2570-2

Preparatory School to the Winter College on Optics: Fundamentals of Photonics - Theory, Devices and Applications

3 - 7 February 2014

ELECTROMAGNETIC WAVES IN MATTER

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Electromagnetic Wave Propagation in Linear Media

EM wave propagation inside matter - in regions with no free charges and no free currents (the medium is an insulator/non-conductor).

For this situation, Maxwell's equations become:

1) $\boxed{\vec{\nabla} \cdot \vec{D}(\vec{r},t) = 0}$

2) $\boxed{\vec{\nabla} \cdot \vec{B}(\vec{r},t) = 0}$

3) $\boxed{\vec{\nabla} \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}}$

4) $\boxed{\vec{\nabla} \times \vec{H}(\vec{r},t) = \frac{\partial \vec{D}(\vec{r},t)}{\partial t}}$

Electromagnetic Wave Propagation in Linear Media

The medium is assumed to be linear, homogeneous and isotropic- thus the following relations are valid in this medium:

$$\boxed{\vec{D}(\vec{r},t) = \varepsilon \vec{E}(\vec{r},t)} \quad \text{and} \quad \boxed{\vec{H}(\vec{r},t) = \frac{1}{\mu}\vec{B}(\vec{r},t)}$$

- ε = electric permittivity of the medium.
- $\varepsilon = \varepsilon_o(1+\chi_e)$, χ_e = electric susceptibility of the medium.
- μ = magnetic permeability of the medium.
- $\mu = \mu_o(1+\chi_m)$, χ_m = magnetic susceptibility of the medium.
- ε_o = electric permittivity of free space = 8.85×10^{-12} Farads/m.
- μ_o = magnetic permeability of free space = $4\pi \times 10^{-7}$ Henrys/m.

Electromagnetic Wave Propagation in Linear Media

Maxwell's equations inside the linear, homogeneous and isotropic non-conducting medium become:

1) $\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r},t)=0}$ 2) $\boxed{\vec{\nabla}\bullet\vec{B}(\vec{r},t)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}}$ 4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

In a linear /homogeneous/isotropic medium, the speed of propagation of EM waves is:

$$\boxed{v'_{prop}=\frac{1}{\sqrt{\varepsilon\mu}}}$$

Electromagnetic Wave Propagation in Linear Media

The *E* and *B* fields in the medium obey the following wave equation:

$$\boxed{\nabla^2 \vec{E}(\vec{r},t) = \varepsilon\mu \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = \frac{1}{v'^2_{prop}} \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}}$$

$$\boxed{\nabla^2 \vec{B}(\vec{r},t) = \varepsilon\mu \frac{\partial \vec{B}(\vec{r},t)}{\partial t} = \frac{1}{v'^2_{prop}} \frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2}}$$

Electromagnetic Wave Propagation in Linear Media

For linear / homogeneous / isotropic media:

$$\varepsilon = K_e\varepsilon_o = (1+\chi_e)\varepsilon_o \qquad K_e = \frac{\varepsilon}{\varepsilon_o} = (1+\chi_e) = \text{relative electric permittivity}$$

$$\mu = K_m\mu_0 = (1+\chi_m)\mu_o \qquad K_m = \frac{\mu}{\mu_o} = (1+\chi_m) = \text{relative magnetic permeability}$$

$$v'_{prop} = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{K_e\varepsilon_o K_m\mu_o}} = \frac{1}{\sqrt{K_e K_m}}\frac{1}{\sqrt{\varepsilon_o\mu_o}} = \frac{1}{\sqrt{K_e K_m}}c$$

If $\boxed{K_e K_m \geq 1}$ thus $\boxed{\frac{1}{\sqrt{K_e K_m}} \leq 1} \Rightarrow \boxed{v'_{prop} = \frac{1}{\sqrt{K_e K_m}}c \leq c}$

Electromagnetic Wave Propagation in Linear Media

Note also that since $K_e = \frac{\varepsilon}{\varepsilon_o}$ and $K_m = \frac{\mu}{\mu_o}$ are dimensionless

quantities, then so is $\frac{1}{\sqrt{K_e K_m}}$

Define the index of refraction - *a dimensionless quantity-* of the linear / homogeneous / isotropic medium as:

$$n \equiv \sqrt{K_e K_m} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_o \mu_o}}$$

Electromagnetic Wave Propagation in Linear Media

For linear / homogeneous / isotropic media:

$\boxed{v'_{prop} = c/n \ (\leq c)}$ because $\boxed{n \geq 1}$

For many (but not all) linear/homogeneous/isotropic materials:

$$\boxed{\mu = \mu_o(1+\chi_m) \simeq \mu_o}$$

(*True for many paramagnetic and diamagnetic-type materials*)

$$\boxed{|\chi_m| \sim \vartheta(10^{-8}) \sim 0}$$

Thus $\boxed{K_m = \frac{\mu}{\mu_o} = (1+\chi_m) \simeq 1} \Rightarrow \boxed{n \simeq \sqrt{K_e}}$ and $\boxed{v'_{prop} = \frac{c}{n} \simeq \frac{c}{\sqrt{K_e}}}$.

Electromagnetic Wave Propagation in Linear Media

The instantaneous EM energy density associated with a linear/homogeneous/isotropic material

$$\boxed{u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon E^2(\vec{r},t) + \frac{1}{\mu}B^2(\vec{r},t)\right) = \frac{1}{2}\left(\vec{E}(\vec{r},t)\cdot\vec{D}(\vec{r},t) + \vec{B}(\vec{r},t)\cdot\vec{H}(\vec{r},t)\right)} \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

with $\boxed{\vec{D}(\vec{r},t) = \varepsilon\vec{E}(\vec{r},t)}$ and $\boxed{\vec{H}(\vec{r},t) = \frac{1}{\mu}\vec{B}(\vec{r},t)}$

Electromagnetic Wave Propagation in Linear Media

The instantaneous Poynting's vector associated with a linear/homogeneous/isotropic material

$$\boxed{\vec{S}(\vec{r},t)=\frac{1}{\mu}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)=\left(\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right)}\left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The intensity of an EM wave propagating in this medium is:

$$\boxed{I(\vec{r})\equiv\left\langle\left|\vec{S}(\vec{r},t)\right|\right\rangle=v'_{prop}\left\langle u_{EM}(\vec{r},t)\right\rangle=\frac{1}{2}v'_{prop}\varepsilon E_o^2(\vec{r})=\frac{1}{2}\left(\frac{c}{n}\right)\varepsilon E_o^2(\vec{r})=\left(\frac{c}{n}\right)\varepsilon E_{o_{rms}}^2(\vec{r})}\left(\frac{\text{Watts}}{\text{m}^2}\right)$$

Where

$$\boxed{E_{o_{rms}}\equiv\frac{1}{\sqrt{2}}E_o}$$

Electromagnetic Wave Propagation in Linear Media

The instantaneous linear momentum density associated with an EM wave propagating in a linear/homogeneous/isotropic medium is:

$$\boxed{\vec{\wp}_{EM}(\vec{r},t) = \varepsilon\mu\vec{S}(\vec{r},t) = \frac{1}{v'^2_{prop}}\vec{S}(\vec{r},t) = \varepsilon\cancel{\mu}\frac{1}{\cancel{\mu}}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right) = \varepsilon\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)}\left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

The instantaneous angular momentum density associated with an EM wave propagating in this medium is:

$$\boxed{\vec{\ell}_{EM}(\vec{r},t) = \vec{r}\times\vec{\wp}_{EM}(\vec{r},t) = \varepsilon\,\vec{r}\times\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)}\left(\frac{\text{kg}}{\text{m-sec}}\right)$$

Electromagnetic Wave Propagation in Linear Media

Total instantaneous EM energy: $\boxed{U_{EM}(t)=\int_v u_{EM}(\vec{r},t)\,d\tau}$ (Joules)

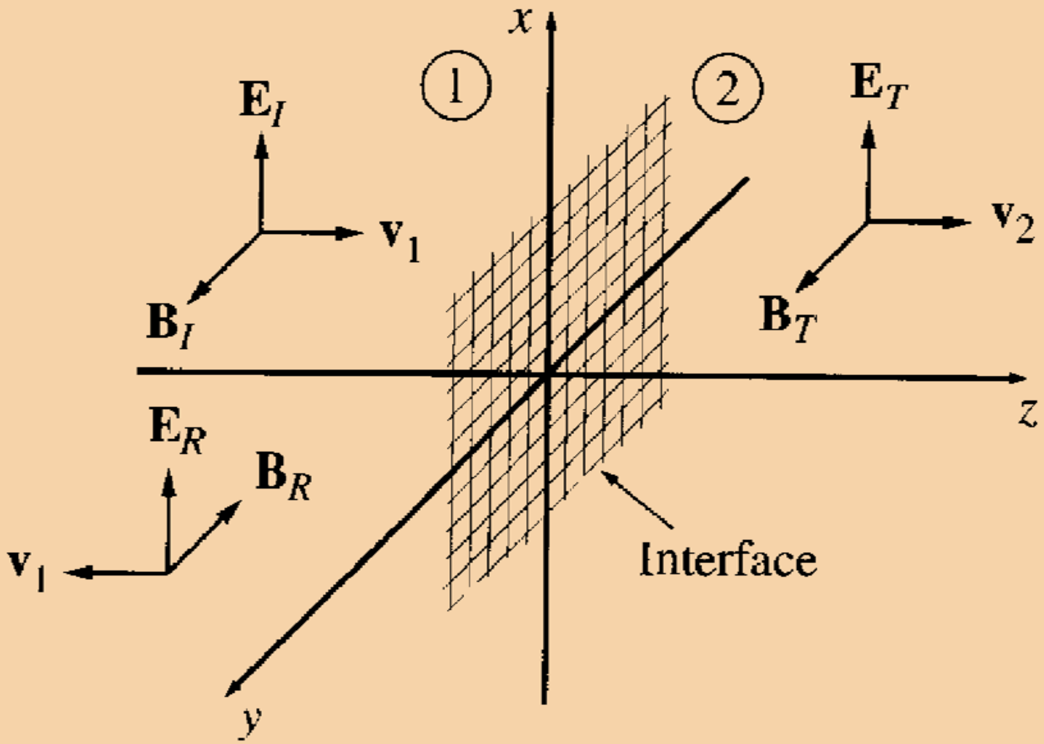
Total instantaneous linear momentum: $\boxed{\vec{p}_{EM}(t)=\int_v \vec{\wp}_{EM}(\vec{r},t)\,d\tau}$ $\left(\dfrac{\text{kg-m}}{\text{sec}}\right)$

Instantaneous *EM* Power: $\boxed{P_{EM}(t)=\dfrac{\partial U_{EM}(t)}{\partial t}=-\oint_S \vec{S}(\vec{r},t)\cdot d\vec{a}}$ (Watts)

Total instantaneous angular momentum: $\boxed{\vec{\mathcal{L}}_{EM}(t)=\int_v \vec{\ell}_{EM}(\vec{r},t)\,d\tau}$ $\left(\dfrac{\text{kg-m}^2}{\text{sec}}\right)$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Suppose the x-y plane forms the boundary between two linear media. A plane wave of frequency ω- travelling in the z- direction and polarized in the x- direction- approaches the interface from the left



Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Incident EM plane wave (in medium 1):

Propagates in the $+\hat{z}$-direction (*i.e.* $\hat{k}_{inc} = +\hat{k}_1 = +\hat{z}$), with polarization $\boxed{\hat{n}_{inc} = +\hat{x}}$

$\boxed{\tilde{\vec{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{x}}$ with: $\boxed{k_{inc} = |\vec{k}_{inc}| = k_1 = |\vec{k}_1| = 2\pi/\lambda_1 = \omega/v_1}$

$\boxed{\tilde{\vec{B}}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}_{inc}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{y}}$ <u>since</u>: $\boxed{\hat{k}_{inc} \times \hat{n}_{inc} = +\hat{z} \times \hat{x} = +\hat{y}}$

Reflected EM plane wave (in medium 1):

Propagates in the $-\hat{z}$-direction (*i.e.* $\hat{k}_{refl} = -\hat{k}_1 = -\hat{z}$), with polarization $\boxed{\hat{n}_{refl} = +\hat{x}}$

$\boxed{\tilde{\vec{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)} \hat{x}}$ with: $\boxed{k_{refl} = |\vec{k}_{refl}| = k_1 = |\vec{k}_1| = 2\pi/\lambda_1 = \omega/v_1}$

$\boxed{\tilde{B}_{refl}(z,t) = \frac{1}{v_1}\hat{k}_{refl} \times \tilde{\vec{E}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)} \hat{y}}$ <u>since</u>: $\boxed{\hat{k}_{refl} \times \hat{n}_{refl} = -\hat{z} \times \hat{x} = -\hat{y}}$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Transmitted EM plane wave (in medium 2):

Propagates in the $+\hat{z}$-direction (*i.e.* $\hat{k}_{trans} = +\hat{k}_2 = +\hat{z}$), with polarization $\boxed{\hat{n}_{trans} = +\hat{x}}$

$\boxed{\tilde{\vec{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)} \hat{x}}$ with: $\boxed{k_{trans} = \left|\vec{k}_{trans}\right| = k_2 = \left|\vec{k}_2\right| = 2\pi/\lambda_2 = \omega/v_2}$

$\boxed{\tilde{\vec{B}}_{trans}(z,t) = \frac{1}{v_2}\hat{k}_{trans} \times \tilde{\vec{E}}_{trans}(z,t) = \frac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)} \hat{y}}$ <u>since</u>: $\boxed{\hat{k}_{trans} \times \hat{n}_{trans} = +\hat{z} \times \hat{x} = +\hat{y}}$

In this situation the E -field - polarization vectors are all oriented in the same direction

$\boxed{\hat{n}_{inc} = \hat{n}_{refl} = \hat{n}_{trans} = +\hat{x}}$ or equivalently:

$$\boxed{\vec{E}_{inc}(\vec{r},t) \parallel \vec{E}_{refl}(\vec{r},t) \parallel \vec{E}_{trans}(\vec{r},t)}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

At the interface between the two linear / homogeneous / isotropic media -at z = 0 in the x-y plane- the boundary conditions 1 – 4 must be satisfied for the total E and B -fields immediately present on either side of the interface:

BC 1) Normal $\vec{D}$ continuous: $\boxed{\varepsilon_1 E^{\perp}_{1_{Tot}} = \varepsilon_2 E^{\perp}_{2_{Tot}}}$

(*n.b.* $\perp$ refers to the *x*-*y* boundary, *i.e.* in the $+\hat{z}$ direction)

BC 2) Tangential $\vec{E}$ continuous: $\boxed{E^{\parallel}_{1_{Tot}} = E^{\parallel}_{2_{Tot}}}$

(*n.b.* || refers to the *x*-*y* boundary, *i.e.* in the *x*-*y* plane)

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 3) Normal $\vec{B}$ continuous: $\boxed{B_{1_{Tot}}^{\perp} = B_{2_{Tot}}^{\perp}}$

(⊥ to x-y boundary, i.e. in the +z^ direction)

BC 4) Tangential $\vec{H}$ continuous: $\boxed{\frac{1}{\mu_1} B_{1_{Tot}}^{\parallel} = \frac{1}{\mu_2} B_{2_{Tot}}^{\parallel}}$

(‖ to x-y boundary, i.e. in x-y plane)

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For plane EM waves at normal incidence on the boundary at z = 0- lying in the x-y plane- no components of **E or B** (incident, reflected or transmitted waves) - allowed to be along the ±zˆ propagation direction(s) - the E and *B*-field are transverse fields - constraints imposed by Maxwell's equations.

