currents: The flow of charges that has been going on forever - never increasing - never decreasing.**

# Magnetostatic and Current Distributions

**Biot and Savart Law**




$$\vec{B}(p) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\vec{r}}{|\vec{r}|^3}dl$$

$dl$ is an element of length.

$\vec{r}$ vector from source to point p.

$\mu_0$ Permeability of free space.

Unit of B = N/Am = Tesla (T)

# Biot and Savart Law for Surface and Volume Currents

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \vec{r}}{|\vec{r}|^3} da$$

**For Surface Currents**

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \vec{r}}{|\vec{r}|^3} d\tau$$

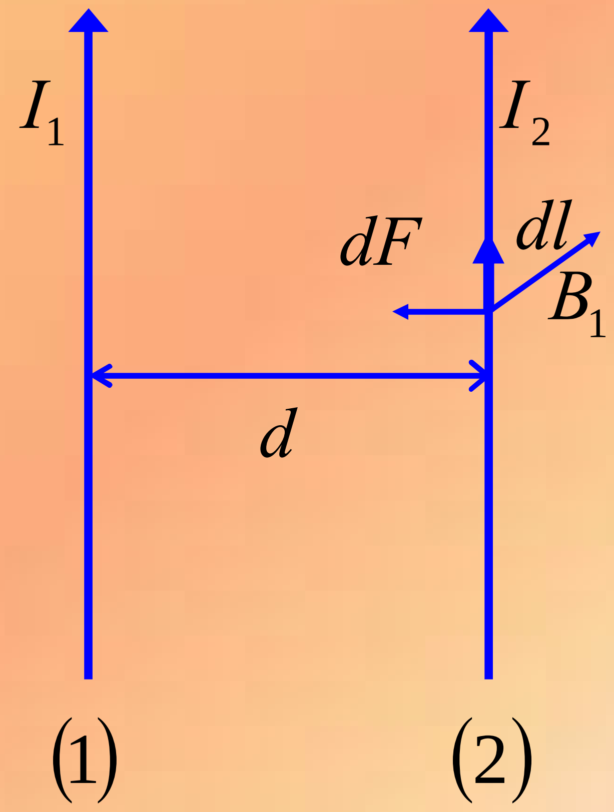
**For Volume Currents**

# Force between two parallel wires

The magnetic field at (2) due to current $I_1$ is

$$B_1 = \frac{\mu_0 I_1}{2\pi d}$$

Points inside




Magnetic force law

$$dF = \int I_2 \left( d\vec{l}_2 \times \vec{B}_1 \right)$$

$$dF = \int I_2 \left( d\vec{l}_2 \times \frac{\mu_0 I_1}{2\pi d} \hat{k} \right)$$

# Force between two parallel wires

$$dF = \frac{\mu_0 I_1 I_2}{2\pi d} dl_2$$

**The total force is infinite but force per unit length is**

$$\frac{dF}{dl_2} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

**If currents are anti-parallel the force is repulsive.**

# Straight line currents

**The integral of B around a circular path of radius s, centered at the wire is**

$$\oint \vec{B} \bullet d\vec{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \mu_0 I$$

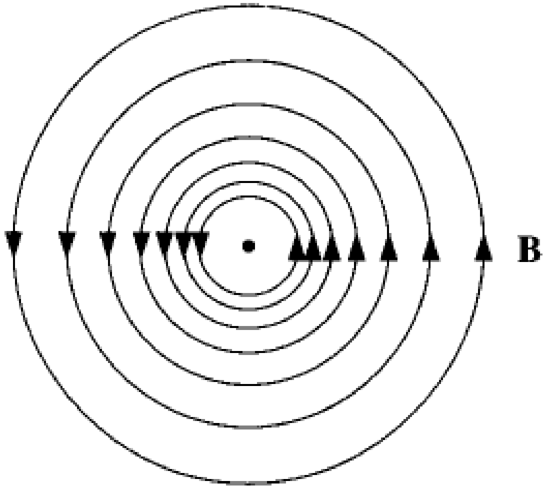
**For bundle of straight wires. Wire that passes through loop contributes only.**

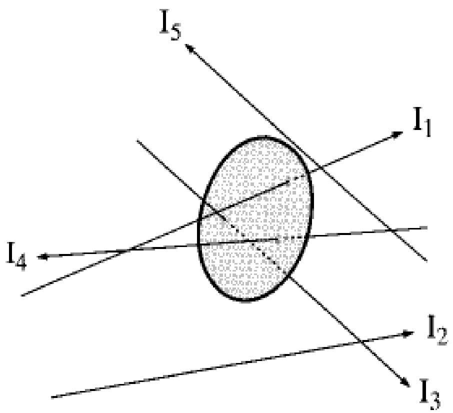
$$\oint \vec{B} \bullet d\vec{l} = \mu_0 I_{enc}$$

**Applying Stokes' theorem**

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

The current is out of the page

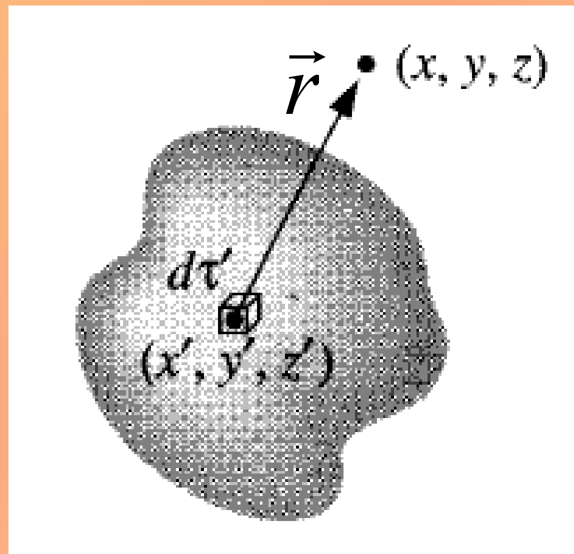


# Divergence and Curl of B

Biot-Savart law for the general case of a volume current reads

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(r')\times\vec{r}}{r^3}\, d\tau'$$




**B** is a function of $(x, y, z)$,

**J** is a function of $(x', y', z')$,

$$\vec{r} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

$\vec{\nabla}\bullet\vec{B} = 0$ **and** $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

# Ampere's Law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$ **Ampere's law**

**Integral form of Ampere's law**

**Using Stokes' theorem**

$$\int \left( \vec{\nabla} \times \vec{B} \right) \bullet d\vec{a} = \oint \vec{B} \bullet d\vec{l} = \mu_0 \int \vec{J} \bullet d\vec{a}$$

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 I_{enc}$$

# Vector Potential

**The basic differential law of Magnetostatics**

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

**B is curl of some vector field called vector potential** $A(P)$

$$\vec{B}(P) = \vec{\nabla} \times \vec{A}(P)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_0 J$$

**Coulomb's gauge**

$$\vec{\nabla} \bullet \vec{A} = 0$$

$$\nabla^2 A = -\mu_0 J$$

# Magnetostatic Field in Matter

- Magnetic fields- due to electrical charges in motion.
- Examine a magnet on atomic scale we would find tiny currents.
- Two reasons for atomic currents.
  - Electrons orbiting around nuclei.
  - Electrons spinning on their axes.
- Current loops form magnetic dipoles - they cancel each other due to random orientation of the atoms.
- Under an applied Magnetic field- a net alignment of - magnetic dipole occurs- and medium becomes magnetically polarized or magnetized

# Magnetization

**If m is the average magnetic dipole moment per unit atom and N is the number of atoms per unit volume, the magnetization is define as**

$$\overrightarrow{M} = N\,\overrightarrow{m}$$

$$\overrightarrow{m} = I\,\overrightarrow{a} = Am^2$$

or

$$m = M\,d\tau$$

$$M = \frac{Am^2}{m^3} = \frac{A}{m}$$

# Magnetic Materials

## Paramagnetic Materials

**The materials having magnetization parallel to B are called paramagnets.**

## Diamagnetic Materials

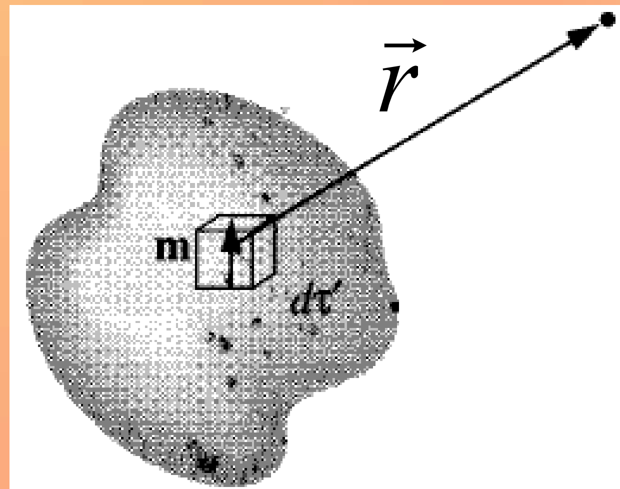
**The elementary moment are not permanent but are induced according to Faraday's law of induction. In these materials magnetization is opposite to B.**

## Ferromagnetic Materials

**Have large magnetization due to electron spin. Elementary moments are aligned in form of groups called domain**

# The Field of Magnetized Object

**Using the vector potential of current loop**




$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{M}\times\hat{n}}{r}da + \frac{\mu_0}{4\pi}\int\frac{\vec{\nabla}\times\vec{M}}{r}d\tau$$

$$\vec{K_b} = \vec{M}\times\hat{n}$$ **Bound Surface Current**

$$\vec{J_b} = \vec{\nabla}\times\vec{M}$$ **Bound Volume Current**

# Ampere's Law in Magnetized Material

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{J} = \vec{J}_b + \vec{J}_f$$

$$\frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right) = \vec{J}_b + \vec{J}_f = \vec{J}_f + \left(\vec{\nabla} \times \vec{M}\right)$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f$$

**where**

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

**Integral form**

$$\oint \vec{H} \bullet d\vec{l} = I_{fenc}$$

# Faraday's Law of Induction

- **Faraday's Law - a changing -magnetic flux through circuit induces an electromotive force around the circuit.**

$$\varepsilon = \oint \vec{E} \bullet \vec{dl} = -\frac{d\phi}{dt} = -\frac{d}{dt}\int \vec{B} \bullet \vec{da}$$

**$\epsilon$ – Induced emf**

**E – Induced electric field intensity**

**Differential form of Faraday's law**

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

# Faraday's Law of Induction

**Induced Electric field intensity in terms of vector potential**

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$$

**For steady currents**

$$\vec{E} = -\vec{\nabla} V$$

**V – Scalar potential**

**Induced emf in a system moving in a changing magnetic field**

$$\varepsilon = \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right)$$

# MAXWELL'S EQUATIONS

Imrana Ashraf Zahid
Quaid-i-Azam University Islamabad
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# Introduction to Maxwell's Equation

- In electrodynamics Maxwell's equations are a set of four equations, that describes the behavior of both the electric and magnetic fields as well as their interaction with matter
- Maxwell's four equations express
  - How electric charges produce electric field (Gauss's law)
  - The absence of magnetic monopoles
  - How currents and changing electric fields produces magnetic fields (Ampere's law)
  - How changing magnetic fields produces electric fields (Faraday's law of induction)

# Historical Background

- 1864 Maxwell in his paper "A Dynamical Theory of the Electromagnetic Field" collected all four equations

- 1884 Oliver Heaviside and Willard Gibbs gave the modern mathematical formulation using vector calculus.