BC -1) and BC- 3) impose no restrictions on such *EM* waves since:

$$\{ E_{1_{Tot}}^{\perp} = \mathrm{E}_{1_{Tot}}^{z} = 0 ;\; E_{2_{Tot}}^{\perp} = E_{2_{Tot}}^{z} = 0 \} \text{ and } \{ B_{1_{Tot}}^{\perp} = B_{1_{Tot}}^{z} = 0 ;\; B_{2_{Tot}}^{\perp} = B_{2_{Tot}}^{z} = 0 \}$$

⇒ The only restrictions on plane EM waves propagating with normal incidence on the boundary at z = 0 are imposed by BC-2) and BC- 4).

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

At z = 0 in medium 1) (i.e. z ≤ 0) we must have:

$$\boxed{\vec{\tilde{E}}^{\parallel}_{1_{Tot}}(z=0,t)=\vec{\tilde{E}}_{inc}(z=0,t)+\vec{\tilde{E}}_{refl}(z=0,t)} \quad \text{and}$$

$$\boxed{\frac{1}{\mu_1}\vec{\tilde{B}}^{\parallel}_{1_{Tot}}(z=0,t)=\frac{1}{\mu_1}\vec{\tilde{B}}_{inc}(z=0,t)+\frac{1}{\mu_1}\vec{\tilde{B}}_{refl}(z=0,t)}$$

While at z = 0 in medium 2) (i.e. z ≥ 0) we must have:

$$\boxed{\vec{\tilde{E}}^{\parallel}_{2_{Tot}}(z=0,t)=\vec{\tilde{E}}_{trans}(z=0,t)} \quad \text{and}$$

$$\boxed{\frac{1}{\mu_2}\vec{\tilde{B}}^{\parallel}_{2_{Tot}}(z=0,t)=\frac{1}{\mu_2}\vec{\tilde{B}}_{trans}(z=0,t)}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) -Tangential *E* is continuous @ z = 0) requires that:

$$\boxed{\vec{\tilde{E}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \vec{\tilde{E}}^{\parallel}_{2_{Tot}}\Big|_{z=0}} \quad \text{or:} \quad \boxed{\vec{\tilde{E}}_{inc}(z=0,t) + \vec{\tilde{E}}_{refl}(z=0,t) = \vec{\tilde{E}}_{trans}(z=0,t)}.$$

BC 4) -Tangential *H* is continuous @ z = 0) requires that:

$$\boxed{\frac{1}{\mu_1}\vec{\tilde{B}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \frac{1}{\mu_2}\vec{\tilde{B}}^{\parallel}_{2_{Tot}}\Big|_{z=0}}$$

$$\text{or:} \quad \boxed{\frac{1}{\mu_1}\vec{\tilde{B}}_{inc}(z=0,t) + \frac{1}{\mu_1}\vec{\tilde{B}}_{refl}(z=0,t) = \frac{1}{\mu_2}\vec{\tilde{B}}_{trans}(z=0,t)}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Using explicit expressions for the complex **E** and **B** fields

$$\boxed{\vec{\tilde{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}}\, e^{i(k_1 z - \omega t)}\,\hat{x}}$$

$$\boxed{\vec{\tilde{B}}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \vec{\tilde{E}}_{inc}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}}\, e^{i(k_1 z - \omega t)}\,\hat{y}}$$

$$\boxed{\vec{\tilde{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}}\, e^{i(-k_1 z - \omega t)}\,\hat{x}}$$

$$\boxed{\tilde{B}_{refl}(z,t) = \frac{1}{v_1}\hat{k}_{refl} \times \vec{\tilde{E}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}}\, e^{i(-k_1 z - \omega t)}\,\hat{y}}$$

$$\boxed{\vec{\tilde{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}}\, e^{i(k_2 z - \omega t)}\,\hat{x}}$$

$$\boxed{\tilde{B}_{trans}(z,t) = \frac{1}{v_2}\hat{k}_{trans} \times \vec{\tilde{E}}_{trans}(z,t) = \frac{1}{v_2}\tilde{E}_{o_{trans}}\, e^{i(k_2 z - \omega t)}\,\hat{y}}$$

The above boundary condition relations become

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} \cancel{e^{-i\omega t}} + \tilde{E}_{o_{refl}} \cancel{e^{-i\omega t}} = \tilde{E}_{o_{trans}} \cancel{e^{-i\omega t}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\frac{1}{\mu_1 v_1} \tilde{E}_{o_{inc}} \cancel{e^{-i\omega t}} - \frac{1}{\mu_1 v_1} \tilde{E}_{o_{refl}} \cancel{e^{-i\omega t}} = \frac{1}{\mu_2 v_2} \tilde{E}_{o_{trans}} \cancel{e^{-i\omega t}}}$

Cancelling the common $e^{-i\omega t}$ factors on the LHS & RHS of above equations - we have at z = 0 (everywhere in the x-y plane- must be independent of any time t):

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) (Tangential $\vec{E}$ continuous @ $z=0$): $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z=0$): $\boxed{\frac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}} - \frac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}} = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}}$

Assuming that $\{\mu_1$ and $\mu_2\}$ and $\{v_1$ and $v_2\}$ are known / given for the two media, we have two equations {from BC 2) and BC 4)} and *three* unknowns $\{\tilde{E}_{o_{inc}}, \tilde{E}_{o_{refl}}, \tilde{E}_{o_{trans}}\}$

→ Solve above equations simultaneously for

$\{\tilde{E}_{o_{refl}}$ and $\tilde{E}_{o_{trans}}\}$ in terms of / scaled to $\tilde{E}_{o_{inc}}$.

Let us define: $\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}}$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 4) -Tangential $\boldsymbol{H}$ continuous @ z = 0- relation becomes:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\, \tilde{E}_{o_{trans}}}$$

BC 2) -Tangential **E** continuous @ z = 0 - gives:

$$\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$$

BC 4) -Tangential $\boldsymbol{H}$ continuous @ z = 0- reduces to

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\, \tilde{E}_{o_{trans}}} \quad \text{with} \quad \boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Add and Subtract BC -2) and BC- 4) relations:

$$\boxed{2\tilde{E}_{o_{inc}} = (1+\beta)\,\tilde{E}_{o_{trans}}} \Rightarrow \boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}}} \quad (2+4)$$

$$\boxed{2\tilde{E}_{o_{refl}} = (1-\beta)\tilde{E}_{o_{trans}}} \Rightarrow \boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{2}\right)\tilde{E}_{o_{trans}}} \quad (2-4)$$

Insert the result of eqn. (2+4) into eqn. (2−4):

$$\boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{\cancel{2}}\right)\left(\frac{\cancel{2}}{1+\beta}\right)\tilde{E}_{o_{inc}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

$$\boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}}} \quad \text{and} \quad \boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}}}$$

<u>Now</u>: $\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}}$ and: $\boxed{v_1 = \frac{c}{n_1}}$, $\boxed{v_2 = \frac{c}{n_2}}$ where: $\boxed{n_1 = \sqrt{\frac{\varepsilon_1\mu_1}{\varepsilon_o\mu_o}}}$ and $\boxed{n_2 = \sqrt{\frac{\varepsilon_2\mu_2}{\varepsilon_o\mu_o}}}$

$$\boxed{\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 (c/n_1)}{\mu_2 (c/n_2)} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\mu_1\sqrt{\varepsilon_2\mu_2/\varepsilon_o\mu_o}}{\mu_2\sqrt{\varepsilon_1\mu_1/\varepsilon_o\mu_o}} = \frac{\mu_1}{\mu_2}\frac{\sqrt{\varepsilon_2\mu_2}}{\sqrt{\varepsilon_1\mu_1}} = \sqrt{\left(\frac{\varepsilon_2}{\mu_2}\right)\Big/\left(\frac{\varepsilon_1}{\mu_1}\right)} = \sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Now if the two media are both paramagnetic or diamagnetic- such that $\boxed{\left|\chi_{m_{1,2}}\right| \ll 1}$

$$i.e.\ \boxed{\mu_1 = \mu_o\left(1+\chi_{m_1}\right) \approx \mu_o} \text{ and: } \boxed{\mu_2 = \mu_o\left(1+\chi_{m_2}\right) \approx \mu_o}$$

Common for many (but not all) non-conducting linear/ homogeneous/isotropic media

Then $$\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} \simeq \left(\frac{v_1}{v_2}\right) = \left(\frac{n_2}{n_1}\right)} \text{ for } \boxed{\mu_1 \approx \mu_2 \approx \mu_o} \text{ or } \boxed{\left|\chi_{m_{1,2}}\right| \ll 1}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Then

$$\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}} \simeq \left(\frac{1-(v_1/v_2)}{1+(v_1/v_2)}\right)\tilde{E}_{o_{inc}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{E}_{o_{inc}}$$

$$\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}} \simeq \left(\frac{2}{1+(v_1/v_2)}\right)\tilde{E}_{o_{nc}} = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{E}_{o_{inc}}$$

We can alternatively express these relations in terms of the indices of refraction n_1 and n_2:

$$\tilde{E}_{o_{refl}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)\tilde{E}_{o_{inc}} \quad \text{and} \quad \tilde{E}_{o_{trans}} = \left(\frac{2n_1}{n_1 + n_2}\right)\tilde{E}_{o_{inc}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Now since:

$$\boxed{\tilde{E}_{o_{inc}} = E_{o_{inc}} e^{i\delta}}$$

$$\boxed{\tilde{E}_{o_{refl}} = E_{o_{refl}} e^{i\delta}}$$

$$\boxed{\tilde{E}_{o_{trans}} = E_{o_{trans}} e^{i\delta}}$$

δ = phase angle (in radians) defined at the zero of time - $t = 0$

Then for the purely real amplitudes $(E_{o_{inc}}, E_{o_{refl}}, E_{o_{trans}})$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

The relations between real amplitudes become:

for $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$

$$\boxed{E_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}} \qquad \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

$$\boxed{E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1 + v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{o_{inc}}}$$

for $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Monochromatic plane EM wave at normal incidence on a boundary between two linear / homogeneous / isotropic media for $\mu_1 \simeq \mu_2 \simeq \mu_o$ for the following cases:

If $v_2 > v_1$ $(i.e.\ n_2 < n_1)$ $\{e.g.$ medium 1) = glass $\Rightarrow$ medium 2) = air$\}$:

$$\boxed{E_{o_{refl}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}} \Rightarrow$$

$E_{o_{refl}}$ is precisely in-phase with $E_{o_{inc}}$ because $(v_2 - v_1) > 0$.

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If $v_2 < v_1$ $(i.e.\ n_2 > n_1)$ $\{e.g.$ medium 1) = air $\Rightarrow$ medium 2) = glass$\}$:

$$\boxed{E_{0_{refl}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{0_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{0_{inc}}} \Rightarrow$$

$E_{0_{refl}}$ is 180° out-of-phase with $E_{0_{inc}}$ because $(v_2 - v_1) < 0$.

$$i.e. \quad \boxed{E_{0_{refl}} = -\left|\frac{v_2 - v_1}{v_2 + v_1}\right| E_{0_{inc}} = -\left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0_{inc}}} \Rightarrow$$

The minus sign indicates a 180° phase shift occurs upon reflection for $v_2 < v_1$ (i.e. $n_2 > n_1$) !!!

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

$E_{o_{trans}}$ is always in-phase with $E_{o_{inc}}$ for all possible v_1 & v_2 $\left(n_1 \,\&\, n_2\right)$ because:

$$\boxed{E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1+v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1+n_2}\right) E_{o_{inc}}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

What fraction of the incident *EM* wave energy is reflected ?

What fraction of the incident *EM* wave energy is transmitted?

In a given linear/homogeneous/isotropic medium with

$$\boxed{v = \sqrt{\frac{\varepsilon_o \mu_o}{\varepsilon \mu}}\, c = c/n}$$

The time-averaged energy density in the EM wave is:

$$\boxed{\langle u_{EM}(\vec{r},t)\rangle = \frac{1}{2}\varepsilon E_o^2(\vec{r}) = \varepsilon E_{o_{rms}}^2(\vec{r})} \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

The time-averaged Poynting's vector is:

$$\boxed{\left\langle \vec{S}(\vec{r},t)\right\rangle = \frac{1}{\mu}\left\langle \vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right\rangle}\ \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The intensity of the EM wave is:

$$\boxed{I(\vec{r}) \equiv \left\langle \left|\vec{S}(\vec{r},t)\right|\right\rangle = v\left\langle u_{EM}(\vec{r},t)\right\rangle = v\left(\frac{1}{2}\varepsilon E_o^2(\vec{r})\right) = \frac{1}{2}\varepsilon v E_o^2(\vec{r}) = \varepsilon v E_{o_{rms}}^2(\vec{r})}\ \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The three Poynting's vectors associated with this problem are such that

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

The three Poynting's vectors associated with this problem are such that

$$\vec{S}_{inc} \parallel (+\hat{z}), \quad \vec{S}_{refl} \parallel (-\hat{z}) \text{ and } \vec{S}_{trans} \parallel (+\hat{z})$$

For a monochromatic plane EM wave at normal incidence with $\mu_1 \simeq \mu_2 \simeq \mu_o$

$$\boxed{E_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right)E_{o_{inc}} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)E_{o_{inc}}} \qquad \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

$$\boxed{E_{o_{trans}} = \left(\frac{2}{1+\beta}\right)E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1 + v_1}\right)E_{o_{inc}} = \left(\frac{2n_1}{n_1 + n_2}\right)E_{o_{inc}}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Take the ratios $\left(E_{o_{refl}} / E_{o_{inc}}\right)$ and $\left(E_{o_{trans}} / E_{o_{inc}}\right)$ - then square them:

$$\boxed{\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \approx \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}$$

and

$$\boxed{\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 \approx \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Define the reflection coefficient as:

$$R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})} \right) = \frac{\left\langle \left| \vec{S}_{refl}(\vec{r},t) \right| \right\rangle}{\left\langle \left| \vec{S}_{inc}(\vec{r},t) \right| \right\rangle} = \frac{v_1 \left\langle u_{EM}^{refl}(\vec{r},t) \right\rangle}{v_1 \left\langle u_{EM}^{inc}(\vec{r},t) \right\rangle} = \frac{\left\langle u_{EM}^{refl}(\vec{r},t) \right\rangle}{\left\langle u_{EM}^{inc}(\vec{r},t) \right\rangle} = \frac{\frac{1}{2}\varepsilon_1 v_1 E_{o_{refl}}^2(\vec{r})}{\frac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})} = \frac{E_{o_{refl}}^2(\vec{r})}{E_{o_{inc}}^2(\vec{r})}$$

Define the transmission coefficient as:

$$T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})} \right) = \frac{\left\langle \left| \vec{S}_{trans}(\vec{r},t) \right| \right\rangle}{\left\langle \left| \vec{S}_{inc}(\vec{r},t) \right| \right\rangle} = \frac{v_2 \left\langle u_{EM}^{trans}(\vec{r},t) \right\rangle}{v_1 \left\langle u_{EM}^{inc}(\vec{r},t) \right\rangle} = \frac{\left(\frac{1}{2}\varepsilon_2 v_2 E_{o_{trans}}^2(\vec{r})\right)}{\left(\frac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})\right)} = \frac{\varepsilon_2 v_2 E_{o_{trans}}^2(\vec{r})}{\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For a linearly-polarized monochromatic plane EM wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, with $\mu_1 \simeq \mu_2 \simeq \mu_o$

Reflection coefficient:

$$R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})}\right) = \left(\frac{E_{o_{refl}}(\vec{r})}{E_{o\,inc}(\vec{r})}\right)^2$$

Transmission coefficient:

$$T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

But:

$$\left(\frac{E_{o_{refl}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \quad \&$$

$$\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 \simeq \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2$$

Thus Reflection and Transmission coefficient:

$$\boxed{R(\vec{r}) \equiv \left(\frac{1-\beta}{1+\beta}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2} \qquad \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

$$\boxed{T(\vec{r}) \equiv \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 \simeq \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2v_2}{v_2 + v_1}\right)^2 = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Now: $\boxed{\dfrac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \dfrac{\dfrac{\varepsilon_2 \mu_2 v_2}{\mu_2}}{\dfrac{\varepsilon_1 \mu_1 v_1}{\mu_1}}}$ but: $\boxed{\begin{array}{l} v_2^2 = \dfrac{1}{\varepsilon_2 \mu_2} \Rightarrow \varepsilon_2 \mu_2 = \dfrac{1}{v_2^2} \\ v_1^2 = \dfrac{1}{\varepsilon_1 \mu_1} \Rightarrow \varepsilon_1 \mu_1 = \dfrac{1}{v_1^2} \end{array}}$

$$\boxed{\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\left(\frac{1}{v_2{}^2} \cdot v_2\right) \Big/ \mu_2}{\left(\frac{1}{v_1{}^2} \cdot v_1\right) \Big/ \mu_2} = \frac{1 \big/ \mu_2 v_2}{1 \big/ \mu_1 v_1} = \frac{\mu_1 v_1}{\mu_2 v_2} \equiv \beta} \;!!! \quad i.e. \quad \boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

for $\boxed{\mu_1 \approx \mu_2 \approx \mu_o}$

$$\boxed{T(\vec{r}) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 = \beta\left(\frac{2}{1+\beta}\right)^2 = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4 v_2 v_1}{(v_2+v_1)^2} = \frac{4 n_1 n_2}{(n_1+n_2)^2}}$$

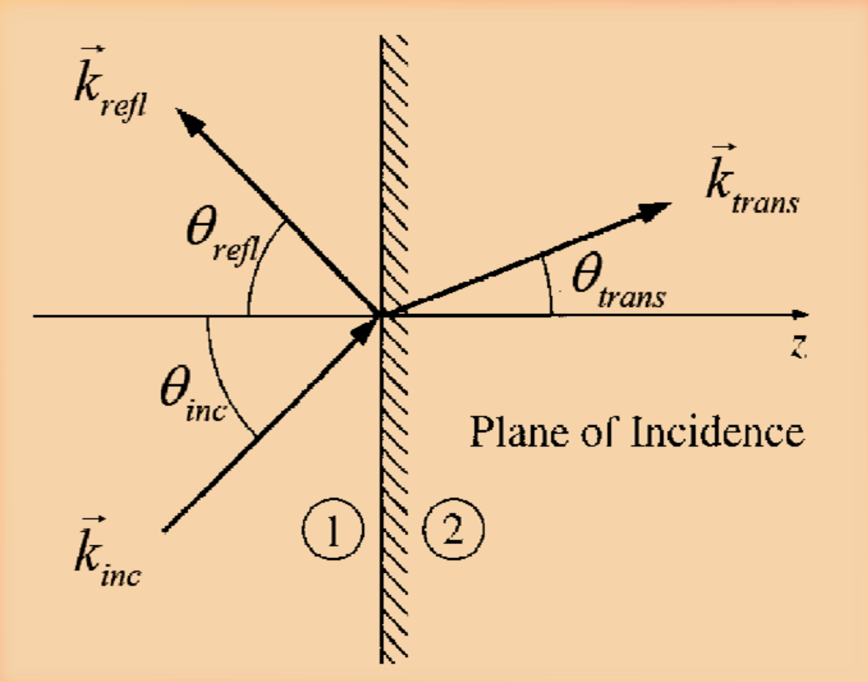
Thus:

$$\boxed{R(\vec{r})+T(\vec{r}) = \frac{(1-\beta)^2}{(1+\beta)^2} + \frac{4\beta}{(1+\beta)^2} = \frac{(1-\beta)^2+4\beta}{(1+\beta)^2} = \frac{1-2\beta+\beta^2+4\beta}{(1+\beta)^2} = \frac{1+2\beta+\beta^2}{(1+\beta)^2} = \frac{(1+\beta)^2}{(1+\beta)^2} = 1}$$

$\boxed{R(\vec{r})+T(\vec{r})=1}$ ⇒EM energy is conserved at the interface/boundary between two L/H/I media

Reflection & Transmission of Monochromatic Plane EM Waves at Oblique Incidence

A monochromatic plane EM wave incident at an oblique angle θ_{inc} on a boundary between two linear/ homogeneous/isotropic media, defined with respect to the normal to the interface- as shown in the figure below:



Reflection & Transmission of Monochromatic Plane EM Waves at Oblique Incidence

The incident EM wave is:

$$\boxed{\vec{\tilde{E}}_{inc}(\vec{r},t)=\vec{\tilde{E}}_{o_{inc}}e^{i(\vec{k}_{inc}\cdot\vec{r}-\omega t)}}\quad \text{and}\quad \boxed{\vec{\tilde{B}}_{inc}(\vec{r},t)=\frac{1}{v_1}\hat{k}_{inc}\times\vec{\tilde{E}}_{inc}(\vec{r},t)}$$

The reflected EM wave is:

$$\boxed{\vec{\tilde{E}}_{refl}(\vec{r},t)=\vec{\tilde{E}}_{o_{refl}}e^{i(\vec{k}_{refl}\cdot\vec{r}-\omega t)}}\quad \text{and}\quad \boxed{\vec{\tilde{B}}_{refl}(\vec{r},t)=\frac{1}{v_1}\hat{k}_{refl}\times\vec{\tilde{E}}_{refl}(\vec{r},t)}$$

The transmitted EM wave is:

$$\boxed{\vec{\tilde{E}}_{trans}(\vec{r},t)=\vec{\tilde{E}}_{o_{trans}}e^{i(\vec{k}_{trans}\cdot\vec{r}-\omega t)}}\quad \text{and}\quad \boxed{\vec{\tilde{B}}_{trans}(\vec{r},t)=\frac{1}{v_2}\hat{k}_{trans}\times\vec{\tilde{E}}_{trans}(\vec{r},t)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

All three EM waves have the same frequency- $f = \omega/2\pi$

$$\omega = k_{inc} v_1 = k_{refl} v_1 = k_{trans} v_2$$

$$\boxed{k_{inc} = k_{refl} = k_1 = \left(\frac{v_2}{v_1}\right) k_{trans} = \left(\frac{v_2}{v_1}\right) k_2 = \left(\frac{n_1}{n_2}\right) k_{trans} = \left(\frac{n_1}{n_2}\right) k_2}$$

$$\boxed{v_i = c/n_i \quad i = 1,2}$$

The total EM fields in medium 1

$$\boxed{\vec{\tilde{E}}_{Tot_1}(\vec{r},t) = \vec{\tilde{E}}_{inc}(\vec{r},t) + \vec{\tilde{E}}_{refl}(\vec{r},t)} \quad \text{and} \quad \boxed{\vec{\tilde{B}}_{Tot_1}(\vec{r},t) = \vec{\tilde{B}}_{inc}(\vec{r},t) + \vec{\tilde{B}}_{refl}(\vec{r},t)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Must match to the total EM fields in medium 2:

$$\boxed{\vec{\tilde{E}}_{Tot_2}(\vec{r},t)=\vec{\tilde{E}}_{trans}(\vec{r},t)} \quad \text{and} \quad \boxed{\vec{\tilde{B}}_{Tot_2}(\vec{r},t)=\vec{\tilde{B}}_{trans}(\vec{r},t)}$$

Using the boundary conditions BC1) → BC4) at z = 0.

At $z = 0$- four boundary conditions are of the form:

$$\boxed{(\sim\!\sim)\, e^{i\left(\vec{k}_{inc}\cdot\vec{r}-\omega t\right)}+(\sim\!\sim)\; e^{i\left(\vec{k}_{refl}\cdot\vec{r}-\omega t\right)}=(\sim\!\sim)\; e^{i\left(\vec{k}_{trans}\cdot\vec{r}-\omega t\right)}}$$

They must hold for all (x,y) on the interface at z = 0 - and also must hold for all times t. The above relation is already satisfied for arbitrary time, t - the factor $e^{-i\omega t}$ is common to all terms.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The following relation must hold for all (x,y) on interface at at z = 0:

$$\boxed{(\sim\!\sim)\, e^{i\left(\vec{k}_{inc}\cdot\vec{r}\right)} + (\sim\!\sim)\; e^{i\left(\vec{k}_{refl}\cdot\vec{r}\right)} = (\sim\!\sim)\; e^{i\left(\vec{k}_{trans}\cdot\vec{r}\right)}}$$

When *z = 0* - at interface we must have:

$$\vec{k}_{inc}\bullet\vec{r} = \vec{k}_{refl}\bullet\vec{r} = \vec{k}_{trans}\bullet\vec{r}$$

$$k_{inc_x}x + k_{inc_y}y = k_{refl_x}x + k_{refl_y}y = k_{trans_x}x + k_{trans_y}y \qquad @\ z = 0$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The above relation can only hold for arbitrary (x, y, z = 0) **iff (= if and only if):**

$$k_{inc_x} x = k_{refl_x} x = k_{trans_x} x \quad \Rightarrow \quad k_{inc_x} = k_{refl_x} = k_{trans_x}$$

$$k_{inc_y} y = k_{refl_y} y = k_{trans_y} y \quad \Rightarrow \quad k_{inc_y} = k_{refl_y} = k_{trans_y}$$