- The change to vector notation produced a symmetric mathematical representation, that reinforced the perception of physical symmetries between the various fields.

# Electrodynamics Before Maxwell

Gauss's Law

$$(i)\ \vec{\nabla}\bullet\vec{E} = \frac{\rho}{\varepsilon_o}$$

No name

$$(ii)\ \vec{\nabla}\bullet\vec{B} = 0$$

Faraday's Law

$$(iii)\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Ampere's Law

$$(iv)\ \vec{\nabla}\times\vec{B} = \mu_o\vec{J}$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

# Electrodynamics Before Maxwell (Cont'd)

**Apply divergence to (iii)**

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\bullet\left(-\frac{\partial\vec{B}}{\partial t}\right)=-\frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{B}\right)$$

The left hand side is zero, because divergence of a curl is zero.

The right hand side is zero because $\vec{\nabla}\bullet\vec{B}=0$.

**Apply divergence to (iv)**

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{B}\right)=\mu_o\left(\vec{\nabla}\bullet\vec{J}\right)$$

# Electrodynamics Before Maxwell (Cont'd)

- **The left hand side is zero, because divergence of a curl is zero.**
- **The right hand side is zero for steady currents i.e.,**

$$\vec{\nabla} \bullet \vec{J} = 0$$

- **In electrodynamics from conservation of charge**

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = 0$$

$\rho$ **is constant at any point in space which is wrong.**

# Maxwell's Correction to Ampere's Law

**Consider Gauss's Law**

$$\vec{\nabla}\bullet\varepsilon_o\vec{E}=\rho$$

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\varepsilon_o\vec{E}\right)=\frac{\partial\rho}{\partial t}$$

$$\Rightarrow\frac{\partial\rho}{\partial t}=\vec{\nabla}\bullet\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

$$\frac{\partial\vec{D}}{\partial t}=\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$ **Displacement current**

**This result along with Ampere's law and the conservation of charge equation suggest that there are actually two sources of magnetic field.**

**The current density and displacement current.**

# Maxwell's Correction to Ampere's Law (Cont'd)

**Amperes law with Maxwell's correction**

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

# General Form of Maxwell's Equations

**Differential Form**

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_o}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

**Integral Form**

$$\oint_S \vec{E} \bullet d\vec{a} = \frac{1}{\varepsilon_o} \int_V \rho dV$$

$$\oint_S \vec{B} \bullet d\vec{a} = 0$$

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \bullet d\vec{a}$$

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_o I_{enc} + \mu_o \varepsilon_o \frac{d}{dt} \oint_S \vec{E} \bullet d\vec{a}$$

# Maxwell's Equations in vacuum

- The vacuum is a linear, homogeneous, isotropic and dispersion less medium
- Since there is no current or electric charge is present in the vacuum, hence Maxwell's equations reads as
- These equations have a simple solution in terms of traveling sinusoidal waves, with the electric and magnetic fields direction orthogonal to each other and the direction of travel

$$\vec{\nabla} \bullet \vec{E} = 0$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

# Maxwell's Equations Inside Matter

**Maxwell's equations are modified for polarized and magnetized materials. For linear materials the polarization *P* and magnetization *M* is given by**

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

$$\vec{M} = \chi_m \vec{H}$$

**And the *D* and *B* fields are related to *E* and *H* by**

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P} = \left(1 + \chi_e\right) \varepsilon_o \vec{E} = \varepsilon \vec{E}$$

$$\vec{B} = \mu_o \left(\vec{H} + \vec{M}\right) = \left(1 + \chi_m\right) \mu_o \vec{H} = \mu \vec{H}$$

Where $\chi_e$ is the electric susceptibility of material,

$\chi_m$ is the magnetic susceptibility of material.

# Maxwell's Equations Inside Matter (Cont'd)

- For polarized materials we have bound charges in addition to free charges

$$\sigma_b = \vec{P} \bullet \hat{n}$$

$$\rho_b = -\vec{\nabla} \bullet \vec{P}$$

- For magnetized materials we have bound currents

$$\overrightarrow{K_b} = \overrightarrow{M} \times \hat{n}$$

$$\overrightarrow{J_b} = \vec{\nabla} \times \overrightarrow{M}$$

# Maxwell's Equations Inside Matter (Cont'd)

- **In electrodynamics any change in the electric polarization involves a flow of bound charges resulting in polarization current $J_P$**

$$\vec{J}_P = \frac{\partial \vec{P}}{\partial t}$$

**Polarization current density is due to linear motion of charge when the Electric polarization changes**

Total charge density

$$\rho_t = \rho_f + \rho_b$$

Total current density

$$J_t = J_f + J_b + J_p$$

# Maxwell's Equations Inside Matter (Cont'd)

- **Maxwell's equations inside matter are written as**

$$\vec{\nabla}\bullet\vec{E}=\frac{\rho_t}{\varepsilon_o}$$

$$\vec{\nabla}\bullet\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B}=\mu_o\vec{J}_f+\mu_o\vec{J}_p+\mu_o\vec{J}_b+\mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\frac{\vec{B}}{\mu_o}=\vec{J}_f+\frac{\partial\vec{P}}{\partial t}+\vec{\nabla}\times\vec{M}+\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_o}-\vec{M}\right)=\vec{J}_f+\frac{\partial}{\partial t}\left(\varepsilon_o\vec{E}+\vec{P}\right)$$

$$\vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial}{\partial t}\vec{D}$$

# Maxwell's Equations Inside Matter (Cont'd)

- In non-dispersive, isotropic media ε and μ are time-independent scalars, and Maxwell's equations reduces to

$$\vec{\nabla} \bullet \varepsilon \vec{E} = \rho$$

$$\vec{\nabla} \bullet \mu \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

# Maxwell's Equations Inside Matter (Cont'd)

- **In uniform (homogeneous) medium ε and μ are independent of position- Maxwell's equations reads as**

$$\vec{\nabla}\bullet\vec{D}=\rho_f \qquad \oint_S \vec{D}\bullet d\vec{a}=Q_{f\ enc}$$

$$\vec{\nabla}\bullet\vec{H}=0 \qquad \oint_S \vec{H}\bullet d\vec{a}=0$$

$$\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t} \qquad \oint_C \vec{E}\bullet d\vec{l}=-\mu\frac{d}{dt}\int_S \vec{H}\bullet d\vec{a}$$

$$\vec{\nabla}\times\vec{H}=\vec{J}_f+\varepsilon\frac{\partial\vec{E}}{\partial t} \qquad \oint_C \vec{H}\bullet d\vec{l}=I_{f\ enc}+\frac{d}{dt}\int_S \vec{D}\bullet d\vec{a}$$

**Generally, ε and μ can be rank-2 tensor (3X3 matrices) describing bi-refringent anisotropic materials.**

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

**Maxwell's equations in integral form:**

**Gauss' Law:**

$$\int_v \vec{\nabla}\cdot\vec{E}(\vec{r},t)\,d\tau' = \frac{1}{\varepsilon_0}\int_v \rho^E_{Tot}(\vec{r},t)\,d\tau' = \frac{1}{\varepsilon_0}\int_v \left(\rho^E_{free}(\vec{r},t)+\rho^E_{bound}(\vec{r},t)\right)d\tau'$$

$$= \oint_S \vec{E}(\vec{r},t)\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q^{enclosed}_{ToT}(t) = \frac{1}{\varepsilon_0}\left(Q^{enclosed}_{free}(t)+Q^{enclosed}_{bound}(t)\right)$$

$$\oint_S \vec{D}(\vec{r},t)\,d\vec{a} = Q^{enclosed}_{free}(t)$$

$$\oint_S \vec{P}(\vec{r},t)\cdot d\vec{a} \equiv -Q^{enclosed}_{bound}(t)$$

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Auxiliary Relation: $\vec{D}(\vec{r},t) = \varepsilon_o \vec{E}(\vec{r},t) + \vec{P}(\vec{r},t)$

$$\rho_{Bound}(\vec{r},t) \equiv -\vec{\nabla} \cdot \vec{P}(\vec{r},t) \qquad \sigma_{Bound}(\vec{r},t) \equiv \vec{P}(\vec{r},t) \cdot \hat{n}\Big|_{intf}$$

No Magnetic Monopoles: $\int_v \vec{\nabla} \cdot \vec{B}(\vec{r},t)\, d\tau' = \oint_S \vec{B}(\vec{r},t) \cdot d\vec{a} = 0$

Faraday's Law:

$$\int_S \vec{\nabla} \times \vec{E}(\vec{r},t) \cdot d\vec{a} = \oint_C \vec{E}(\vec{r},t) \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}(\vec{r},t)}{\partial t} \cdot d\vec{a} = -\frac{d}{dt}\left[\int_S \vec{B}(\vec{r},t) \cdot d\vec{a}\right]$$

$$\varepsilon mf \;\; \varepsilon(t) \equiv \oint_C \vec{E}(\vec{r},t) \cdot d\vec{\ell} = -\frac{d}{dt}\left[\int_S \vec{B}(\vec{r},t) \cdot d\vec{a}\right] = -\frac{d\Phi_M^{enclosed}(t)}{dt}$$

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Ampere's Law:

$$\int_S \vec{\nabla}\times\vec{B}(\vec{r},t)\bullet d\vec{a} = \oint_C \vec{B}(\vec{r},t)\bullet d\vec{\ell} = \mu_0\int_S\left(\vec{J}_{ToT}(\vec{r},t)+\vec{J}_D(\vec{r},t)\right)\bullet d\vec{a}$$

$$= \oint_C \vec{B}(\vec{r},t)\bullet d\vec{\ell} = \mu_0\left(I_{ToT}^{encl}(t)+I_D^{encl}(t)\right) = \mu_0\left(I_{free}^{encl}(t)+I_{bound}^{m\ encl}(t)+I_{P_{bound}}^{encl}(t)+I_D^{encl}(t)\right)$$

Auxiliary Relation: $$\vec{H}(\vec{r},t)=\frac{1}{\mu_0}\vec{B}(\vec{r},t)-\vec{\mathrm{M}}(\vec{r},t)$$

$$\vec{J}_{bound}^{m}(\vec{r},t)\equiv\vec{\nabla}\times\vec{\mathrm{M}}(\vec{r},t)$$

$$\vec{K}_{bound}^{m}(\vec{r},t)\equiv\vec{\mathrm{M}}(\vec{r},t)\times\hat{n}\Big|_{intf}$$

$$\vec{J}_{P_{bound}}(\vec{r},t)\equiv\frac{\partial\vec{P}(\vec{r},t)}{\partial t}$$

$$\rho_m^{Bound}(\vec{r},t)\equiv-\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r},t)$$

$$\sigma_m^{Bound}(\vec{r},t)\equiv\vec{\mathrm{M}}(\vec{r},t)\bullet\hat{n}\Big|_{intf}$$