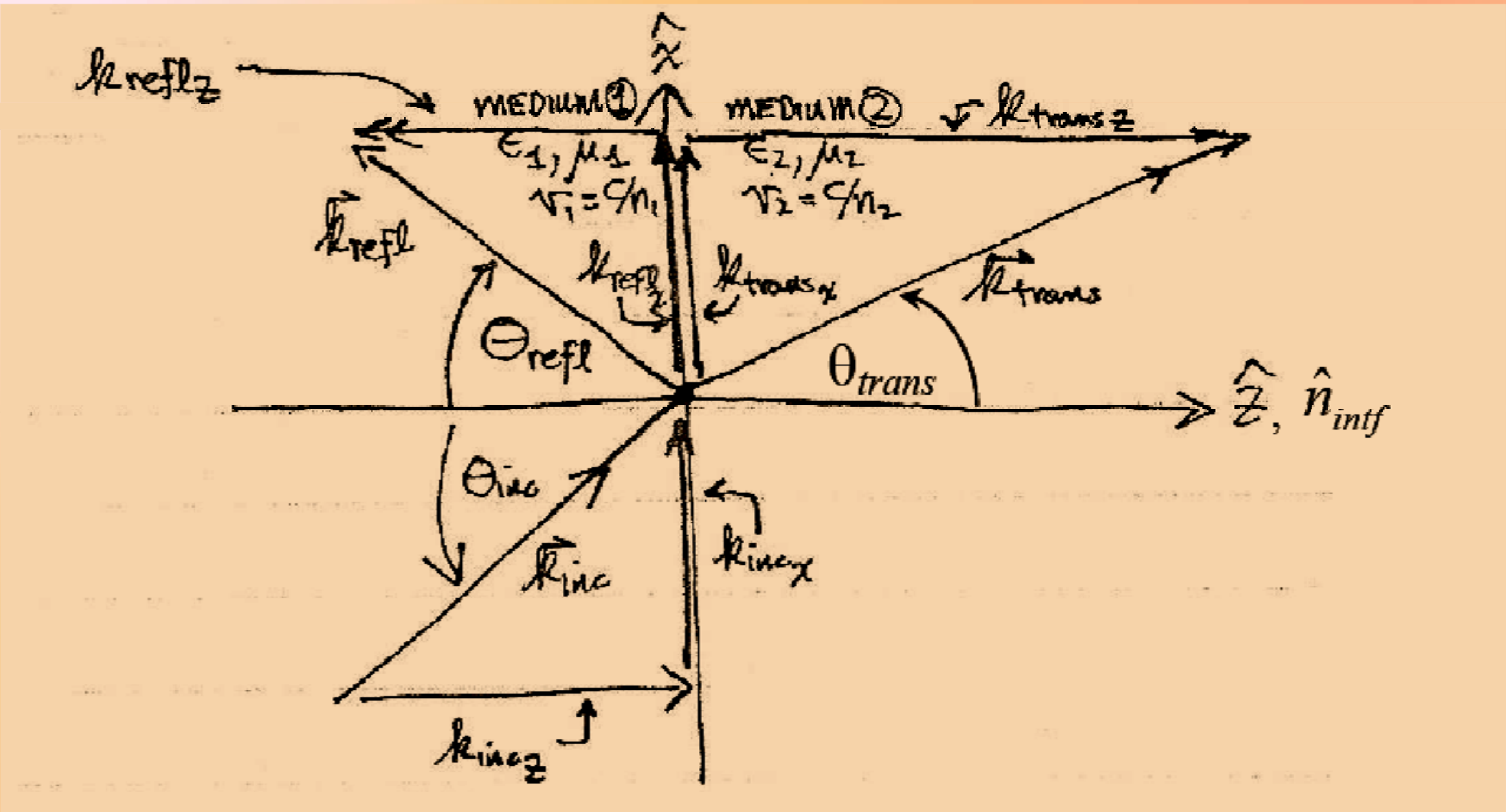
Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The problem has rotational symmetry about the z –axis- without any loss of generality - choose k to lie entirely within the x-z plane- that is no component of k in y-direction as shown in the figure on next slide

$$k_{inc_y} = k_{refl_y} = k_{trans_y} = 0 \quad \text{and thus:} \quad k_{inc_x} = k_{refl_x} = k_{trans_x}$$

The transverse components of $\vec{k}_{inc}, \vec{k}_{refl}, \vec{k}_{trans}$ are all equal and point in the +x^ direction.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*



Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The First Law of Geometrical Optics:

The incident, reflected, and transmitted wave vectors form a plane - called the plane of incidence- which also includes the normal to the surface -here the z axis.

The Second Law of Geometrical Optics (Law of Reflection):

From the figure- we see that:

$$\boxed{k_{inc_x} = k_{inc}\sin\theta_{inc}} = \boxed{k_{refl_x} = k_{refl}\sin\theta_{refl}} = \boxed{k_{trans_x} = k_{trans}\sin\theta_{trans}}$$

$$\boxed{k_{inc} = k_{refl} = k_1} \Rightarrow \boxed{\sin\theta_{inc} = \sin\theta_{refl}}$$

Angle of Incidence = Angle of Reflection $\boxed{\theta_{inc} = \theta_{refl}}$ Law of Reflection!

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Third Law of Geometrical Optics (Law of Refraction – Snell's Law):

For the transmitted angle - θ_{trans} we see that:

$$\boxed{k_{inc} \sin\theta_{inc} = k_{trans} \sin\theta_{trans}}$$

In medium 1): $\boxed{k_{inc} = k_1 = \omega/v_1 = n_1\omega/c = n_1 k_o}$

where $\boxed{k_o = \text{ vacuum wave number } = 2\pi/\lambda_o}$

and $\boxed{\lambda_o = \text{ vacuum wave length}}$

In medium 2): $\boxed{k_{trans} = k_2 = \omega/v_2 = n_2\omega/c = n_2 k_o}$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

$$\boxed{k_{inc}\sin\theta_{inc} = k_{trans}\sin\theta_{trans}} \Rightarrow \boxed{k_1\sin\theta_{inc} = k_2\sin\theta_{trans}}$$

$$\boxed{k_{inc} = k_1 = n_1 k_o} \text{ and } \boxed{k_{trans} = k_2 = n_2 k_o}$$

$$\boxed{k_1\sin\theta_{inc} = k_2\sin\theta_{trans}} \Rightarrow \boxed{n_1\sin\theta_{inc} = n_2\sin\theta_{trans}}$$

Law of Refraction
(Snell's Law)

Which can also be written as: $\boxed{\dfrac{\sin\theta_{trans}}{\sin\theta_{inc}} = \dfrac{n_1}{n_2}}$

Since θ_{trans} refers to medium 2) and θ_{inc} refers to medium 1)

$$\boxed{n_1\sin\theta_1 = n_2\sin\theta_2} \quad \text{or:} \quad \boxed{\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}}$$

(incident) (transmitted)

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Using three laws of geometrical optics we can see that :

$$\left.\vec{k}_{inc}\bullet\vec{r}\right|_{z=0}=\left.\vec{k}_{refl}\bullet\vec{r}\right|_{z=0}=\left.\vec{k}_{trans}\bullet\vec{r}\right|_{z=0}$$

everywhere on the interface at $z = 0$ -in the x-y plane

Thus $$\left.e^{i\left(\vec{k}_{inc}\bullet\vec{r}-\omega t\right)}\right|_{z=0}=\left.e^{i\left(\vec{k}_{refl}\bullet\vec{r}-\omega t\right)}\right|_{z=0}=\left.e^{i\left(\vec{k}_{trans}\bullet\vec{r}-\omega t\right)}\right|_{z=0}$$

everywhere on the interface at z = 0 -in the x-y plane and valid also for all time(s) t, since ω is the same in either medium (1 or 2).

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The BC 1) → BC 4) for a monochromatic plane *EM* wave incident on an interface at an oblique angle between two linear/homogeneous/isotropic media become:

BC 1): Normal (z-) component of ***D*** continuous at z = 0 (no free surface charges):

$$\boxed{\varepsilon_1\left(\tilde{E}_{o_{inc_z}}+\tilde{E}_{o_{refl_z}}\right)=\varepsilon_2\tilde{E}_{o_{trans_z}}} \qquad \left\{\text{using } \vec{D}=\varepsilon\vec{E}\right\}$$

BC 2): Tangential (x-, y-) components of ***E*** continuous at z = 0:

$$\boxed{\left(\tilde{E}_{o_{inc_{x,y}}}+\tilde{E}_{o_{refl_{x,y}}}\right)=\tilde{E}_{o_{trans_{x,y}}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3): Normal (z-) component of ***B*** continuous at $z = 0$:

$$\boxed{\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}}$$

BC 4): Tangential (x-, y-) components of **H** continuous at z = 0 (no free surface currents):

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_{x,y}}} + \tilde{B}_{o_{refl_{x,y}}}\right) = \frac{1}{\mu_2}\tilde{B}_{o_{trans_{x,y}}}}$$

Note that in each of the above, we also have the relation

$$\boxed{\vec{\tilde{B}}_o = \frac{1}{v}\hat{k} \times \vec{\tilde{E}}_o}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For a monochromatic plane EM wave incident on a boundary between two L / H/ I media at an oblique angle of incidence - three possible polarization cases to consider:

Case I): $\vec{E}_{inc} \perp$ plane of incidence

$\{\vec{B}_{inc} \parallel$ plane of incidence$\}$

Transverse Electric (TE) Polarization

Case II): $\vec{E}_{inc} \parallel$ plane of incidence

$\{\vec{B}_{inc} \perp$ plane of incidence$\}$

Transverse Magnetic (TM) Polarization

Case III): The most general case: $\vec{E}_{inc}$ is neither $\perp$ nor $\parallel$ to the plane of incidence.

$\{\Rightarrow \vec{B}_{inc}$ is neither $\parallel$ nor $\perp$ to the plane of incidence$\}$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Case I): Electric Field Vectors Perpendicular to the Plane of Incidence: Transverse Electric (*TE*) *Polarization*

•A monochromatic plane EM wave is incident on a boundary at z = 0 -in the x-y plane between two L/H/I media - at an oblique angle of incidence.

•The polarization of the incident EM wave is transverse (⊥) to the plane of incidence (containing the three wave-vectors and the unit normal to the boundary $\hat{n} = +\hat{z}$).

•The three B-field vectors are related to their respective E -field vectors by the right hand rule - all three B-field vectors lie in the x-z plane (the plane of incidence)

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The four boundary conditions on the complex *E* and *B* fields on the boundary at z = 0 are:

BC 1) Normal (*z-*) component of D continuous at z = 0 (no free surface charges)

$$\boxed{\varepsilon_1\left(\underbrace{\cancel{\tilde{E}_{o_{inc_z}}}}_{=0}+\underbrace{\cancel{\tilde{E}_{o_{refl_z}}}}_{=0}\right)=\varepsilon_2\,\underbrace{\cancel{\tilde{E}_{o_{trans_z}}}}_{=0}} \Rightarrow \boxed{0+0=0}$$

BC 2) Tangential (*x-, y-*) components of E continuous at *z = 0:*

$$\boxed{\left(\tilde{E}_{o_{inc_y}}+\tilde{E}_{o_{refl_y}}\right)=\tilde{E}_{o_{trans_y}}} \Rightarrow \boxed{\tilde{E}_{o_{inc}}+\tilde{E}_{o_{refl}}=\tilde{E}_{o_{trans}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3) Normal (*z-)* component of B continuous at z = 0:

$$\boxed{\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}}$$

$$\hat{k}_{inc} = \hat{k}_{inc_x} + \hat{k}_{inc_z} = \sin\theta_{inc}\hat{x} + \cos\theta_{inc}\hat{z}$$

$$\hat{k}_{refl} = \hat{k}_{refl_x} + \hat{k}_{refl_z} = \sin\theta_{refl}\hat{x} - \cos\theta_{refl}\hat{z}$$

$$\hat{k}_{trans} = \hat{k}_{trans_x} + \hat{k}_{trans_z} = \sin\theta_{trans}\hat{x} + \cos\theta_{trans}\hat{z}$$

$$\boxed{\left(\tilde{B}_{o_{inc_z}}\hat{z} + \tilde{B}_{o_{refl_z}}\hat{z}\right) = \tilde{B}_{o_{trans_z}}\hat{z}} = \boxed{\frac{1}{v_1}\left(\tilde{E}_{o_{inc}}\sin\theta_{inc} + \tilde{E}_{o_{refl}}\sin\theta_{refl}\right)\hat{z} = \frac{1}{v_2}\tilde{E}_{o_{trans}}\sin\theta_{trans}\hat{z}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Using the Law of Reflection on the BC 3) result:

$$\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \left(\frac{v_1}{v_2}\cdot\frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$$

Using Snell's Law / Law of Refraction:

$$\boxed{n_1\sin\theta_{inc} = n_2\sin\theta_{trans}} \Rightarrow \boxed{\frac{n_1}{c}\sin\theta_{inc} = \frac{n_2}{c}\sin\theta_{trans}} \Rightarrow \boxed{\frac{1}{v_1}\sin\theta_{inc} = \frac{1}{v_2}\sin\theta_{trans}}$$

or: $\boxed{v_2\sin\theta_{inc} = v_1\sin\theta_{trans}}$ or: $\boxed{\left(\frac{v_1}{v_2}\cdot\frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right) = 1}$

Reduces to BC2) $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 4) Tangential (*x*-, *y*-) components of H continuous at z = 0 (no free surface currents):

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_x}}\hat{x}+\tilde{B}_{o_{refl_x}}\hat{x}\right)=\frac{1}{\mu_2}\tilde{B}_{o_{trans_x}}\hat{x}}$$

$$=\boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}}(-\cos\theta_{inc})+\tilde{E}_{o_{refl}}\cos\theta_{refl}\right)\hat{x}=\frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}(-\cos\theta_{trans})\hat{x}}$$

Using the Law of Reflection on the BC 4) result:

$$\boxed{\left(\tilde{E}_{o_{inc}}-\tilde{E}_{o_{refl}}\right)=\left(\frac{\mu_1 v_1}{\mu_2 v_2}\cdot\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$$

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From BC 1) → BC 4) actually have only two independent relations for the case of transverse electric (TE) polarization:

1) $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$ 2) $\boxed{\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \left(\frac{\mu_1 v_1}{\mu_2 v_2} \cdot \frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$

Define: $\boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$ $\boxed{\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)}$

Then eqn. 2) becomes:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \alpha\beta\,\tilde{E}_{o_{trans}}}$$

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Adding and subtracting Eqn's 1 &2 to get:

$$\boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}} \quad \text{eqn. (1+2)} \qquad \boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\alpha\beta}{2}\right)\tilde{E}_{o_{trans}}} \quad \text{eqn. (2−1)}$$

Plug eqn. (2+1) into eqn. (2−1) to obtain:

$$\boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\alpha\beta}{2}\right)\left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}}$$

$$\boxed{\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)} \quad \text{and} \quad \boxed{\frac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}} = \left(\frac{2}{1+\alpha\beta}\right)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Fresnel Equations for $\vec{E}\,||$ to Interface

$=\vec{E}\perp$ Plane of Incidence = Transverse Electric (TE) Polarization

$$\boxed{E^{TE}_{o_{refl}}=\left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)E^{TE}_{o_{inc}}} \text{ and } \boxed{E^{TE}_{o_{trans}}=\left(\frac{2}{1+\alpha\beta}\right)E^{TE}_{o_{inc}}}$$

with $$\boxed{\alpha\equiv\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)} \text{ and } \boxed{\beta\equiv\left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For *TE polarization:*

Incident Intensity

$$I_{inc}^{TE} = \left|\left\langle \vec{S}_{inc}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E_{o_{inc}}^{TE}\right)^2\right)\left|\hat{k}_{inc}\bullet\hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E_{o_{inc}}^{TE}\right)^2\right)\cos\theta_{inc} = \frac{1}{2}\varepsilon_1 v_1\left(E_{o_{inc}}^{TE}\right)^2\cos\theta_{inc}$$

Reflection Intensity

$$I_{refl}^{TE} = \left|\left\langle \vec{S}_{refl}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E_{o_{refl}}^{TE}\right)^2\right)\cos\theta_{refl} = \frac{1}{2}\varepsilon_1 v_1\left(E_{o_{refl}}^{TE}\right)^2\cos\theta_{inc}$$

Transmission Intensity

$$I_{trans}^{TE} = \left|\left\langle \vec{S}_{trans}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_2\varepsilon_2\left(E_{o_{trans}}^{TE}\right)^2\right)\cos\theta_{trans} = \frac{1}{2}\varepsilon_2 v_2\left(E_{o_{trans}}^{TE}\right)^2\cos\theta_{trans}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission coefficients for transverse electric (*TE*) *polarization*

$$\boxed{R_{TE} \equiv \frac{I_{refl}^{TE}}{I_{inc}^{TE}} = \frac{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{refl}}^{TE}\right)^2 \cos \overbrace{\theta_{inc}}^{=\theta_{refl}}}{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos \theta_{inc}} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2}$$

$$\boxed{T_{TE} \equiv \frac{I_{trans}^{TE}}{I_{inc}^{TE}} = \frac{\frac{1}{2}\varepsilon_2 v_2 \left(E_{o_{trans}}^{TE}\right)^2 \cos \theta_{trans}}{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos \theta_{inc}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{\cos \theta_{trans}}{\cos \theta_{inc}}\right)\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The reflection and transmission coefficients for transverse electric (*TE*) polarization

$$\boxed{R_{TE} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2}$$

$$\boxed{T_{TE} = \alpha\beta\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

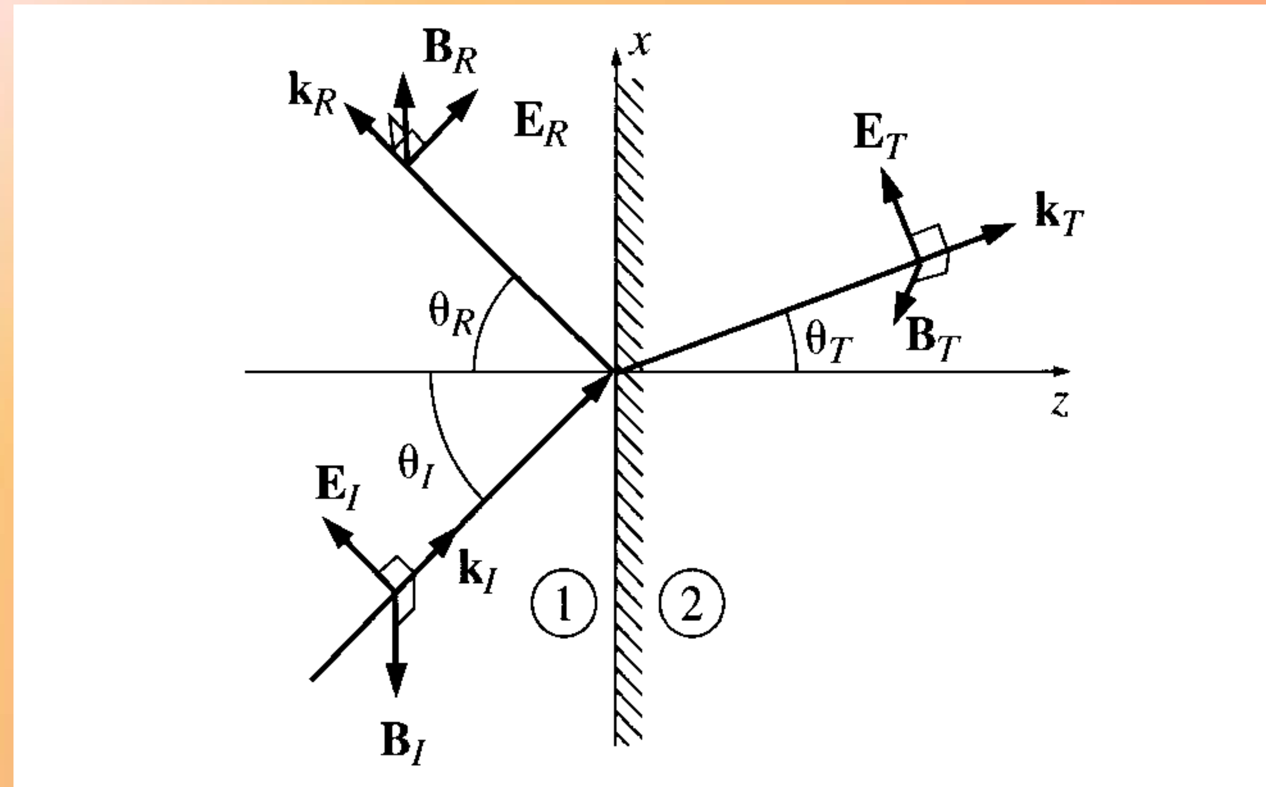
Case II): Electric Field Vectors Parallel to the Plane of Incidence: Transverse Magnetic (*TM*) *Polarization*

•A monochromatic plane EM wave is incident on a boundary at z = 0 in the x-y plane between two L / H/ I media at an oblique angle of incidence.

•The polarization of the incident EM wave is now parallel to the plane of incidence –(containing the three wave-vectors and the unit normal to the boundary $\hat{n} = +\hat{z}$).

• The three B -field vectors are related to E -field vectors by the right hand rule –then all three B-field vectors are ⊥ to the plane of incidence {hence the origin of the name transverse magnetic polarization}.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*



Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The four boundary conditions on the complex E and B-fields on the boundary at z = 0 are:

BC 1) Normal (z-) component of D continuous at z = 0 (no free surface charges)

$$\boxed{\varepsilon_1\left(\tilde{E}_{o_{inc_z}}+\tilde{E}_{o_{refl_z}}\right)=\varepsilon_2\tilde{E}_{o_{trans_z}}}$$

$$\boxed{\varepsilon_1\left(-\tilde{E}_{o_{inc}}\sin\theta_{inc}+\tilde{E}_{o_{refl}}\sin\theta_{refl}\right)=\varepsilon_2\left(-\tilde{E}_{o_{trans}}\sin\theta_{trans}\right)}$$

BC 2) Tangential (x-, y-) components of E continuous at *z = 0:*

$$\boxed{\left(\tilde{E}_{o_{inc_x}}+\tilde{E}_{o_{refl_x}}\right)=\tilde{E}_{o_{trans_x}}}$$

$$\boxed{\left(\tilde{E}_{o_{inc}}\cos\theta_{inc}+\tilde{E}_{o_{refl}}\cos\theta_{refl}\right)=\tilde{E}_{o_{trans}}\cos\theta_{trans}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3) Normal (**z-**) component of *B* continuous at *z = 0:*

$$\left(\underbrace{\cancel{\tilde{B}_{o_{inc_z}}}}_{}^{=0} + \cancel{\tilde{B}_{o_{refl_z}}}^{=0}\right) = \cancel{\tilde{B}_{o_{trans_z}}}^{=0} \;\Rightarrow\; \boxed{0+0=0}$$

BC 4) Tangential (x-, y-) components of H continuous at z = 0 (no free surface currents):

$$\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_y}} + \tilde{B}_{o_{refl_y}}\right) = \frac{1}{\mu_2}\left(\tilde{B}_{o_{trans_y}}\right) \;\Rightarrow\; \frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

From BC 1) at $z = 0$:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\varepsilon_2}{\varepsilon_1}\frac{n_1}{n_2}\right)\tilde{E}_{o_{trans}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\tilde{E}_{o_{trans}} = \beta\,\tilde{E}_{o_{trans}}}$$

From BC 4) at $z = 0$:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)\tilde{E}_{o_{trans}} = \beta\tilde{E}_{o_{trans}}}$$

where:

$$\boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

From BC 2) at $z = 0$:

$$\left(\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}}\right) = \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}} = \alpha\tilde{E}_{o_{trans}}$$

where:

$$\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}$$

Thus for the case of transverse magnetic (TM) polarization:

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\tilde{E}_{o_{trans}} \quad \text{and} \quad \tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \alpha\tilde{E}_{o_{trans}}$$

Solving these two above equations simultaneously, we obtain:

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

$$\boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{o_{inc}}} \qquad \boxed{\tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{2}\right)\tilde{E}_{o_{trans}}} \qquad \boxed{\tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{o_{inc}}}$$

The Fresnel Equations for $\vec{B} \parallel$ to Interface

$= \vec{B} \perp$ Plane of Incidence = Transverse Magnetic (*TM*) Polarization

$$\boxed{\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)} \quad \text{and} \quad \boxed{\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right) = \left(\frac{2}{\alpha+\beta}\right)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflected & transmitted intensities at oblique incidence for the *TM case*

$$I_{inc}^{TM} = v_1 \left| \left\langle \vec{S}_{inc}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_1 \varepsilon_1 \left(E_{o_{inc}}^{TM} \right)^2 \right) \cos\theta_{inc} = \frac{1}{2} \varepsilon_1 v_1 \left(E_{o_{inc}}^{TM} \right)^2 \cos\theta_{inc}$$

$$I_{refl}^{TM} = v_1 \left| \left\langle \vec{S}_{refl}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_1 \varepsilon_1 \left(E_{o_{refl}}^{TM} \right)^2 \right) \cos\theta_{refl} = \frac{1}{2} \varepsilon_1 v_1 \left(E_{o_{refl}}^{TM} \right)^2 \cos\theta_{inc}$$

$$I_{trans}^{TM} = v_2 \left| \left\langle \vec{S}_{trans}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_2 \varepsilon_2 \left(E_{o_{trans}}^{TM} \right)^2 \right) \cos\theta_{trans} = \frac{1}{2} \varepsilon_2 v_2 \left(E_{o_{trans}}^{TM} \right)^2 \cos\theta_{trans}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission coefficients

$$R_{TM} = \left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}} \right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^2$$

$$T_{TM} = \alpha\beta \left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}} \right)^2 = \frac{4\alpha\beta}{(\alpha + \beta)^2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Fresnel Equations

TE Polarization

$$\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)$$

$$\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{2}{(1+\alpha\beta)}$$

$$\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}$$

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}$$

TM Polarization

$$\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)$$

$$\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{2}{(\alpha+\beta)}$$

$$v_1 = c/n_1 = 1/\sqrt{\varepsilon_1\mu_1}$$

$$v_2 = c/n_2 = 1/\sqrt{\varepsilon_2\mu_2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission Coefficients *R & T*

$$R+T=1$$

TE Polarization

$$R_{TE}\equiv\frac{I_{refl}^{TE}}{I_{inc}^{TE}}=\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2=\left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$$

$$T_{TE}\equiv\left(\frac{I_{trans}^{TE}}{I_{inc}^{TE}}\right)=\alpha\beta\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2=\frac{4\alpha\beta}{(1+\alpha\beta)^2}$$

$$\alpha\equiv\frac{\cos\theta_{trans}}{\cos\theta_{inc}}$$

$$\beta\equiv\frac{\mu_1 v_1}{\mu_2 v_2}=\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}=\frac{\mu_1 n_2}{\mu_2 n_1}=\frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}$$

TM Polarization

$$R_{TM}\equiv\frac{I_{refl}^{TM}}{I_{inc}^{TM}}=\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$T_{TM}\equiv\left(\frac{I_{trans}^{TM}}{I_{inc}^{TM}}\right)=\alpha\beta\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2=\frac{4\alpha\beta}{(\alpha+\beta)^2}$$

$$v_1={c}/{n_1}={1}/{\sqrt{\varepsilon_1\mu_1}}$$

$$v_2={c}/{n_2}={1}/{\sqrt{\varepsilon_2\mu_2}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

- Now explore the physics associated with the Fresnel Equations -the reflection and transmission coefficients.
- Comparing results for TE vs. TM polarization for the cases of external reflection ($n_1 < n_2$) and internal reflection $n_1 > n_2$)

Comment 1):

- When $(E_{refl} / E_{inc}) < 0$ - E_{orefl} *is* 180^o out-of-phase with E_{oinc} since the numerators of the original Fresnel Equations for TE & TM polarization are $(1 - \alpha\beta)$ and $(\alpha - \beta)$ respectively.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Comment 2):

•For TM Polarization (only)- there exists an angle of incidence where $(E_{refl} / E_{inc}) = 0$ - no reflected wave occurs at this angle for TM polarization!

•This angle is known as Brewster's angle θ_B (also known as the polarizing angle θ_P - because an incident wave which is a linear combination of TE and TM polarizations will have a reflected wave which is 100% pure-TE polarized for an incidence angle $\theta_{inc} = \theta_B = \theta_P$!!).

•Brewster's angle θ_B exists for both external ($n_1 < n_2$) & internal reflection ($n_1 > n_2$) for TM polarization (only).

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Brewster's Angle θ_B / *the Polarizing Angle* θ_P *for Transverse Magnetic (TM) Polarization*

From the numerator of $\left(E_{o_{refl}}^{TM} / E_{o_{inc}}^{TM}\right) = \left(\dfrac{\alpha - \beta}{\alpha + \beta}\right)$ -the originally-derived expression for TM polarization- when this ratio = 0 at Brewster's angle θ_B = polarizing angle θ_P - this occurs when $(\alpha - \beta) = 0$, i.e. when $\alpha = \beta$.