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

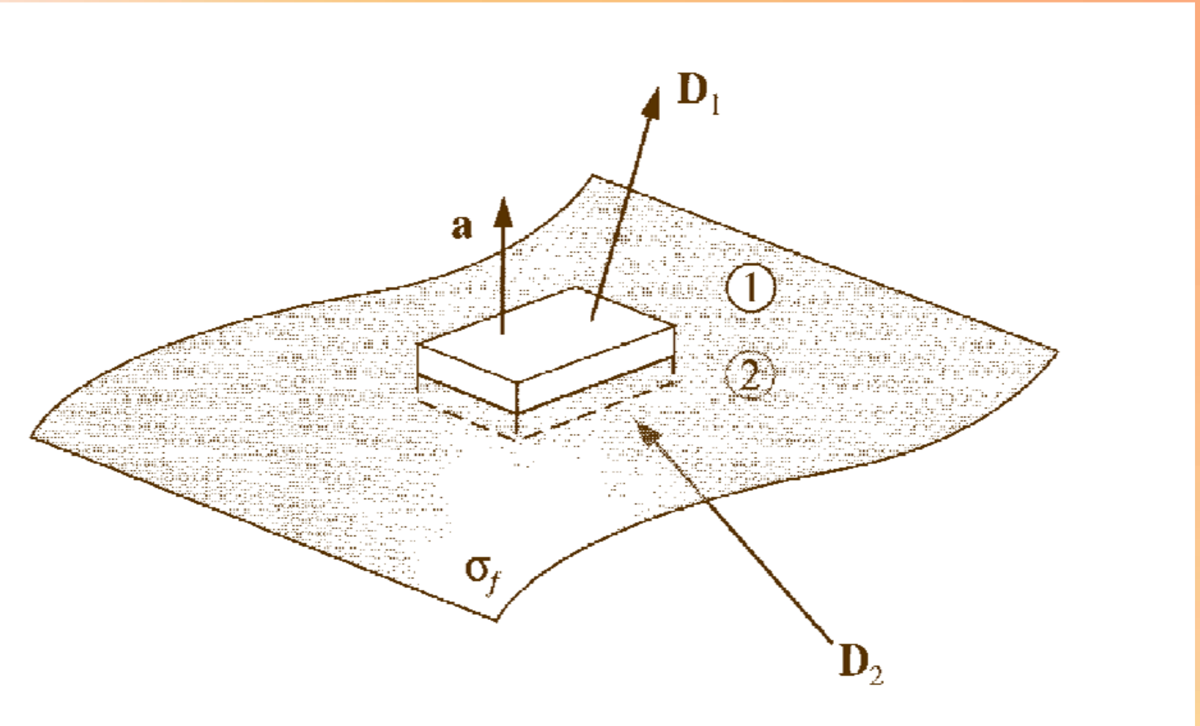
$$\int_S \vec{\nabla}\times\vec{H}(\vec{r},t)\cdot d\vec{a} = \oint_C \vec{H}(\vec{r},t)\cdot d\vec{\ell} = I_{free}^{enclosed}(t) + \int_S \frac{\partial\vec{D}(\vec{r},t)}{\partial t}\cdot d\vec{a} = I_{free}^{enclosed}(t) + \frac{d}{dt}\left[\int_S \vec{D}(\vec{r},t)\cdot d\vec{a}\right]$$

1) Apply the integral form of Gauss' Law at a dielectric interface/boundary using infinitesimally thin Gaussian pillbox extending slightly into dielectric material on either side of interface:

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}Q_{ToT}^{enclosed} = \frac{1}{\varepsilon_0}Q_{free}^{enclosed} + \frac{1}{\varepsilon_0}Q_{bound}^{enclosed} = \frac{1}{\varepsilon_0}\oint_S \sigma_{free}\,da + \frac{1}{\varepsilon_0}\oint_S \sigma_{bound}\,da$$

Gives: $$\underset{above}{E_2^{\perp}} - \underset{below}{E_1^{\perp}} = \frac{1}{\varepsilon_0}\sigma_{ToT} = \frac{1}{\varepsilon_0}\left(\sigma_{free} + \sigma_{bound}\right)$$ (at interface)

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter



# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

The positive direction is from medium 2 (below) to medium 1 (above)

$$\oint_S \vec{D} \cdot d\vec{a} = Q_{free}^{enclosed} = \oint_S \sigma_{free} da \quad \Rightarrow \quad \boxed{\underset{above}{D_2^{\perp}} - \underset{below}{D_1^{\perp}} = \sigma_{free}}$$ (at interface)

Likewise: $$\oint_S \vec{\mathrm{P}} \cdot d\vec{a} = Q_{bound}^{enclosed} = -\oint_S \sigma_{bound} da \Rightarrow \boxed{\underset{above}{P_2^{\perp}} - \underset{below}{P_1^{\perp}} = \sigma_{bound}}$$ (at interface)

$$\vec{E} \equiv -\vec{\nabla} V$$

Since: $$\boxed{\left(\frac{\partial V_2^{above}}{\partial n} - \frac{\partial V_1^{below}}{\partial n}\right)\Bigg|_{interface} = -\frac{1}{\varepsilon_0}\sigma_{ToT} = -\frac{1}{\varepsilon_0}\left(\sigma_{free} + \sigma_{bound}\right)}$$ (at interface)

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

$$\vec{D} = \varepsilon\vec{E} = -\varepsilon\vec{\nabla}V$$

Since: $$\left.\left(\varepsilon_2 \frac{\partial V_2^{above}}{\partial n} - \varepsilon_1 \frac{\partial V_1^{below}}{\partial n}\right)\right|_{interface} = -\sigma_{free}$$ (at interface)

Similarly, for $\int_v \vec{\nabla}\bullet\vec{B}\,d\tau' = \oint_S \vec{B}\bullet d\vec{a} = 0$ (no magnetic monopoles), then at an interface:

$$\vec{B}_2^{above}\bullet\vec{a} - \vec{B}_1^{above}\bullet\vec{a} = 0 \Rightarrow \boxed{B_{2\,above}^{\perp} - B_{1\,below}^{\perp} = 0} \text{ or: } \boxed{B_{2\,above}^{\perp} = B_{1\,below}^{\perp}}$$ (at interface)

Since:

$$\vec{H} = \left(\frac{1}{\mu_o}\right)\vec{B} - \vec{M} \qquad \underline{\text{Then}}: \qquad \vec{B} = \mu_o\left(\vec{H} + \vec{M}\right)$$

$$\oint_S \vec{B}\bullet d\vec{a} = \mu_o \oint_S \left(\vec{H} + \vec{M}\right)\bullet d\vec{a} = 0 \qquad \underline{\text{or}}: \qquad \oint_S \vec{H}\bullet d\vec{a} = -\oint_S \vec{M}\bullet d\vec{a}$$

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Then: $\vec{H}_2^{above}\cdot\vec{a} - \vec{H}_1^{below}\cdot\vec{a} = -\left(\vec{M}_2^{above}\cdot\vec{a} - \vec{M}_1^{below}\cdot\vec{a}\right)$ (at interface)

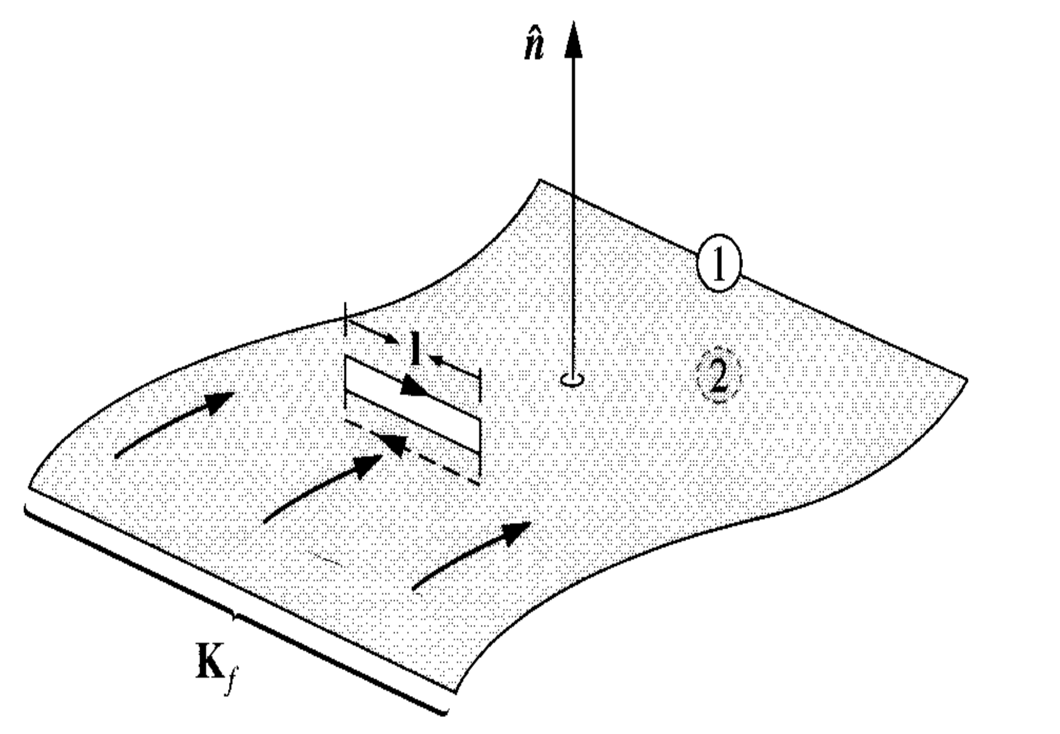
Or: $$\boxed{\left(H^{\perp}_{\underset{above}{2}} - H^{\perp}_{\underset{below}{1}}\right) = -\left(M^{\perp}_{\underset{above}{2}} - M^{\perp}_{\underset{below}{1}}\right) = -\sigma^{bound}_{magnetic}}$$ (at interface)

Effective bound magnetic charge at interface

3) For Faraday's Law: EMF, $\varepsilon = \oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\left(\oint_S \vec{B}\cdot d\vec{a}\right) = -\frac{d\Phi_m}{dt}$

At interface between two different media, taking a closed contour C of width l extending slightly into the material on either side of interface.

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter



# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

$$\vec{E}_2^{above}\bullet\vec{\ell} - \vec{E}_1^{below}\bullet\vec{\ell} = -\frac{d}{dt}\oint_S \vec{B}\bullet d\vec{a} = 0$$

(in limit area of contour loop → 0, magnetic flux enclosed → 0)

$$\boxed{\underset{above}{E_2^{\parallel}} - \underset{below}{E_1^{\parallel}} = 0}$$ (<u>at interface</u>) <u>or</u>: $$\boxed{\underset{above}{E_2^{\parallel}} = \underset{below}{E_1^{\parallel}}}$$ (<u>at interface</u>)

<u>Since</u>: $\vec{D} = \varepsilon_o\vec{E} + \vec{P}$ And: $\varepsilon_o\vec{E} = \vec{D} - \vec{P}$

<u>Thus</u>: $$\left(\vec{E}_2^{above}\bullet\vec{\ell} - \vec{E}_1^{below}\bullet\vec{\ell}\right) = \left(\vec{D}_2^{above}\bullet\vec{\ell} - \vec{D}_1^{below}\bullet\vec{\ell}\right) - \left(\vec{P}_2^{above}\bullet\vec{\ell} - \vec{P}_1^{below}\bullet\vec{\ell}\right) = 0$$

In limit area of contour loop → 0 magnetic flux enclosed → 0

$$\Rightarrow \boxed{\left(\underset{above}{\vec{D}_2^{\parallel}} - \underset{below}{\vec{D}_1^{\parallel}}\right) = \left(\underset{above}{\vec{P}_2^{\parallel}} - \underset{below}{\vec{P}_1^{\parallel}}\right)}$$ (<u>at interface</u>)

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

4) Finally, for Ampere's Law: $\oint_C \vec{B} \bullet d\vec{\ell} = \mu_o \left( I_{TOT}^{encl} + I_D^{encl} \right)$

$$\vec{B}_2^{above} \bullet \vec{\ell} - \vec{B}_1^{below} \bullet \vec{\ell} = \mu_o I_{TOT}^{encl} + \mu_o I_D^{encl}$$

$$I_D^{encl} = \int_S \vec{J}_D \bullet d\vec{a} = \varepsilon_o \int_S \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a}$$

$$I_{TOT}^{encl} = I_{free}^{encl} + I_{bound}^{encl} + I_{P_{bound}}^{encl}$$

$$I_{P_{bound}}^{encl} = \int_S \vec{J}_{P_{bound}} \bullet d\vec{a} = \int_S \frac{\partial \vec{\mathrm{P}}}{\partial t} \bullet d\vec{a}$$

$$I_{bound}^{encl} = \int_S \vec{J}_m^{bound} \bullet d\vec{a} = \int_S \vec{\nabla} \times \vec{\mathrm{M}} \bullet d\vec{a}$$

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Where $I_{TOT}^{encl}$ = TOTAL current (free + bound + polarization) passing through enclosing Amperian loop contour C

No volume current density $\vec{J}_{TOT}, \vec{J}_{free}, \vec{J}_{bound}^{m}$ or $\vec{J}_{p}$ contributes to $I_{TOT}^{encl}$ in the limit area of contour loop $\rightarrow 0$, however a surface current $\vec{K}_{TOT}, \vec{K}_{free}, \vec{K}_{bound}^{m} = \vec{M} \times \hat{n}$ can contribute!