$\cos\theta_{trans} = \sqrt{1 - \sin^2\theta_{trans}}$ and Snell's Law: $\sin\theta_{trans} = \left(\dfrac{n_1}{n_2}\right)\sin\theta_{inc}$

$$\boxed{\alpha = \frac{\sqrt{1 - \left(\dfrac{n_1}{n_2}\right)^2 \sin^2\theta_{inc}}}{\cos\theta_{inc}} \simeq \left(\frac{n_2}{n_1}\right) = \beta}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Brewster's Angle θ_B / *the Polarizing Angle* θ_P *for Transverse Magnetic (TM) Polarization*

$$\boxed{1-\frac{1}{\beta^2}\sin^2\theta_{inc}=\beta^2\cos^2\theta_{inc}=\beta^2\left(1-\sin^2\theta_{inc}\right)} \leftarrow \text{Solve for } \sin^2\theta_{inc}$$

$$\boxed{1-\beta^2=\left(\frac{1}{\beta^2}-\beta^2\right)\sin^2\theta_{inc}} \Rightarrow \boxed{\sin^2\theta_{inc}=\frac{1-\beta^2}{1/\beta^2-\beta^2}=\frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^4\right)}}$$

$$\boxed{1-\beta^4=\left(1-\beta^2\right)\left(1+\beta^2\right)}$$

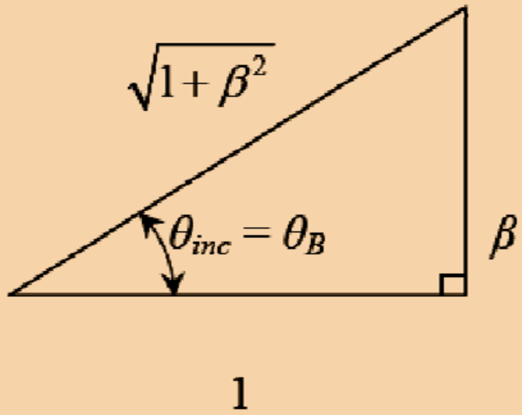
$$\boxed{\sin^2\theta_{inc}=\frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^2\right)\left(1+\beta^2\right)}=\frac{\beta^2}{1+\beta^2}} \Rightarrow \boxed{\sin\theta_{inc}=\frac{\beta}{\sqrt{1+\beta^2}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Geometrically: $\boxed{\sin\theta_{inc} = \frac{\beta}{\sqrt{1+\beta^2}}} = \boxed{\frac{\text{opp. side}}{\text{hypotenuse}}}$

$\boxed{\cos\theta_{inc} = \frac{1}{\sqrt{1+\beta^2}}} = \boxed{\frac{\text{adjacent}}{\text{hypotenuse}}}$

$\boxed{\tan\theta_{inc} = \beta} = \boxed{\frac{\text{opp. side}}{\text{adjacent}}} \simeq \left(\frac{n_2}{n_1}\right)$



Thus, at an angle of incidence $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ = Brewster's angle / the polarizing angle for a *TM* polarized incident wave, where *no reflected* wave exists, we have:

$$\boxed{\tan\theta_B^{inc} \equiv \tan\theta_P^{inc} \simeq \left(\frac{n_2}{n_1}\right)} \quad \text{for} \quad \mu_1 \simeq \mu_2 \simeq \mu_o$$

From Snell's Law: $n_1 \sin\theta_{inc} = n_2 \sin\theta_{trans}$ we also see that: $\tan\theta_B^{inc} = \frac{\sin\theta_B^{inc}}{\cos\theta_B^{inc}} \simeq \frac{n_2}{n_1}$

or: $n_1 \sin\theta_B^{inc} \simeq n_2 \cos\theta_B^{inc}$ for $\mu_1 \simeq \mu_2 \simeq \mu_o$.

Thus, from Snell's Law we see that: $\cos\theta_B^{inc} = \sin\theta_{trans}$ when $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

So what's so interesting about this???

Well: $\boxed{\cos\theta_B^{inc} = \sin\left(\tfrac{\pi}{2}-\theta_B^{inc}\right) = \sin\left(\tfrac{\pi}{2}\right)\cos\theta_B^{inc} - \cancel{\overset{=0}{\cos\left(\tfrac{\pi}{2}\right)}}\sin\theta_B^{inc} = \sin\theta_{trans}}$ *i.e.* $\boxed{\sin\left(\frac{\pi}{2}-\theta_B^{inc}\right) = \sin\theta_{trans}}$

$\therefore$ When $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ for an incident *TM*-polarized *EM* wave, we see that $\theta_{trans} = \pi/2 - \theta_B^{inc}$

Thus: $\theta_B^{inc} + \theta_{trans} = \pi/2$, *i.e.* $\theta_B^{inc} \equiv \theta_P^{inc}$ and θ_{trans} are *complimentary* angles !!!

Comment 3):

For internal reflection ($n_1 > n_2$) there exists a critical angle of incidence past which no transmitted beam exists for either TE or TM polarization. The critical angle does not depend on polarization – it is actually defined by Snell's Law:

$\boxed{n_1\sin\theta_{critical}^{inc} = n_2\sin\theta_{trans}^{\max} = n_2\sin\left(\frac{\pi}{2}\right) = n_2}$ or: $\sin\theta_{critical}^{inc} = \left(\frac{n_2}{n_1}\right)$ or: $\theta_{critical}^{inc} = \sin^{-1}\left(\frac{n_2}{n_1}\right)$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For $\theta_{inc} \geq \theta_{critical}^{inc}$, no transmitted beam exists → incident beam is totally internally reflected.

For $\theta_{inc} > \theta_{critical}^{inc}$, the transmitted wave is actually exponentially damped – becomes a so-called:

Evanescent Wave:

$$\boxed{\tilde{\vec{E}}_{trans}(\vec{r},t) = \tilde{\vec{E}}_{o_{trans}}\underbrace{e^{-\alpha z}}\underbrace{e^{i\left(k_2 x \sin\theta_{inc}\left(\frac{n_1}{n_2}\right)-\omega t\right)}}}$$

$$\boxed{\alpha = k_2\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_{inc} - 1}}$$

Exp. damping in z Oscillatory along <u>interface</u> in x-direction

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Brewster's angle for *TE polarization:*

$$\theta^{B}_{\substack{inc\\TE}} = \sin^{-1}\sqrt{\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right)-\left(\frac{\mu_2}{\mu_1}\right)}} = \sin^{-1}\sqrt{A}$$

$$\sin\theta^{B}_{inc} = \sqrt{\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right)-\left(\frac{\mu_2}{\mu_1}\right)}} \equiv \sqrt{A} \quad i.e. \quad A \equiv \left[\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right)-\left(\frac{\mu_2}{\mu_1}\right)}\right]$$

ELECTROMAGNETIC WAVES IN CONDUCTORS

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ELECTROMAGNETIC WAVES IN CONDUCTORS

- Free charge and free currents are zero for propagation through a vacuum or insulating materials such as glass or pure water.
- Inside a conductor- free charges can move around in response to *EM* fields contained therein- free current is not zero.
- Assume that the conductor is linear/homogeneous/ isotropic media.
- From Ohm's Law $$\vec{J}_{free}(\vec{r},t) = \sigma_C \vec{E}(\vec{r},t)$$

where σ_C = conductivity of the metal conductor (Ohm^{-1}/m) *and* $\sigma_C = 1/\rho_C$ where ρ_C = resistivity of the metal conductor (*Ohm-m*).

ELECTROMAGNETIC WAVES IN CONDUCTORS

Assume that the linear/ homogeneous/isotropic conducting medium has electric permittivity ε and magnetic permeability μ. Maxwell's equations inside such a conductor are thus:

1) $\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r},t)=\rho_{free}(\vec{r},t)/\varepsilon}$

2) $\boxed{\vec{\nabla}\bullet\vec{B}(\vec{r},t)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}}$

Using Ohm's Law:
$\vec{J}_{free}(\vec{r},t)=\sigma_C\vec{E}(\vec{r},t)$

4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu\vec{J}_{free}(\vec{r},t)+\mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}=\mu\sigma_C\vec{E}(\vec{r},t)+\mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

ELECTROMAGNETIC WAVES IN CONDUCTORS

Electric charge is conserved- thus the continuity equation inside the conductor is:

$$\boxed{\vec{\nabla}\bullet\vec{J}_{free}(\vec{r},t) = -\frac{\partial \rho_{free}(\vec{r},t)}{\partial t}}$$

but: $\boxed{\vec{J}_{free}(\vec{r},t) = \sigma_C \vec{E}(\vec{r},t)}$

$$\boxed{\sigma_C\left(\vec{\nabla}\bullet\vec{E}(\vec{r},t)\right) = -\frac{\partial \rho_{free}(\vec{r},t)}{\partial t}}$$

but: $\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r},t) = \rho_{free}(\vec{r},t)\Big/\varepsilon}$

thus:

$$\boxed{\frac{\sigma_C \rho_{free}(\vec{r},t)}{\varepsilon} = -\frac{\partial \rho_{free}(\vec{r},t)}{\partial t}}$$

or: $\boxed{\frac{\partial \rho_{free}(\vec{r},t)}{\partial t} + \left(\frac{\sigma_C}{\varepsilon}\right)\rho_{free}(\vec{r},t) = 0}$

1st order linear, homogeneous differential equation

ELECTROMAGNETIC WAVES IN CONDUCTORS

The general solution of this differential equation for the free charge density is of the form:

$$\boxed{\rho_{free}(\vec{r},t)=\rho_{free}(\vec{r},t=0)e^{-\sigma_C t/\varepsilon}=\rho_{free}(\vec{r},t=0)e^{-t/\tau_{relax}}}$$

A damped exponential!!!

The continuity equation inside a conductor tells us that any free charge density initially present at time t = 0 is exponentially damped in a characteristic time $\tau_{relax} \equiv \varepsilon/\sigma_C$ = charge relaxation time.

ELECTROMAGNETIC WAVES IN CONDUCTORS

Maxwell's equations for a ***charge-equilibrated conductor***

1) $\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r},t)=0}$

2) $\boxed{\vec{\nabla}\bullet\vec{B}(\vec{r},t)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}}$

4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu\sigma_C\vec{E}(\vec{r},t)+\mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}=\mu\left(\sigma_C\vec{E}(\vec{r},t)+\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}\right)}$

ELECTROMAGNETIC WAVES IN CONDUCTORS

These equations are different from the previous derivation(s) of monochromatic plane EM waves propagating in free space and in linear/homogeneous/ isotropic non-conducting materials. Re-derive the wave equations for ***E and B.*** Apply ∇ × ( ) to equations 3) and 4):

We get $$\boxed{\nabla^2\vec{E}(\vec{r},t) = \mu\varepsilon\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2} + \mu\sigma_c\frac{\partial\vec{E}(\vec{r},t)}{\partial t}}$$

and $$\boxed{\nabla^2\vec{B}(\vec{r},t) = \mu\varepsilon\frac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2} + \mu\sigma_c\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$$

ELECTROMAGNETIC WAVES IN CONDUCTORS

General solution(s) - are usually in the form of an oscillatory function times a damping term (*a decaying exponential)* – in the direction of the propagation of the EM wave. A complex plane-wave type solutions for E and B associated with the above wave equation(s) are of the general form:

$$\boxed{\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{i(\tilde{k}z-\omega t)}}$$

$$\boxed{\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_o e^{i(\tilde{k}z-\omega t)} = \left(\frac{\tilde{k}}{\omega}\right)\hat{k}\times\tilde{\vec{E}}(z,t) = \frac{1}{\omega}\tilde{\vec{k}}\times\tilde{\vec{E}}(z,t)}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

With (frequency-dependent) complex wave number:

$$\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$$

$$\boxed{k(\omega) = \Re e\left(\tilde{k}(\omega)\right) = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}}$$

$$\boxed{\kappa(\omega) = \Im m\left(\tilde{k}(\omega)\right) = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2}}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The imaginary part of k that is - $\kappa = \Im m(k)$ results in an exponential damping of the monochromatic plane *EM* wave with increasing z*:*

$$\boxed{\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz-\omega t)}}$$

$$\boxed{\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_o e^{-\kappa z} e^{i(kz-\omega t)} = \frac{1}{\omega}\tilde{\vec{k}} \times \tilde{\vec{E}}(z,t) = \frac{1}{\omega}\tilde{\vec{k}} \times \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz-\omega t)}}$$

These solutions satisfy the above wave equations for any choice $\tilde{\vec{E}}_o$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The characteristic distance over which E and B are reduced to 1/e=0.3679- of their initial values (at $z = 0$) is known as the skin depth

$$\boxed{\delta_{sc}(\omega) \equiv 1/\kappa(\omega)}$$

$$\boxed{\delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} = \frac{1}{\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2}}} \Rightarrow \boxed{\begin{array}{l} \tilde{\vec{E}}(z=\delta_{sc},t) = \tilde{\vec{E}}_o e^{-1} e^{i(kz-\omega t)} \\ \tilde{\vec{B}}(z=\delta_{sc},t) = \tilde{\vec{B}}_o e^{-1} e^{i(kz-\omega t)} \end{array}}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

- The above plane wave solutions satisfy the above wave equations(s).
- Maxwell's equations rule out the presence of any longitudinal i.e, z- component of E and B.
- E and B are purely transverse waves (as before)- even in a conductor.
- Consider a linearly polarized monochromatic plane EM wave propagating in the +zˆ -direction in a conducting medium.

$$\tilde{\vec{E}}(z,t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{x}$$

then

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

$$\boxed{\tilde{\vec{B}}(z,t) = \frac{1}{\omega}\tilde{\vec{k}} \times \tilde{\vec{E}}(z,t) = \left(\frac{\tilde{k}}{\omega}\right)\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y} = \left(\frac{k+i\kappa}{\omega}\right)\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}}$$

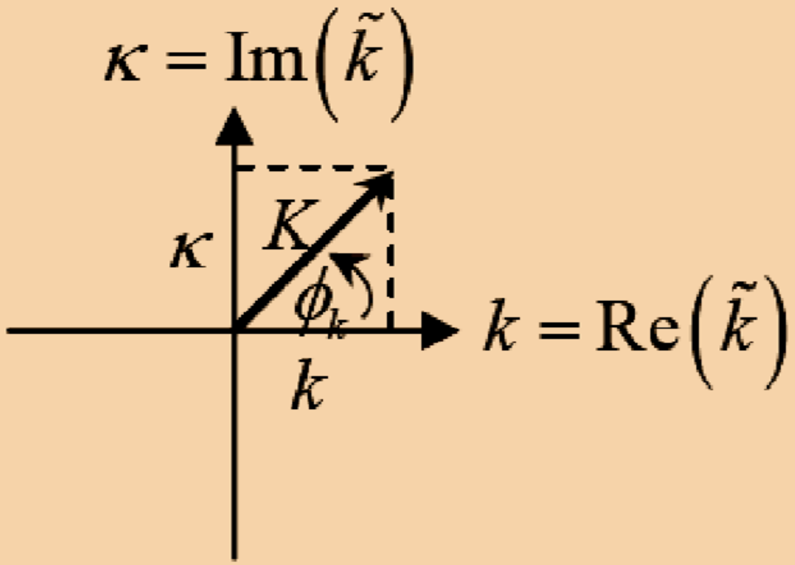
$$\Rightarrow \quad \tilde{\vec{E}}(z,t) \perp \tilde{\vec{B}}(z,t) \perp \hat{z} \quad (+\hat{z} = \text{propagation direction})$$

The complex wave-number $\tilde{k} = k + ik = Ke^{i\phi}$

$$\text{where: } K \equiv \left|\tilde{k}\right| = \sqrt{k^2+\kappa^2} \quad \text{and} \quad \phi_k \equiv \tan^{-1}\left(\kappa/k\right)$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

In the complex $\tilde{k}$-plane:



$$K = \left|\tilde{k}\right| = \sqrt{k^2 + \kappa^2}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

Then we see that: $$\boxed{\tilde{\vec{E}}(z,t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{x}}$$

has $$\boxed{\tilde{E}_o = E_o e^{i\delta_E}}$$

$$\boxed{\tilde{\vec{B}}(z,t) = \tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)}\hat{y} = \frac{\tilde{k}}{\omega}\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}}$$

$$\boxed{\tilde{k} = K e^{i\phi_k}}$$

has $$\boxed{\tilde{B}_o = B_o e^{i\delta_B} = \frac{\tilde{k}}{\omega}\tilde{E}_o = \frac{K e^{i\phi_k}}{\omega} E_o e^{i\delta_E}}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

$$\boxed{B_o e^{i\delta_B} = \frac{K e^{i\phi_k}}{\omega} E_o e^{i\delta_B} = \frac{K}{\omega} E_o e^{i(\delta_E + \phi_k)} = \frac{\sqrt{k^2 + \kappa^2}}{\omega} E_o e^{i(\delta_E + \phi_k)}}$$

inside a conductor, **E** and **B** are no longer in phase with each other!!!

Phases of ***E*** *and* ***B*** $\boxed{\delta_B = \delta_E + \phi_k}$

With phase difference: $\boxed{\Delta\varphi_{B-E} \equiv \delta_B - \delta_E = \phi_k}$

We also see that: $$\boxed{\frac{B_o}{E_o} = \frac{K}{\omega} = \left[\varepsilon\mu\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} \neq \frac{1}{c}}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The real/physical ***E*** and ***B*** fields associated with linearly polarized monochromatic plane *EM* waves propagating in a conducting medium are exponentially damped:

$$\boxed{\vec{E}(z,t) = \Re e\left(\tilde{\vec{E}}(z,t)\right) = E_o e^{-\kappa z}\cos\left(kz - \omega t + \delta_E\right)\hat{x}} \qquad \boxed{\delta_B = \delta_E + \phi_k}$$

$$\boxed{\vec{B}(z,t) = \Re e\left(\tilde{\vec{B}}(z,t)\right) = B_o e^{-\kappa z}\cos\left(kz - \omega t + \delta_B\right)\hat{y} = B_o e^{-\kappa z}\cos\left(kz - \omega t + \{\delta_E + \phi_k\}\right)\hat{y}}$$

$$\boxed{\frac{B_o}{E_o} = \frac{K(\omega)}{\omega} = \left[\varepsilon\mu\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{\frac{1}{2}}}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

where $$\boxed{K(\omega) \equiv \left|\tilde{k}(\omega)\right| = \sqrt{k^2(\omega) + \kappa^2(\omega)} = \omega\left[\varepsilon\mu\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\,\right]^{1/2}}$$

$$\boxed{\delta_B = \delta_E + \phi_k}, \quad \boxed{\phi_k(\omega) \equiv \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right)}$$

and $$\boxed{\tilde{k}(\omega) = \left|\tilde{\vec{k}}(\omega)\right| = k(\omega) + i\kappa(\omega)}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The real part of *k*- determines the spatial wavelength λ (ω)-the propagation speed v(ω) and also the index of refraction

$$\boxed{\lambda(\omega) = \frac{2\pi}{k(\omega)} = \frac{2\pi}{\Re e\left(\tilde{k}(\omega)\right)}}$$

$$\boxed{v(\omega) = \frac{\omega}{k(\omega)} = \frac{\omega}{\Re e\left(\tilde{k}(\omega)\right)}}$$

$$\boxed{n(\omega) = \frac{c}{v(\omega)} = \frac{ck(\omega)}{\omega} = \frac{c\Re e\left(\tilde{k}(\omega)\right)}{\omega}}$$

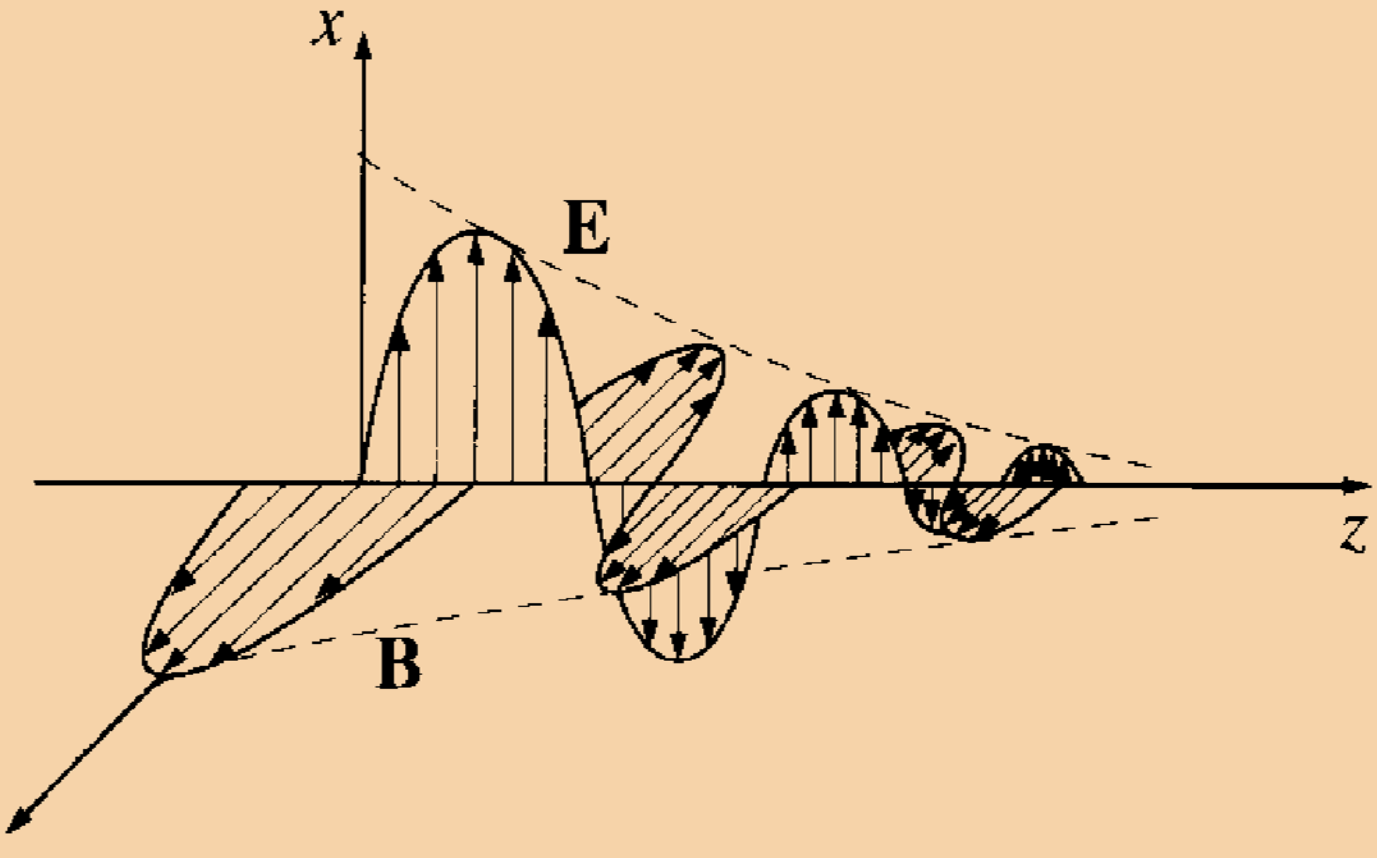
MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

Definition of the ***skin depth in a conductor:***

$$\boxed{\delta_{sc}(\omega) \equiv \frac{1}{\kappa(\omega)} = \frac{1}{\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2}}}$$

$=$ Distance over which the $\vec{E}$ and $\vec{B}$ fields fall to $1/e = e^{-1} = 0.3679$ of their initial values.

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA



Reflection of EM Waves at Normal Incidence from a Conducting Surface

In the presence of free surface charges σ and free surface currents- the BC's for reflection and refraction at *e.g.* a dielectric-conductor interface become:

BC 1): (normal ***D*** at interface): $\boxed{\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_{free}}$

BC 2): (tangential ***E*** at interface): $\boxed{E_1^{\parallel} - E_2^{\parallel} = 0} \Rightarrow \boxed{E_1^{\parallel} = E_2^{\parallel}}$

BC 3): (normal ***B*** at interface): $\boxed{B_1^{\perp} - B_2^{\perp} = 0} \Rightarrow \boxed{B_1^{\perp} = B_2^{\perp}}$

BC 4): (tangential ***H*** at interface): $\boxed{\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = \vec{K}_{free} \times \hat{n}_{\underset{21}{\rightarrow}}}$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

$\perp$ = normal to plane of interface
$||$ = parallel to plane of interface

Where $n_{21}\rightarrow$ is a unit vector $\perp$ to the interface - pointing from medium (2) into medium (1).

Incident *EM* wave [medium (1)]:

$$\tilde{\vec{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{x}$$

and

$$\tilde{\vec{B}}_{inc}(z,t) = \frac{1}{v_1} \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{y}$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Reflected *EM* wave [medium (1)]*:*

$$\boxed{\tilde{\vec{E}}_{refl}\left(z,t\right)=\tilde{E}_{o_{refl}}e^{i(-k_1 z-\omega t)}\hat{x}}$$

and

$$\boxed{\tilde{\vec{B}}_{refl}\left(z,t\right)=-\frac{1}{v_1}\tilde{E}_{o_{refl}}e^{i(-k_1 z-\omega t)}\hat{y}}$$

Transmitted *EM* wave [medium (2)]*:*

$$\boxed{\tilde{\vec{E}}_{trans}\left(z,t\right)=\tilde{E}_{o_{trans}}e^{i\left(\tilde{k}_2 z-\omega t\right)}\hat{x}}$$

and

$$\boxed{\tilde{\vec{B}}_{trans}\left(z,t\right)=\frac{\tilde{k}_2}{\omega}\tilde{E}_{o_{trans}}e^{i\left(\tilde{k}_2 z-\omega t\right)}\hat{y}}$$

complex wave-number in (conducting) medium (2) is:

$$\boxed{\tilde{k}_2=k_2+i\kappa_2}$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

In medium (1) EM fields are:

$$\boxed{\vec{\tilde{E}}_{Tot_1}(z,t)=\vec{\tilde{E}}_{inc}(z,t)+\vec{\tilde{E}}_{refl}(z,t)} \quad \boxed{\vec{\tilde{B}}_{Tot_1}(z,t)=\vec{\tilde{B}}_{inc}(z,t)+\vec{\tilde{B}}_{refl}(z,t)}$$

In medium (2) EM fields are:

$$\boxed{\vec{\tilde{E}}_{Tot_2}(z,t)=\vec{\tilde{E}}_{trans}(z,t)} \quad \underline{\text{and}}\text{:} \quad \boxed{\vec{\tilde{B}}_{Tot_2}(z,t)=\vec{\tilde{B}}_{trans}(z,t)}$$

Reflection of EM Waves at Normal Incidence from a Conducting Surface

Apply BC′s at the z = 0 interface in the x-y plane:

BC 1): $\boxed{\varepsilon_1 E_1^\perp - \varepsilon_2 E_2^\perp = \sigma_{free}}$ but $\boxed{E_1^\perp = \tilde{E}_{1_z} = 0}$ **and**: $\boxed{E_2^\perp = \tilde{E}_{2_z} = 0}$

$$\boxed{0 - 0 = \sigma_{free}} \Rightarrow \boxed{\sigma_{free} = 0}$$

BC 2): $\boxed{E_1^{\parallel} = E_2^{\parallel}}$ $\therefore$ $\boxed{\tilde{E}_{0_{inc}} + \tilde{E}_{0_{refl}} = \tilde{E}_{0_{trans}}}$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

BC 3): $\boxed{B_1^{\perp} = B_2^{\perp}}$ but: $\boxed{B_1^{\perp} = B_{1_z} = 0}$ and: $\boxed{B_2^{\perp} = B_{2_z} = 0} \Rightarrow \boxed{0 = 0}$

BC 4): $\boxed{\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = \vec{K}_{free} \times \hat{n}_{\vec{21}}}$ but: $\boxed{\vec{K}_{free} = 0}$ $\therefore$ $\boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) - \frac{\tilde{k}_2}{\mu_2 \omega} \tilde{E}_{o_{trans}} = 0}$

or: $\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \tilde{\beta} \tilde{E}_{o_{trans}}}$ with: $\boxed{\tilde{\beta} \equiv \left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2 \omega}\right) = \left(\frac{\mu_1 v_1}{\mu_2 \omega}\right) \tilde{k}_2}$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Thus we obtain:

$$\boxed{\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)=\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)} \quad \underline{\text{and}}: \quad \boxed{\left(\frac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}}\right)=\frac{2}{\left(1+\tilde{\beta}\right)}}$$

with

$$\boxed{\tilde{\beta}\equiv\left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2 \omega}\right)=\left(\frac{\mu_1 v_1}{\mu_2 \omega}\right)\tilde{k}_2}$$

The relations for reflection/transmission of EMW at normal incidence on a non-conductor/conductor boundary are identical to those obtained for reflection / transmission of EMW at normal incidence on a boundary/interface between two non-conductors- except for the replacement of β with a complex β.