In the limit that the enclosing Amperian loop contour C shrinks to zero height above/below interface- the enclosed area of loop contour → 0,

Then: $$I_{D}^{encl} = \varepsilon_o \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \varepsilon_o \frac{d}{dt}\left[\int_S \vec{E} \cdot d\vec{a}\right] = \varepsilon \frac{d\Phi_E}{dt} \rightarrow 0$$

($\Phi_E \equiv \int_S \vec{E} \cdot d\vec{a}$ = enclosed flux of electric field lines)

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Similarly:

$$I^{encl}_{P_{bound}} = \int_S \frac{\partial \vec{P}}{\partial t} \cdot d\vec{a} = \frac{d}{dt}\left[\int_S \vec{P} \cdot d\vec{a}\right] = \frac{d\Phi_P}{dt} \rightarrow 0$$

$$\left(\Phi_P \equiv \int_S \vec{P} \cdot d\vec{a} = \text{enclosed flux of electric polarization field lines}\right)$$

If $\hat{n}$ is unit normal/perpendicular to interface, note that $\left(\hat{n} \times \vec{\ell}\right)$ is normal/perpendicular to plane of the Amperian loop contour.

$$I^{encl}_{TOT} = \vec{K}_{TOT} \cdot \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}_{TOT} \times \hat{n}\right) \cdot \vec{\ell}$$

$$I^{encl}_{free} = \vec{K}_{free} \cdot \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}_{free} \times \hat{n}\right) \cdot \vec{\ell}$$

$$I^{encl}_{bound} = \vec{K}_{bound} \cdot \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}^{m}_{bound} \times \hat{n}\right) \cdot \vec{\ell}$$

Using: $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right)$

$= \vec{C} \cdot \left(\vec{A} \times \vec{B}\right)$

$\left\{= \left(\vec{A} \times \vec{B}\right) \cdot \vec{C}\right\}$

$$I_{TOT} = I_{free} + I_{bound} \qquad \vec{K}_{TOT} = \vec{K}_{free} + \vec{K}_{bound}$$

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

In the limit that the enclosing Amperian loop contour C (of width l) shrinks to zero height above/below interface, causing area of enclosed loop contour → 0, then:

$$\vec{B}_2^{above} \bullet \vec{\ell} - \vec{B}_1^{below} \bullet \vec{\ell} = \mu_o I_{TOT}^{encl} + \overbrace{\cancel{\mu_o I_D^{encl}}}^{=0} = \mu_o I_{TOT}^{encl} = \left(\vec{K}_{TOT} \times \hat{n}\right) \bullet \vec{\ell}$$

$$\boxed{\underset{above}{B_2^{\parallel}} - \underset{below}{B_1^{\parallel}} = \mu_o \vec{K}_{TOT} \times \hat{n} = \mu_o \left(\vec{K}_{free} + \vec{K}_{bound}^{m}\right) \times \hat{n}} \quad \text{(\underline{at interface})}$$

Since: $\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}$ and: $\frac{1}{\mu_o}\vec{B} = \vec{H} + \vec{M}$ then:

$$\boxed{\frac{1}{\mu_o}\left(\vec{B}_2^{above} \bullet \vec{\ell} - \vec{B}_1^{below} \bullet \vec{\ell}\right) = \left(\vec{H}_2^{above} \bullet \vec{\ell} - \vec{H}_1^{below} \bullet \vec{\ell}\right) + \left(\vec{M}_2^{above} \bullet \vec{\ell} - \vec{M}_1^{below} \bullet \vec{\ell}\right) = \left[\left(\vec{K}_{free} \times \hat{n}\right) + \left(\vec{K}_{bound} \times \hat{n}\right)\right]}$$

(at interface)

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

We also see that: $$\underset{above}{H_2^{\parallel}} - \underset{below}{H_1^{\parallel}} = \vec{K}_{free} \times \hat{n}$$ (at interface)

and: $$\underset{above}{M_2^{\parallel}} - \underset{below}{M_1^{\parallel}} = \vec{K}_{bound}^{m} \times \hat{n}$$ (at interface)

∥- components of ***B*** are discontinuous at interface by $\mu_o \vec{K}_{TOT} \times \hat{n}$

∥- components of ***H*** are discontinuous at interface by $\vec{K}_{free} \times \hat{n}$

∥- components of **M** are discontinuous at interface by $\vec{K}_{bound}^{m} \times \hat{n}$

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

If $\vec{B} = \vec{\nabla} \times \vec{A}$

where *A* is the magnetic vector potential - then:

$$\left(\frac{1}{\mu_0}\right)\left[B_{\substack{2\\above}}^{\parallel} - B_{\substack{1\\below}}^{\parallel}\right] = \vec{K}_{TOT} \times \hat{n}$$ (at interface) is equivalent to:

$$\left(\frac{1}{\mu_o}\right)\left(\frac{\partial \vec{A}_2^{above}}{\partial n} - \frac{\partial \vec{A}_1^{below}}{\partial n}\right)\Bigg|_{interface} = -\vec{K}_{TOT}$$ (at interface)

# Maxwell's Equations and Boundary Conditions at Interfaces in Matter

For linear magnetic media:

$$\vec{B} = \mu \vec{H} \quad \text{or:} \quad \vec{H} = \frac{1}{\mu}\vec{B}$$

$$\boxed{\left[ H_{\underset{above}{2}}^{\parallel} - H_{\underset{below}{1}}^{\parallel} \right] = \vec{K}_{free} \times \hat{n}}$$

(<u>at interface</u>) is <u>equivalent</u> to:

$$\boxed{\left(\frac{1}{\mu_{\underset{above}{2}}}\right)\frac{\partial \vec{A}_2^{above}}{\partial n}\bigg|_{\text{interface}} - \left(\frac{1}{\mu_{\underset{below}{1}}}\right)\frac{\partial \vec{A}_1^{below}}{\partial n}\bigg|_{\text{interface}} = -\vec{K}_{free}}$$

(<u>at interface</u>)

# Potential Formulation of Electrodynamics 1

- In electrostatic

$$\vec{\nabla}\times\vec{E}(\vec{r},t)=0$$

$$\vec{E}(\vec{r},t)=-\vec{\nabla}V(\vec{r},t)$$

In electrodynamics

$$\vec{\nabla}\times\vec{E}(\vec{r},t)\neq 0$$

But

$$\vec{\nabla}\bullet\vec{B}(\vec{r},t)=0$$

$$\vec{B}(\vec{r},t)=\vec{\nabla}\times\vec{A}(\vec{r},t)$$

Putting this in Faraday's Law

$$\vec{\nabla}\times\vec{E}(\vec{r},t)=-\vec{\nabla}\times\left(\frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right)$$

$$\vec{\nabla}\times\left[\vec{E}(\vec{r},t)+\frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right]=0$$

$$\Rightarrow \left[\vec{E}(\vec{r},t)+\frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right]\equiv-\vec{\nabla}V(\vec{r},t)$$

$$\Rightarrow \vec{E}(\vec{r},t)=-\vec{\nabla}V(\vec{r},t)-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}$$

# Potential Formulation of Electrodynamics 2

$$\vec{\nabla}\bullet\vec{B}(\vec{r},t)=\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{A}(\vec{r},t)\right)$$

and from

$$\vec{\nabla}\times\vec{E}(\vec{r},t)=-\vec{\nabla}\times\vec{\nabla}V(\vec{r},t)-\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}(\vec{r},t)$$

$$\vec{\nabla}\times\vec{E}(\vec{r},t)=0-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}$$

Explain Maxwell's ii and iii Equations

# Potential Formulation of Electrodynamics 3

Now consider Maxwell's i and iv equations

As $$\vec{\nabla}\bullet\vec{E}(\vec{r},t)=\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$ Gauss's Law

$$\vec{\nabla}\bullet\left[-\vec{\nabla}V(\vec{r},t)-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right]=\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$

$$\nabla^2V(\vec{r},t)+\frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{A}(\vec{r},t)\right)=-\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$

This replaces Poisson's Equation in electrodynamics

# Potential Formulation of Electrodynamics 4

Now consider Ampere's Law:

$$\vec{\nabla} \times \vec{B}(\vec{r},t) = \mu_o \vec{J}_{tot}(\vec{r},t) + \mu_o \varepsilon_o \frac{\partial \vec{E}(\vec{r},t)}{\partial t}$$

with: $$\vec{E}(\vec{r},t) = -\vec{\nabla} V(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$$

and: $$\vec{B}(\vec{r},t) = \vec{\nabla} \times \vec{A}(\vec{r},t)$$

$$\vec{\nabla} \times \left( \vec{\nabla} \times \vec{A}(\vec{r},t) \right) = \mu_o \vec{J}_{tot}(\vec{r},t) - \mu_o \varepsilon_o \vec{\nabla} \left( \frac{\partial V(\vec{r},t)}{\partial t} \right) - \mu_o \varepsilon_o \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2}$$

# Potential Formulation of Electrodynamics 5

Using vector identity:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)\equiv\vec{\nabla}\left(\vec{\nabla}\bullet\vec{A}\right)-\nabla^2\vec{A}$$

Re-arranging:

$$\left(\nabla^2\vec{A}(\vec{r},t)-\mu_o\varepsilon_o\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2}\right)-\vec{\nabla}\left(\vec{\nabla}\bullet\vec{A}(\vec{r},t)+\mu_o\varepsilon_o\frac{\partial V(\vec{r},t)}{\partial t}\right)=-\mu_o\vec{J}_{tot}(\vec{r},t)$$

These equation carry all information in Maxwell's Equations

# Potential Formulation of Electrodynamics 6

$$\nabla^2 V(\vec{r},t) + \frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{A}(\vec{r},t)\right) = -\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$

$$\left(\nabla^2\vec{A}(\vec{r},t) - \mu_o\varepsilon_o\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\bullet\vec{A}(\vec{r},t) + \mu_o\varepsilon_o\frac{\partial V(\vec{r},t)}{\partial t}\right) = -\mu_o\vec{J}_{tot}(\vec{r},t)$$

Four Maxwell's equations reduced to two equations using potential formulation.
Potentials V and A are not uniquely defined by above equations.

# Gauge Transformations

Suppose we have e.g. two sets of potentials

$$\left\{V(\vec{r},t),\vec{A}(\vec{r},t)\right\} \text{ and } \left\{V'(\vec{r},t),\vec{A}'(\vec{r},t)\right\}$$

That correspond to the same physical fields

$$\vec{E}(\vec{r},t) \quad \text{and} \quad \vec{B}(\vec{r},t)$$

These two sets of potentials must be related to each other by:

$$\vec{A}'(\vec{r},t)=\vec{A}(\vec{r},t)+\vec{\alpha}(\vec{r},t)$$

and

$$V'(\vec{r},t)=V(\vec{r},t)+\beta(\vec{r},t)$$

Because:

$$\vec{B}(\vec{r},t)=\vec{\nabla}\times\vec{A}(\vec{r},t)=\vec{\nabla}\times\vec{A}'(\vec{r},t)=\vec{\nabla}\times\left(\vec{A}(\vec{r},t)+\vec{\alpha}(\vec{r},t)\right)$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{\alpha}(\vec{r},t)\equiv 0$$

# Gauge Transformations

But if: $\left(\vec{\nabla}\times\vec{\alpha}(\vec{r},t)\right)=0$, then since $\left(\vec{\nabla}\times\vec{\nabla}f(\vec{r},t)\right)\equiv 0$

we can always write $\vec{\alpha}(\vec{r},t)\equiv\vec{\nabla}\lambda(\vec{r},t)$

And if:

$$\vec{E}(\vec{r},t)=-\vec{\nabla}V(\vec{r},t)-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}$$

$$=-\vec{\nabla}V'(\vec{r},t)-\frac{\partial\vec{A}'(\vec{r},t)}{\partial t}=-\vec{\nabla}V(\vec{r},t)-\vec{\nabla}\beta(\vec{r},t)-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}-\frac{\partial\vec{\alpha}(\vec{r},t)}{\partial t}$$

Then we see that $\vec{\nabla}\beta(\vec{r},t)+\dfrac{\partial\vec{\alpha}(\vec{r},t)}{\partial t}=0$

# Gauge Transformations

But:

$$\vec{\alpha}(\vec{r},t) \equiv \vec{\nabla}\lambda(\vec{r},t) \Rightarrow \vec{\nabla}\beta(\vec{r},t) + \frac{\partial\left(\vec{\nabla}\lambda(\vec{r},t)\right)}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla}\left(\beta(\vec{r},t) + \frac{\partial\lambda(\vec{r},t)}{\partial t}\right) = 0$$

which must hold for arbitrary all space-time points $(\vec{r},t)$

$$\Rightarrow \beta(\vec{r},t) + \frac{\partial\lambda(\vec{r},t)}{\partial t} = 0$$

# Gauge Transformations

Note that

$$\vec{\nabla}\left(\beta(\vec{r},t)+\frac{\partial\lambda(\vec{r},t)}{\partial t}\right)=0$$

can also be satisfied if

$$\left(\beta(\vec{r},t)+\frac{\partial\lambda(\vec{r},t)}{\partial t}\right)=\kappa(t)$$

i.e. the scalar function κ (t )depends only on time, t.

Thus we see that:

$$\beta(\vec{r},t)=-\frac{\partial\lambda(\vec{r},t)}{\partial t}+\kappa(t)$$

But we can always "absorb" κ (t )

$$\lambda'(\vec{r},t)=\lambda(\vec{r},t)+\int_{t'=0}^{t'=t}\kappa(t)\,dt'$$

# Gauge Transformations

Note also that since the scalar function κ (t ) depends only on time t, this will not affect the gradient of $\vec{\nabla}\lambda(\vec{r},t)$ in any way, and hence $\vec{\alpha}(\vec{r},t)=\vec{\nabla}\lambda(\vec{r},t)$ is completely unaffected by this!

Thus: $$\vec{A}'(\vec{r},t)=\vec{A}(\vec{r},t)+\alpha(\vec{r},t)=\vec{A}(\vec{r},t)+\vec{\nabla}\lambda(\vec{r},t)$$

or $$\vec{\nabla}\lambda(\vec{r},t)=\vec{A}'(\vec{r},t)-\vec{A}(\vec{r},t)\equiv\Delta\vec{A}(\vec{r},t)$$

And: $$V'(\vec{r},t)=V(\vec{r},t)-\frac{\partial\lambda(\vec{r},t)}{\partial t}$$

Such changes in V and A are called Gauge Transformations

or $$-\frac{\partial\lambda(\vec{r},t)}{\partial t}=V'(\vec{r},t)-V(\vec{r},t)\equiv\Delta V(\vec{r},t)$$

# Coulomb's and Lorentz Gauges

Coulomb Gauge $\vec{\nabla}\bullet\vec{A}(\vec{r},t)=0$

Using this we get $\nabla^2 V(\vec{r},t) = -\dfrac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$

It is Poisson's equation, setting $V(\vec{r}=\infty,t)=0$

we get $$V(\vec{r},t)=\frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho_{tot}(\vec{r}',t)}{\boldsymbol{r}}d\tau'$$

$$\vec{\boldsymbol{r}} \equiv \vec{r}-\vec{r}'$$

$$\boldsymbol{r} = |\vec{\boldsymbol{r}}| = \sqrt{r^2 - r'^2}$$

Scalar potential is easy to calculate in Coulomb's gauge but vector potential is difficult to calculate

# Coulomb's Gauge

The differential equations for V and A in Coulombs gauge reads

$$\nabla^2 V(\vec{r},t) = -\frac{1}{\varepsilon_o}\rho_{TOT}(\vec{r},t)$$

$$\nabla^2 \vec{A}(\vec{r},t) - \mu_o\varepsilon_o \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = -\mu_o \vec{J}_{tot}(\vec{r},t) + \mu_o\varepsilon_o \vec{\nabla}\left(\frac{\partial V(\vec{r},t)}{\partial t}\right)$$

# Lorentz Gauge

The Lorentz gauge:

$$\vec{\nabla}\bullet\vec{A}(\vec{r},t) = -\mu_o\varepsilon_o \frac{\partial V(\vec{r},t)}{\partial t}$$

This is design to eliminate the middle term in eqn. for A

$$\nabla^2\vec{A}(\vec{r},t) - \mu_o\varepsilon_o \frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2} = -\mu_o\vec{J}_{tot}(\vec{r},t)$$

And equation for V will become

$$\left(\nabla^2 V(\vec{r},t) - \varepsilon_o\mu_o \frac{\partial^2 V(\vec{r},t)}{\partial t^2}\right) = -\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$

# Lorentz Gauge

The Lorentz gauge treats V and A on equal footing. The same differential operator

$$\Box^2 \equiv \nabla^2 - \varepsilon_o \mu_o \frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

called the d'Alembertian

$$\Box^2 \vec{A}(\vec{r},t) = -\mu_o \vec{J}_{tot}(\vec{r},t)$$

and

$$\Box^2 V(\vec{r},t) = -\frac{1}{\varepsilon_o}\rho_{tot}(\vec{r},t)$$

# ELECTROMAGNETIC WAVES IN VACUUM

Imrana Ashraf Zahid

Quaid-i-Azam University Islamabad Pakistan

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# ELECTROMAGNETIC WAVES IN VACUUM

## THE WAVE EQUATION

- In regions of free space (i.e. the vacuum) - where no electric charges - no electric currents and no matter of any kind are present - Maxwell's equations (in differential form) are:

1) $\vec{\nabla}\cdot\vec{E}(\vec{r},t)=0$

2) $\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0$

3) $\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}$

4) $\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu_o\varepsilon_o\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}=\dfrac{1}{c^2}\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}$

$\left(c^2=1/\varepsilon_o\mu_o\right)$

Set of coupled first-order partial differential equations

# ELECTROMAGNETIC WAVES IN VACUUM . . .

- We can de-couple Maxwell's equations -by applying the curl operator to equations 3) and 4):

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$=\vec{\nabla}\left(\overbrace{\vec{\nabla}\bullet\vec{E}}^{=0}\right)-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

$$=-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)$$

$$=\boxed{\nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=\vec{\nabla}\times\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)$$

$$=\vec{\nabla}\left(\overbrace{\vec{\nabla}\bullet\vec{B}}^{=0}\right)-\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right)$$

$$=-\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$=\boxed{\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}}$$

# ELECTROMAGNETIC WAVES IN VACUUM . . .

- These are three-dimensional de-coupled wave equations.
- Have exactly the same structure – both are linear, homogeneous, 2nd order differential equations.
- Remember that each of the above equations is explicitly dependent on space and time,

$$i.e.\ \vec{E} = \vec{E}(\vec{r},t) \text{ and } \vec{B} = \vec{B}(\vec{r},t):$$

$$\nabla^2 \vec{E}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = 0$$

$$\nabla^2 \vec{B}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2} = 0$$

# ELECTROMAGNETIC WAVES IN VACUUM . . .

- Maxwell's equations implies that empty space – the vacuum ( not empty at the microscopic scale) – supports the propagation of (macroscopic) electromagnetic waves - which propagate at the speed of light (in vacuum):

$$c = 1/\sqrt{\varepsilon_o \mu_o} = 3\times 10^8 \text{ m/s}$$

# MONOCHROMATIC EM PLANE WAVES

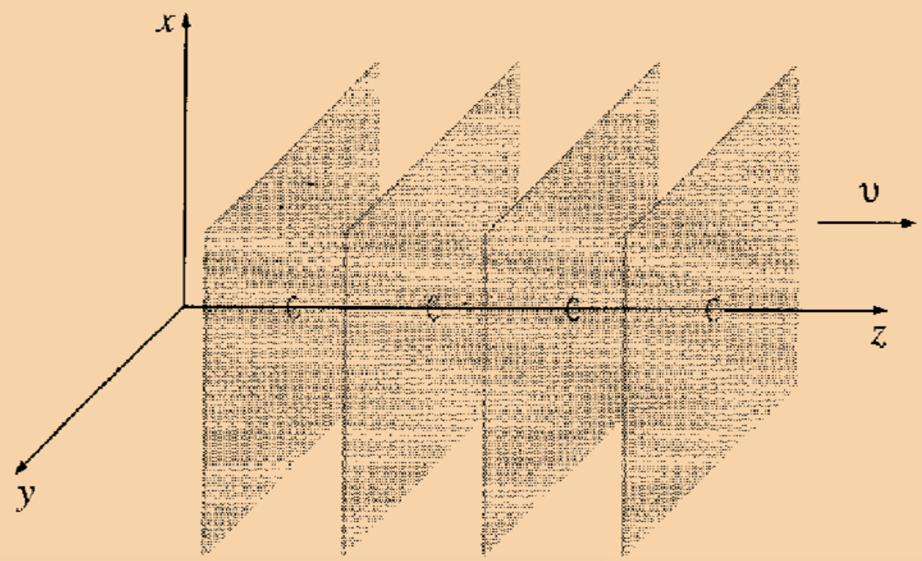
Monochromatic EM plane waves propagating in free space are waves consisting of a single frequency f , wavelength $\lambda = c\, f$ , angular frequency $\omega = 2\pi f$ and wave-number $k = 2\pi / \lambda$ - propagate with speed $c = f\lambda = \omega\, k$.

In the visible region of the EM spectrum [*{~380 nm (violet)* $\leq \lambda \leq$ *~780 nm (red)}-* EM light waves of a given frequency or wavelength are perceived by the human eye as having a specific- single colour.

Single- frequency EM waves are called mono-chromatic.

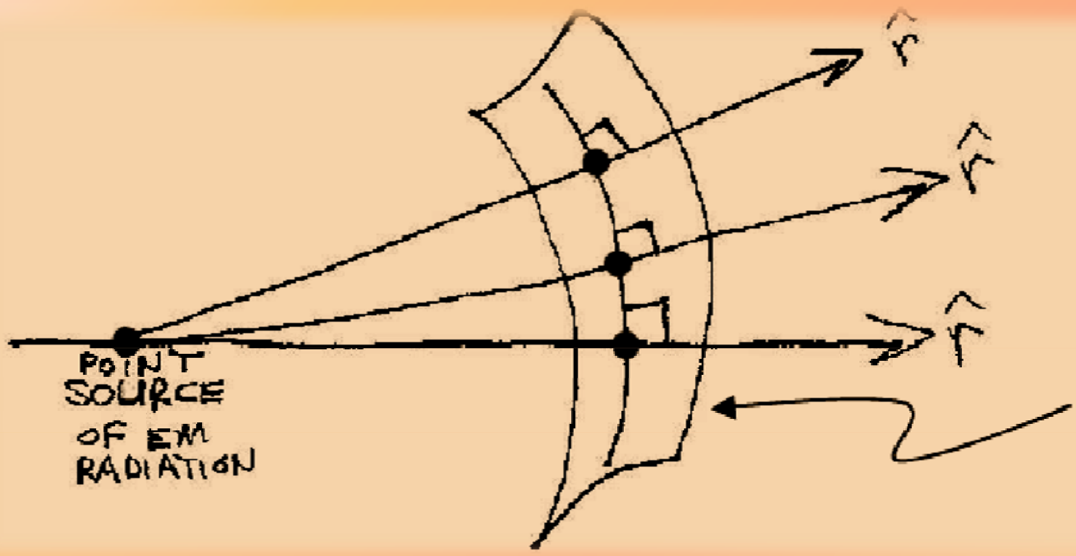
# MONOCHROMATIC EM PLANE WAVES

EM waves that propagate e.g. in the $+\hat{z}$ direction but which additionally have no explicit $x$- or $y$-dependence are known as plane waves- for a given time t the wave fronts of the EM wave lie in a plane- ⊥ to the $\hat{z}$ *-axis*,



# MONOCHROMATIC EM PLANE WAVES

There also exist spherical EM waves – emitted from a point source – the wave-fronts associated with these EM waves are spherical - and thus do not lie in a plane ⊥ to the direction of propagation of the EM wave




Portion of a spherical wavefront associated with a spherical wave

# MONOCHROMATIC EM PLANE WAVES

If the point source is infinitely far away from observer- then a spherical wave → plane wave in this limit, (the radius of curvature → ∞); a spherical surface becomes planar as $R_C \to \infty$.

Criterion for a plane wave: $\lambda \ll R_c$

Monochromatic plane waves associated with $\vec{E}$ and $\vec{B}$

$$\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_o e^{i(kz-\omega t)}$$

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{i(kz-\omega t)}$$

# MONOCHROMATIC EM PLANE WAVES

$$\vec{\tilde{E}}(z,t) = \vec{\tilde{E}}_o e^{\underbrace{i(kz-\omega t)}}$$

$$\vec{\tilde{B}}(z,t) = \vec{\tilde{B}}_o e^{\underbrace{i(kz-\omega t)}}$$

Propagating in $+\hat{z}$ direction

Propagating in $+\hat{z}$ direction

*n.b.* complex vectors:

*e.g.* $\vec{\tilde{E}}_o = E_o e^{i\delta}\,\hat{x}$

*n.b.* complex vectors:

*e.g.* $\vec{\tilde{B}}_o = B_o e^{i\delta}\,\hat{y}$

*n.b.* The *real, physical* (instantaneous) fields are:

$$\left\{ \begin{array}{l} \vec{E}(\vec{r},t) \equiv \mathrm{Re}\left(\vec{\tilde{E}}(\vec{r},t)\right) \\ \vec{B}(\vec{r},t) \equiv \mathrm{Re}\left(\vec{\tilde{B}}(\vec{r},t)\right) \end{array} \right\}$$

Very important to keep in mind!!

# MONOCHROMATIC EM PLANE WAVES

Maxwell's equations for free space impose additional constraints on $\tilde{\vec{E}}_o$ and $\tilde{\vec{B}}_o$

$$\text{Since: } \vec{\nabla}\bullet\vec{E} = 0 = \mathrm{Re}\left(\vec{\nabla}\bullet\tilde{\vec{E}}\right) = 0 \qquad \text{and: } \vec{\nabla}\bullet\vec{B} = 0 = \mathrm{Re}\left(\vec{\nabla}\bullet\tilde{\vec{B}}\right) = 0$$

These two relations can only be satisfied

$$\forall(\vec{r},t) \text{ if } \vec{\nabla}\bullet\tilde{\vec{E}} = 0 \;\; \forall(\vec{r},t) \text{ and } \vec{\nabla}\bullet\tilde{\vec{B}} = 0 \;\; \forall(\vec{r},t)$$

In Cartesian coordinates: $\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

Thus: $\left(\vec{\nabla}\bullet\tilde{\vec{E}}\right) = 0$ and $\left(\vec{\nabla}\bullet\tilde{\vec{B}}\right) = 0$ become:

# MONOCHROMATIC EM PLANE WAVES

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(\tilde{\vec{E}}_o e^{i(kz-\omega t)}\right)=0 \quad \text{and} \quad \left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(\tilde{\vec{B}}_o e^{i(kz-\omega t)}\right)=0$$

Now suppose we do allow:

$$\tilde{\vec{E}}_o=\underbrace{\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)}_{\text{polarization in } \hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{E}_o e^{i\delta}$$

$$\tilde{\vec{B}}_o=\underbrace{\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)}_{\text{polarization in } \hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{B}_o e^{i\delta}$$

Then

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0$$

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0$$

# MONOCHROMATIC EM PLANE WAVES

$E_{ox}$, $E_{oy}$, $E_{oz}$ = Amplitudes (constants) of the electric field components in x, y, z directions respectively.

$B_{ox}$, $B_{oy}$, $B_{oz}$ =Amplitudes (constants) of the magnetic field components in *x, y, z* directions respectively.

$$\frac{\partial}{\partial x}\hat{x}\bullet E_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial y}\hat{y}\bullet E_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial x}\hat{x}\bullet B_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial y}\hat{y}\bullet B_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial z}\left(e^{az}\right)=ae^{az}$$

# MONOCHROMATIC EM PLANE WAVES . . .

$$\frac{\partial}{\partial z}\hat{z}\bullet E_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta} = ikE_{oz}e^{i(kz-\omega t)}e^{i\delta} = 0 \quad \Leftarrow \text{ true } \textbf{\textit{iff}} \ \boxed{E_{oz} \equiv 0} \ !!!$$

$$\frac{\partial}{\partial z}\hat{z}\bullet B_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta} = ik\mathrm{E}_{oz}e^{i(kz-\omega t)}e^{i\delta} = 0 \quad \Leftarrow \text{ true } \textbf{\textit{iff}} \ \boxed{B_{oz} \equiv 0} \ !!!$$

- Maxwell's equations additionally impose the restriction that an electromagnetic plane wave cannot have any component of **E** or **B** ‖ to (or anti- ‖ to) the propagation direction (in this case here, the z -direction)
- Another way of stating this is that an EM wave cannot have any longitudinal components of **E** and **B** (i.e. components of **E** and **B** lying along the propagation direction).

# MONOCHROMATIC EM PLANE WAVES . . .

- Thus, Maxwell's equations additionally tell us that an EM wave is a purely transverse wave (at least for propagation in free space) – the components of **E** and **B** must be ⊥ to propagation direction.
- The plane of polarization of an EM wave is defined (by convention) to be parallel to **E.**

# MONOCHROMATIC EM PLANE WAVES . . .

Maxwell's equations impose another restriction on the allowed form of E and B for an EM wave:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = \mathrm{Re}\left(\vec{\nabla} \times \vec{\tilde{E}}\right) = \mathrm{Re}\left(-\frac{\partial \vec{\tilde{B}}}{\partial t}\right)$$

and/or:

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mathrm{Re}\left(\vec{\nabla} \times \vec{\tilde{B}}\right) = \mathrm{Re}\left(\frac{1}{c^2}\frac{\partial \vec{\tilde{E}}}{\partial t}\right)$$

Can *only* be satisfied $\forall \; (\vec{r}, t)$ ***iff***:

$$\vec{\nabla} \times \vec{\tilde{E}} = -\frac{\partial \vec{\tilde{B}}}{\partial t}$$

and/or:

$$\vec{\nabla} \times \vec{\tilde{B}} = \frac{1}{c^2}\frac{\partial \vec{\tilde{E}}}{\partial t}$$

# MONOCHROMATIC EM PLANE WAVES . . .

$$\vec{\nabla}\times\tilde{\vec{E}} = \left(\overbrace{\frac{\partial \tilde{E}_z}{\partial y}}^{=0} - \frac{\partial \tilde{E}_y}{\partial z}\right)\hat{x} + \left(\frac{\partial \tilde{E}_x}{\partial z} - \overbrace{\frac{\partial \tilde{E}_y}{\partial x}}^{=0}\right)\hat{y} + \left(\overbrace{\frac{\partial \tilde{E}_y}{\partial x}}^{=0} - \overbrace{\frac{\partial \tilde{E}_x}{\partial y}}^{=0}\right)\hat{z} = -\frac{\partial \tilde{B}_x}{\partial t}\hat{x} - \frac{\partial \tilde{B}_y}{\partial t}\hat{y} - \overbrace{\frac{\partial \tilde{B}_z}{\partial t}}^{=0}\hat{z}$$

$$\vec{\nabla}\times\tilde{B} = \left(\overbrace{\frac{\partial \tilde{B}_z}{\partial y}}^{=0} - \frac{\partial \tilde{B}_y}{\partial z}\right)\hat{x} + \left(\frac{\partial \tilde{B}_x}{\partial z} - \overbrace{\frac{\partial \tilde{B}_y}{\partial x}}^{=0}\right)\hat{y} + \left(\overbrace{\frac{\partial \tilde{B}_y}{\partial x}}^{=0} - \overbrace{\frac{\partial \tilde{B}_x}{\partial y}}^{=0}\right)\hat{z} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t}\hat{x} + \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t}\hat{y} + \frac{1}{c^2}\overbrace{\frac{\partial \tilde{E}_z}{\partial t}}^{=0}\hat{z}$$

$$\tilde{\vec{E}} = \tilde{E}_x\hat{x} + \tilde{E}_y\hat{y} + \overbrace{\tilde{E}_z}^{=0}\hat{z} = \left(E_{ox}\hat{x} + E_{oy}\hat{y} + \overbrace{E_{oz}}^{=0}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}$$

$$\tilde{\vec{B}} = \tilde{B}_x\hat{x} + \tilde{B}_y\hat{y} + \overbrace{\tilde{B}_z}^{=0}\hat{z} = \left(B_{ox}\hat{x} + B_{oy}\hat{y} + \overbrace{B_{oz}}^{=0}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}$$

# MONOCHROMATIC EM PLANE WAVES . . .

$$\vec{\tilde{E}} = \tilde{E}_x \hat{x} + \tilde{E}_y \hat{y} = \left( E_{ox} \hat{x} + E_{oy} \hat{y} \right) e^{i(kz - \omega t)} e^{i\delta}$$

$$\vec{\tilde{B}} = \tilde{B}_x \hat{x} + \tilde{B}_y \hat{y} = \left( B_{ox} \hat{x} + B_{oy} \hat{y} \right) e^{i(kz - \omega t)} e^{i\delta}$$

$$\vec{\nabla} \times \vec{\tilde{E}} = -\frac{\partial \tilde{E}_y}{\partial z} \hat{x} + \frac{\partial \tilde{E}_x}{\partial z} \hat{y} = -\frac{\partial \tilde{B}_x}{\partial t} \hat{x} - \frac{\partial \tilde{B}_y}{\partial t} \hat{y}$$

$$\vec{\nabla} \times \vec{\tilde{B}} = -\frac{\partial \tilde{B}_y}{\partial z} \hat{x} + \frac{\partial \tilde{B}_x}{\partial z} \hat{y} = \frac{1}{c^2} \frac{\partial \tilde{E}_x}{\partial t} \hat{x} + \frac{1}{c^2} \frac{\partial \tilde{E}_y}{\partial t} \hat{y}$$

Can only be satisfied / can only be true ***iff*** the $\hat{x}$ and $\hat{y}$ relations are <u>separately</u> / <u>independently</u> satisfied $\forall \ (\vec{r}, t)$!

# MONOCHROMATIC EM PLANE WAVES . . .

$$\vec{\nabla}\times\vec{\tilde{E}}: \quad -\frac{\partial \tilde{E}_y}{\partial z}\hat{x} = -\frac{\partial \tilde{B}_x}{\partial t}\hat{x} \Rightarrow \frac{\partial \tilde{E}_y}{\partial z} = \frac{\partial \tilde{B}_x}{\partial t} \Rightarrow ikE_{oy} = -i\omega B_{ox} \qquad (1)$$

$$+\frac{\partial \tilde{E}_x}{\partial z}\hat{y} = -\frac{\partial \tilde{B}_y}{\partial t}\hat{y} \Rightarrow \frac{\partial \tilde{E}_x}{\partial z} = -\frac{\partial \tilde{B}_y}{\partial t} \Rightarrow ikE_{ox} = +i\omega B_{oy} \qquad (2)$$

$$\vec{\nabla}\times\tilde{B}: \quad -\frac{\partial \tilde{B}_y}{\partial z}\hat{x} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t}\hat{x} \Rightarrow -\frac{\partial \tilde{B}_y}{\partial z} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t} \Rightarrow -ikB_{oy} = -\frac{1}{c^2}i\omega E_{ox} \qquad (3)$$

$$+\frac{\partial \tilde{B}_x}{\partial z}\hat{y} = \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t}\hat{y} \Rightarrow \frac{\partial \tilde{B}_x}{\partial z} = \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t} \Rightarrow ikB_{ox} = -\frac{1}{c^2}i\omega E_{oy} \qquad (4)$$

From (1): $ik\tilde{E}_{oy} = -i\omega B_{ox} \Rightarrow E_{oy} = -\left(\frac{\omega}{k}\right)B_{ox}$ or: $B_{ox} = -\left(\frac{k}{\omega}\right)E_{oy}$

# MONOCHROMATIC EM PLANE WAVES . . .

From (2): $ik\tilde{E}_{ox} = +i\omega B_{oy}$ $\Rightarrow$ $E_{ox} = +\left(\frac{\omega}{k}\right) B_{oy}$ or: $B_{oy} = +\left(\frac{k}{\omega}\right) E_{ox}$

From (3): $-ikB_{oy} = -\frac{1}{c^2} i\omega E_{ox}$ $\Rightarrow$ $B_{oy} = +\frac{1}{c^2}\left(\frac{\omega}{k}\right) E_{ox}$

From (4): $ikB_{ox} = -\frac{1}{c^2} i\omega E_{oy}$ $\Rightarrow$ $B_{ox} = -\frac{1}{c^2}\left(\frac{\omega}{k}\right) E_{oy}$

$c = f\lambda = (2\pi f)\left(\frac{\lambda}{2\pi}\right) = \left(\frac{\omega}{k}\right)$ $\frac{1}{c} = \left(k/\omega\right)$ $\left(k = 2\pi/\lambda\right)$

# MONOCHROMATIC EM PLANE WAVES . . .

$\vec{\nabla}\times\tilde{\vec{E}}$ :

(1) $$B_{ox} = -\frac{1}{c}E_{oy}$$

(2) $$B_{oy} = +\frac{1}{c}E_{ox}$$

$\vec{\nabla}\times\tilde{\vec{B}}$ :

(3) $$B_{oy} = +\frac{1}{c}E_{ox}$$

(4) $$B_{ox} = -\frac{1}{c}E_{oy}$$

Maxwell's Equations also have some redundancy encrypted into them!

Actually we have only two independent relations:

$$B_{ox} = -\frac{1}{c}E_{oy} \quad \text{and} \quad B_{oy} = +\frac{1}{c}E_{ox}$$

But: $\hat{z}\times\hat{y} = -\hat{x}$ $\quad$ $\hat{z}\times\hat{x} = +\hat{y}$

# MONOCHROMATIC EM PLANE WAVES . . .

Very Useful Table:

| | |
|---|---|
| $\hat{x}\times\hat{y}=\hat{z}$ | $\hat{y}\times\hat{x}=-\hat{z}$ |
| $\hat{y}\times\hat{z}=\hat{x}$ | $\hat{z}\times\hat{y}=-\hat{x}$ |
| $\hat{z}\times\hat{x}=\hat{y}$ | $\hat{x}\times\hat{z}=-\hat{y}$ |

Two relations can be written compactly into one relation:

$$\vec{\tilde{B}}_o=\frac{1}{c}\left(\hat{z}\times\vec{\tilde{E}}_o\right)$$

Physically this relation states that E and B are:

- in phase with each other.
- mutually perpendicular to each other - $(\mathbf{E}\perp\mathbf{B})\perp \hat{z}$

# MONOCHROMATIC EM PLANE WAVES . . .

The **E** and **B** fields associated with this monochromatic plane EM wave are purely transverse { n.b. this is as also required by relativity at the microscopic level – for the extreme relativistic particles – the (massless) real photons travelling at the speed of light c that make up the macroscopic monochromatic plane EM wave.}

The real amplitudes of E and B are related to each other by:

$$B_o = \frac{1}{c} E_o$$

with

$$B_o = \sqrt{B_{ox}^2 + B_{oy}^2}$$

and

$$E_o = \sqrt{E_{ox}^2 + E_{oy}^2}$$

# Instantaneous Poynting's Vector for a linearly polarized *EM wave*

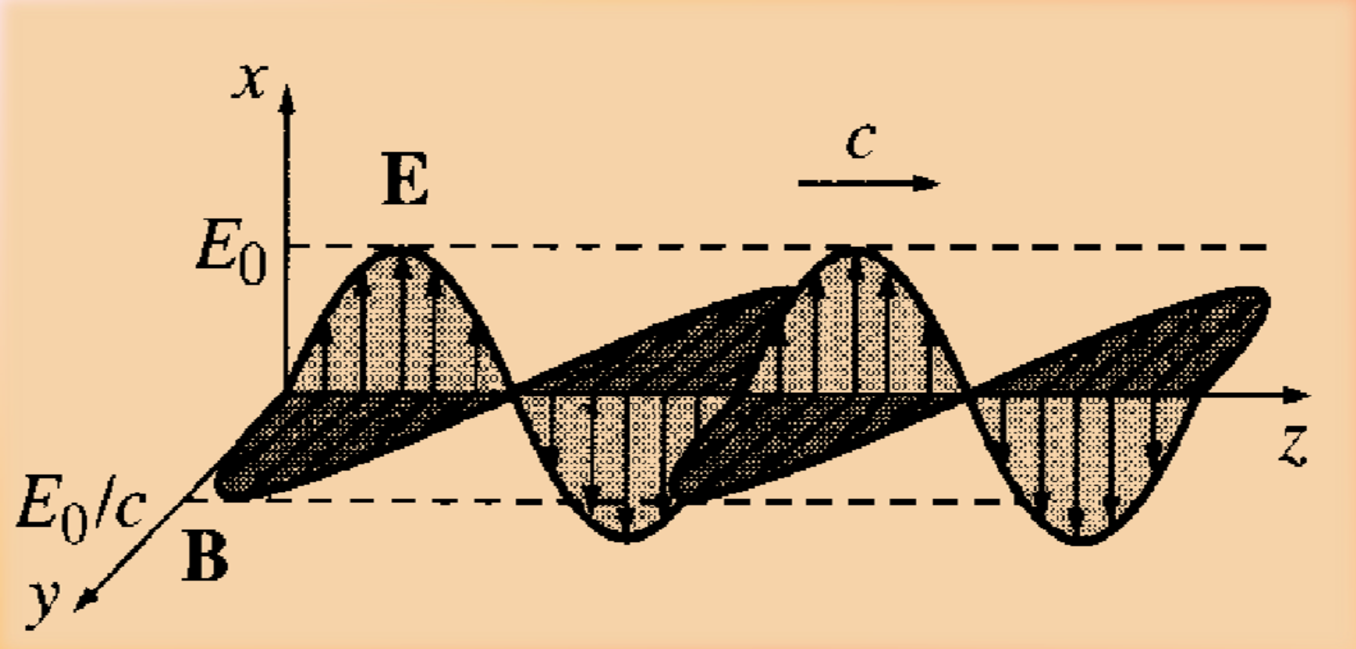
$$\vec{S}(z,t) = \frac{1}{\mu_o}\vec{E}(z,t)\times\vec{B}(z,t) = \frac{1}{\mu_o}\mathrm{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\mathrm{Re}\left\{\tilde{\vec{B}}(z,t)\right\}$$

$$\vec{S}(z,t) = \frac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\underbrace{(\hat{x}\times\hat{y})}_{=\hat{z}}$$

$$\vec{S}(z,t) = \frac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\,\hat{z} \qquad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

⇒EM Power flows in the direction of propagation of the EM wave (here, the +zˆ direction)

# Instantaneous Poynting's Vector for a linearly polarized *EM wave*




This is the paradigm for a monochromatic plane wave. The wave as a whole is said to be polarized in the x direction (by convention, we use the direction of E to specify the polarization of an electromagnetic wave).

# Instantaneous Energy & Linear Momentum & Angular Momentum in *EM Waves*

**Instantaneous Energy Density Associated with an *EM Wave:***

$$u_{EM}(\vec{r},t)=\frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t)+\frac{1}{\mu_o}B^2(\vec{r},t)\right)=u_{elect}(\vec{r},t)+u_{mag}(\vec{r},t)$$

where $$u_{elect}(\vec{r},t)=\frac{1}{2}\varepsilon_o E^2(\vec{r},t)$$

and $$u_{mag}(\vec{r},t)=\frac{1}{2\mu_o}B^2(\vec{r},t)=\frac{1}{2}\varepsilon_o E^2(\vec{r},t)$$

# Instantaneous Energy & Linear Momentum & Angular Momentum in *EM Waves*

But $B^2 = \frac{1}{c^2} E^2$ - EM waves in vacuum, and $\frac{1}{c^2} = \varepsilon_o \mu_o$

$$u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \frac{\varepsilon_o \mu_o}{\mu_o} E^2(\vec{r},t)\right) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \varepsilon_o E^2(\vec{r},t)\right)$$

$$u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2 \cos^2\left(\vec{k}\bullet\vec{r} - \omega t + \delta\right) \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

$u_{elect}(\vec{r},t) = u_{mag}(\vec{r},t)$ - EM waves propagating in the vacuum !!!!

# Instantaneous Poynting's Vector Associated with an *EM Wave*

$$\vec{S}(\vec{r},t) = \frac{1}{\mu_o}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t) = \frac{1}{\mu_o}\mathrm{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\mathrm{Re}\left\{\tilde{\vec{B}}(z,t)\right\} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

For a linearly polarized monochromatic plane EM wave propagating in the vacuum,

$$\vec{S}(\vec{r},t) = c\left(\frac{\varepsilon_o \mu_o}{\mu_o}\right)E_o^2\cos^2(kz-\omega t+\delta)\hat{z} = c\varepsilon_o E_o^2\cos^2(kz-\omega t+\delta)\hat{z}$$

But $u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2\cos^2(kz-\omega t+\delta)$

$$\vec{S}(\vec{r},t) = c u_{EM}(\vec{r},t)\hat{z}$$

# Instantaneous Poynting's Vector Associated with an *EM Wave*

The propagation velocity of energy $\vec{v}_{prop} = c\hat{z}$

Poynting's Vector = Energy Density * Propagation Velocity

$$\vec{S}(\vec{r},t) = u_{EM}(\vec{r},t)\vec{v}_{prop}$$

**Instantaneous Linear Momentum Density Associated with an EM Wave:**

$$\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

# Instantaneous Linear Momentum Density Associated with an *EM Wave*

For linearly polarized monochromatic plane EM waves propagating in the vacuum:

$$\vec{\wp}_{EM} = \frac{1}{c^{\not{2}}} \not{c}\, \varepsilon_o E_o^2 \cos^2(kz - \omega t + \delta)\,\hat{z} = \frac{1}{c} \underbrace{\varepsilon_o E_o^2 \cos^2(kz - \omega t + \delta)}_{=u_{EM}}\,\hat{z}$$

But: $$u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2 \cos^2(kz - \omega t + \delta)$$

$$\boxed{\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) = \frac{1}{c} u_{EM}(\vec{r},t)\,\hat{z}} \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

# Instantaneous Angular Momentum Density Associated with an *EM wave*

$$\vec{\ell}_{EM}(\vec{r},t) = \vec{r} \times \vec{\wp}_{EM}(\vec{r},t) \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

But: $$\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) = \frac{1}{c} u_{EM}(\vec{r},t)\hat{z} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

For an EM wave propagating in the +z^ direction:

$$\vec{\ell}_{EM}(\vec{r},t) = \frac{1}{c^2}\vec{r} \times \vec{S}(\vec{r},t) = \frac{1}{c} u_{EM}(\vec{r},t)(\vec{r} \times \hat{z}) \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

Depends on the choice of origin

# Instantaneous Power Associated with an *EM wave*

The instantaneous EM power flowing into/out of volume v with bounding surface S enclosing volume v (containing EM fields in the volume v) is:

$$P_{EM}(t) = \frac{\partial U_{EM}(t)}{\partial t} = \int_v \frac{\partial u_{EM}(\vec{r},t)}{\partial t} d\tau = -\oint_S \vec{S}(\vec{r},t) \cdot d\vec{a}$$

The instantaneous EM power crossing (imaginary) surface is:

$$P_{EM}(t) = -\int_S \vec{S}(\vec{r},t) \cdot d\vec{a}_\perp$$

The instantaneous total *E*M energy contained in volume v

$$U_{EM}(t) = \int_v u_{EM}(\vec{r},t)\, d\tau \quad \text{(Joules)}$$

# Instantaneous Angular Momentum Density Associated with an *EM wave*

The instantaneous total EM linear momentum contained in the volume v is:

$$\vec{p}_{EM}(t) = \int_v \vec{\wp}_{EM}(\vec{r},t)\,d\tau \qquad \left(\frac{\text{kg-m}}{\text{sec}}\right)$$

The instantaneous total EM angular momentum contained in the volume v is:

$$\vec{\mathcal{L}}_{EM}(t) = \int_v \vec{\ell}_{EM}(\vec{r},t)\;d\tau \qquad \left(\frac{\text{kg-m}^2}{\text{sec}}\right)$$

# Time-Averaged Quantities Associated with EM Waves

Usually we are not interested in knowing the instantaneous power P(t), energy / energy density, Poynting's vector, linear and angular momentum, *etc.*- because experimental measurements of these quantities are very often averages over many extremely fast cycles of oscillation. For example period of oscillation of light wave

$$\tau_{light} = 1/f_{light} \simeq \frac{1}{10^{15}\,\text{cps}} = 10^{-15}\,\text{sec/cycle} = 1\,\text{femto-sec}\,)$$

We need time averaged expressions for each of these quantities - in order to compare directly with experimental data- for monochromatic plane EM light waves:

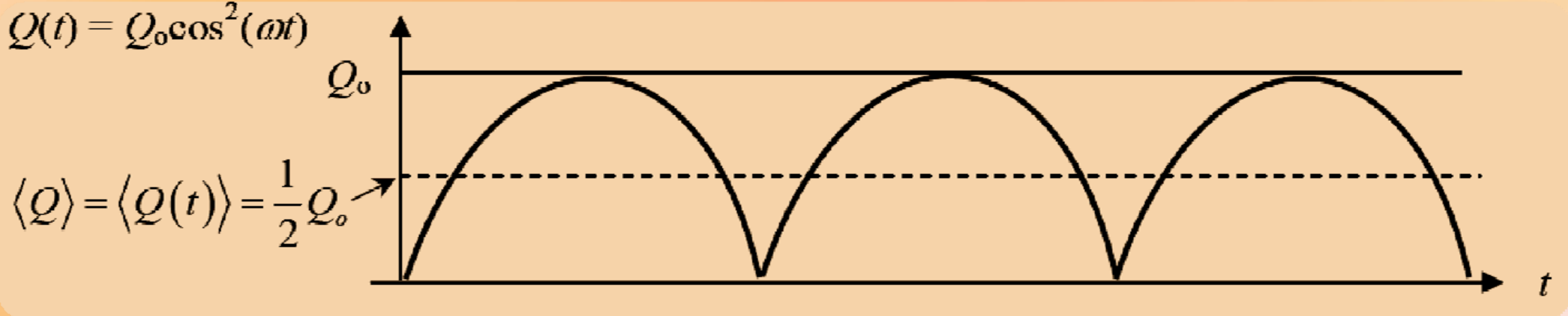
# Time-Averaged Quantities Associated with EM Waves

If we have an instantaneous physical quantity of the form:

$$Q(t) = Q_o \cos^2(\omega t)$$

The time-average of *Q(t)* is defined as:

$$\langle Q(t) \rangle \equiv \langle Q \rangle = \frac{1}{\tau} \int_{t=0}^{t=\tau} Q(t)\, dt = \frac{Q_o}{\tau} \int_{t=0}^{t=\tau} \cos^2(\omega t)\, dt$$



# Time-Averaged Quantities Associated with EM Waves

The time average of the $\cos^2(\omega t)$ function:

$$\frac{1}{\tau}\int_0^{\tau}\cos^2(\omega t)\,dt = \frac{1}{\tau}\left[\frac{t}{2}+\frac{\sin 2\omega t}{4\omega}\right]_{t=0}^{t=\tau} = \frac{1}{2\tau}\left[(\tau-0)+\left(\frac{\sin 2\omega\tau}{2\omega}-0\right)\right] = \frac{1}{2\tau}\left[\tau+\frac{\sin 2\omega\tau}{2\omega}\right]$$

$$\omega\tau = 2\pi f\tau \qquad f = 1/\tau \qquad \omega\tau = 2\pi(\tau/\tau) = 2\pi \qquad \sin(\omega\tau) = \sin(2\pi) = 0$$

$$\frac{1}{\tau}\int_0^{\tau}\cos^2(\omega t)\,dt = \frac{1}{2\not{t}}\left[\not{t}\right] = \frac{1}{2} \qquad \langle Q(t)\rangle = \langle Q\rangle = \frac{1}{2}Q_o$$

Thus, the time-averaged quantities associated with an EM wave propagating in free space are:

# Time-Averaged Quantities Associated with EM Waves

Intensity of an *EM* wave:

$$I(\vec{r}) \equiv \langle S(\vec{r},t) \rangle = \langle |\vec{S}(\vec{r},t)| \rangle = c \langle u_{EM}(\vec{r},t) \rangle = \frac{1}{2} c \varepsilon_o E_o^2 \quad \left( \frac{\text{Watts}}{\text{m}^2} \right)$$

The intensity of an EM wave is also known as the irradiance of the EM wave – it is the radiant power incident per unit area upon a surface.

# Time-Averaged Quantities Associated with EM Waves

EM Energy Density: $u_{EM}(\vec{r},t) \Rightarrow \langle u_{EM}(\vec{r},t) \rangle$

Total EM Energy: $U_{EM}(t) \Rightarrow \langle U_{EM}(t) \rangle$

Poynting's Vector: $\vec{S}(\vec{r},t) \Rightarrow \langle \vec{S}_{EM}(\vec{r},t) \rangle$

EM Power: $P_{EM}(t) \Rightarrow \langle P_{EM}(t) \rangle$

# Time-Averaged Quantities Associated with EM Waves

Linear Momentum Density: $\vec{\wp}_{EM}(\vec{r},t) \Rightarrow \left\langle \vec{\wp}_{EM}(\vec{r},t) \right\rangle$

Linear Momentum: $\vec{p}_{EM}(t) \Rightarrow \left\langle \vec{p}_{EM}(t) \right\rangle$

Angular Momentum Density: $\vec{\ell}_{EM}(\vec{r},t) \Rightarrow \left\langle \vec{\ell}_{EM}(\vec{r},t) \right\rangle$

Angular Momentum: $\vec{\mathcal{L}}_{EM}(t) \Rightarrow \left\langle \vec{\mathcal{L}}_{EM}(t) \right\rangle$

# Time-Averaged Quantities Associated with EM Waves

For a monochromatic EM plane wave propagating in free space / vacuum in ^z direction:

$$\left\langle u_{EM}(\vec{r},t)\right\rangle = \frac{1}{2}\varepsilon_o E_o^2 \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

$$\left\langle \vec{S}(\vec{r},t)\right\rangle = \frac{1}{2}c\varepsilon_o E_o^2\hat{z} = c\left\langle u_{EM}(\vec{r},t)\right\rangle\hat{z} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

$$\left\langle \vec{\wp}_{EM}(\vec{r},t)\right\rangle = \frac{1}{2c}\varepsilon_o E_o^2\hat{z} = \frac{1}{c^2}\left\langle \vec{S}(\vec{r},t)\right\rangle = \frac{1}{c}\left\langle u_{EM}(\vec{r},t)\right\rangle\hat{z} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

$$\left\langle \ell_{EM}(\vec{r},t)\right\rangle = \left(\vec{r}\times\left\langle \vec{\wp}_{EM}(\vec{r},t)\right\rangle\right) = \frac{1}{c^2}\left(\vec{r}\times\left\langle \vec{S}(\vec{r},t)\right\rangle\right) = \frac{1}{c}\left\langle u_{EM}(\vec{r},t)\right\rangle(\hat{r}\times\hat{z}) \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

Story has not finished yet

To be continued…

THANKS FOR TIME BEING