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

For the case of a perfect conductor- the conductivity

$$\sigma_C = \infty \quad \{\text{thus resistivity}, \rho_C = 1/\sigma_C = 0\}$$

$\Rightarrow$ <u>*both*</u> $\boxed{k_2 \simeq \kappa_2 \simeq \sqrt{\dfrac{\omega\mu_2\sigma_C}{2}} = \infty}$ and since: $\boxed{\tilde{k}_2 = k_2 + i\kappa_2}$ then: $\boxed{\tilde{k}_2 = \infty + i\infty = \infty(1+i)}$

and since: $\boxed{\tilde{\beta} \equiv \left(\dfrac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\dfrac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2}$ $\Rightarrow \underline{\underline{\tilde{\beta} = \infty}}$

Thus, for a perfect conductor, we see that:

$$\boxed{\tilde{E}_{o_{refl}} = -\tilde{E}_{o_{inc}}} \text{ and } \boxed{\tilde{E}_{trans} = 0}$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

For a perfect conductor the reflection and transmission coefficients are:

$$\boxed{R \equiv \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)^* = 1} \quad \underline{\text{and}}: \quad \boxed{T = 1 - R = 0}$$

We also see that for a perfect conductor - for normal incidence- the reflected wave undergoes a 180 degree phase shift with respect to the incident wave at the interface at $z = 0$ in the x-y plane. A perfect conductor screens out all *EM* waves from propagating in its interior.

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

For a good conductor- the conductivity is large- but finite. The reflection coefficient R for monochromatic plane EM waves at normal incidence on a good conductor is not unity- but close to it. *{This is why good conductors make good mirrors!}.*

$$R \equiv \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)^* = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^*$$

Where

$$\tilde{\beta} = \left(\frac{\mu_1 v_1}{\mu_2 \omega}\right)\tilde{k}_2 = \left(\frac{\mu_1 v_1}{\mu_2 \omega}\right)\sqrt{\frac{\omega \mu_2 \sigma_C}{2}}(1+i) = \mu_1 v_1 \sqrt{\frac{\sigma_C}{2\mu_2 \omega}}(1+i)$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Define $\boxed{\gamma \equiv \mu_1 v_1 \sqrt{\dfrac{\sigma_C}{2\mu_2 \omega}}}$ Then: $\boxed{\tilde{\beta} = \gamma(1+i)}$

Thus, the reflection coefficient R for monochromatic plane EM waves at normal incidence on a good conductor is:

$$\boxed{R = \left|\frac{\tilde{E}_{0_{refl}}}{\tilde{E}_{0_{inc}}}\right|^2 = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^* = \left(\frac{1-\gamma-i\gamma}{1+\gamma+i\gamma}\right)\left(\frac{1-\gamma+i\gamma}{1+\gamma-i\gamma}\right) = \left[\frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2}\right]}$$

with $\boxed{\gamma \equiv \mu_1 v_1 \sqrt{\dfrac{\sigma_C}{2\mu_2 \omega}}}$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Obviously, only a small fraction of the normally-incident monochromatic plane EM wave *is* transmitted into the good conductor- since $R < 1$ and $\boxed{T = 1 - R}$, *i.e.*:

$$\boxed{T = 1 - R = 1 - \left[\frac{(1-\gamma)^2 + \gamma^2}{(1+\gamma)^2 + \gamma^2}\right] \quad (\ll 1)}$$

Note that the transmitted wave is exponentially attenuated in the *z*-direction; the E and *B* fields in the good conductor fall to 1/e of their initial {z = 0} values (at/on the interface) after the monochromatic plane *EM* wave propagates a distance of one skin depth in z into the conductor:

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

$$\boxed{\delta_{sc}(\omega) \equiv \frac{1}{\kappa_2(\omega)} \simeq \sqrt{\frac{2}{\omega\mu_2\sigma_C}}}$$

Note also that the energy associated with the transmitted monochromatic plane *EM* wave is ultimately dissipated in the conducting medium as heat.

In metals - the transmitted wave is absorbed in the metal- we can only study the reflection coefficient *R*.

A full description of the physics of reflection from the surface of a metal conductor as a function of angle of incidence- requires the use of a complex dispersion relation

Full Maxwell Equations in Matter

The electromagnetic state of matter at a given observation point ***r*** at a given time t is described by four macroscopic quantities:

1.) The volume density of free charge: $\boxed{\rho_{free}(\vec{r},t)}$

2.) The volume density of electric dipoles: $\boxed{\vec{\mathrm{P}}(\vec{r},t)}$ ⇐ electric polarization

3.) The volume density of magnetic dipoles: $\boxed{\vec{\mathrm{M}}(\vec{r},t)}$ ⇐ magnetization

4.) The free electric current / unit area: $\boxed{\vec{J}_{free}(\vec{r},t)}$ ⇐ {free} current density

Full Maxwell Equations in Matter

These four quantities are related to the macroscopic ***E and B*** fields by the four Maxwell equations for matter

1) Gauss' Law: $\boxed{\vec{\nabla}\bullet\vec{E} = \frac{\rho_{Tot}}{\varepsilon_o} = \frac{1}{\varepsilon_o}\left(\rho_{free} + \rho_{bound}\right)}$, where: $\boxed{\rho_{bound} = -\vec{\nabla}\bullet\vec{P}}$

Auxiliary relation: $\boxed{\vec{D} = \varepsilon_o\vec{E} + \vec{P}}$ & constitutive relation: $\boxed{\vec{D} = \varepsilon\vec{E}}$

Electric polarization $\boxed{\vec{P} = (\varepsilon - \varepsilon_o)\vec{E} = \varepsilon_0\chi_e\vec{E}}$, electric susceptibility $\boxed{\chi_e = \left(\frac{\varepsilon}{\varepsilon_o} - 1\right)}$

$$\boxed{\vec{\nabla}\bullet\vec{D} = \varepsilon_o\vec{\nabla}\bullet\vec{E} + \vec{\nabla}\bullet\vec{P} = \rho_{free}}$$

2) No magnetic charges/monopoles: $\boxed{\vec{\nabla}\bullet\vec{B} = 0}$

Auxiliary relation: $\boxed{\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}}$ $\Rightarrow$ $\boxed{\vec{\nabla}\bullet\vec{H} = -\vec{\nabla}\bullet\vec{M}}$ & constitutive relation: $\boxed{\vec{B} = \mu\vec{H}}$

Full Maxwell Equations in Matter

3) <u>Faraday's Law</u>: $$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu_0\frac{\partial\vec{H}}{\partial t} - \mu_0\frac{\partial\vec{M}}{\partial t}}$$

Magnetization: $\boxed{\vec{M} - \left(\frac{\mu}{\mu_o - 1}\right)\vec{H} = \chi_m\vec{H}}$, magnetic susceptibility $\boxed{\chi_m = \left(\frac{\mu}{\mu_o} - 1\right)}$

4) <u>Ampere's Law</u>: $\boxed{\vec{\nabla}\times\vec{B} = \mu_o\vec{J}_{ToT} + \mu_o\vec{J}_D}$ with $\boxed{\vec{J}_D = \varepsilon_o\frac{\partial\vec{E}}{\partial t}}$

Total current density: $\boxed{\vec{J}_{ToT} = \vec{J}_{free} + \vec{J}^{mag}_{bound} + \vec{J}^{P}_{bound}}$ $\boxed{\vec{J}^{mag}_{bound} = \vec{\nabla}\times\vec{M}}$ $\boxed{\vec{J}^{P}_{bound} = \frac{\partial\vec{P}}{\partial t}}$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_o J_{free} + \mu_o\vec{\nabla}\times\vec{M} + \mu_o\frac{\partial\vec{P}}{\partial t} + \mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}}$$

$$\boxed{\vec{\nabla}\times\vec{H} = \mu_o\vec{J}_{free} + \mu_o\frac{\partial\vec{D}}{\partial t}}$$

Full Maxwell Equations in Matter

Then Maxwell's equations in matter, for $\rho_{free} = 0$ and $\vec{M} = 0$

1) Gauss' Law: $\boxed{\vec{\nabla} \bullet \vec{D} = 0}$ or: $\boxed{\vec{\nabla} \bullet \vec{E} = -\frac{1}{\varepsilon_o} \vec{\nabla} \bullet \vec{P} = \rho_{free} / \varepsilon_o}$

2) No magnetic charges: $\boxed{\vec{\nabla} \bullet \vec{B} = 0}$

3) Faraday's Law: $\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

4) Ampere's Law: $\boxed{\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} + \mu_o \frac{\partial \vec{P}}{\partial t} + \mu_o \vec{J}_{free}}$

Full Maxwell Equations in Matter

We also have Ohm's Law $\boxed{\vec{J}_{free} = \sigma_c \vec{E}}$

and the Continuity eqn. $\boxed{\vec{\nabla} \bullet \vec{J}_{free} = 0}$

Then applying the curl operator to Faraday's Law:

We thus obtain the inhomogeneous wave equation:

$$\boxed{\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \underbrace{\frac{1}{\varepsilon_o}\nabla \rho_{bound} + \mu_o \frac{\partial^2 \vec{P}}{\partial t^2} + \mu_o \frac{\partial \vec{J}_{free}}{\partial t}}_{\text{source terms}}}$$

{and a similar one for ***B*** }

Full Maxwell Equations in Matter

For non-conducting or poorly-conducting media, i.e. insulators/ dielectrics- the first two terms on the RHS are important – they explain many optical effects such as dispersion (frequency-dependence of the index of refraction), absorption . . .

Note that the $\vec{\nabla}\rho_{bound} = -\vec{\nabla}\left(\vec{\nabla}\bullet\vec{\mathrm{P}}\right)$ term is often zero- P uniform

$$\vec{\nabla}\bullet\vec{\mathrm{P}} = \frac{\partial \mathrm{P}_x}{\partial x} + \frac{\partial \mathrm{P}_y}{\partial y} + \frac{\partial \mathrm{P}_z}{\partial z} \text{ and } \vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

e.g. for $\vec{\mathrm{P}} \propto \vec{E}$ (i.e. $\vec{\mathrm{P}}$ proportional to $\vec{E}$) where: $\vec{E}(z,t) = E_o \cos(kz - \omega t + \delta)\hat{x}$

Full Maxwell Equations in Matter

For good conductors (e.g. metals), the conduction term

$$\boxed{\mu_o \frac{\partial \vec{J}_{free}}{\partial t} = \mu_o \sigma_C \frac{\partial \vec{E}}{\partial t}}$$

is the most important, because it explains the opacity of metals (e.g. in the visible light region) and also explains the high reflectance of metals.

DISPERSION

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Dispersion Phenomena in Linear Dielectrics

In a non-conducting, linear, homogeneous, isotropic medium there are no *free electrons*

$$(i.e.\ \rho_{free}(\vec{r}) = 0)$$

Atomic electrons are permanently bound to nuclei of atoms comprising the medium. There exist no preferential directions in such an {isotropic} medium.

Suppose each atomic electron (charge –e) in a dielectric is displaced by a small distance $\vec{r}$ from its equilibrium position, e.g. by application of a static electric field $\vec{E}(\vec{r})$.

Dispersion Phenomena in Linear Dielectrics

The resulting macroscopic electric polarization (aka electric dipole moment per unit volume) is:

$$\boxed{\vec{P}(\vec{r}) = n_e \vec{p}(\vec{r})}$$

where:

$n_e =$ (atomic) electron number density $\left(\#e/m^3\right)$

and the induced atomic (molecular) electric dipole moment is:

$\vec{p}(\vec{r}) = -e\vec{r}$ (here- where $\vec{r}$ is the vector displacement of the atomic electron from its equilibrium $\{\vec{E} = 0\}$ position.

Dispersion Phenomena in Linear Dielectrics

Thus:

$$\boxed{\vec{\mathrm{P}}(\vec{r}) = n_e\vec{p}(\vec{r}) = -n_e e\vec{r}}$$

The atomic electrons are each elastically bound to their equilibrium positions with a force constant k_e. The force equation for each atomic electron is thus:

$$\vec{F}_e(\vec{r}) = -e\vec{E}(\vec{r}) = k_e\vec{r}$$

Dispersion Phenomena in Linear Dielectrics

The static polarization is therefore given by:

$$\vec{\mathrm{P}}(\vec{r}) = n_e \vec{p}(\vec{r}) = -n_e e\vec{r} = -n_e e\left(\frac{-e\vec{E}(\vec{r})}{k_e}\right) = +\frac{n_e e^2}{k_e}\vec{E}(\vec{r})$$

If the **E**-field varies with time

$$\tilde{\vec{E}} = \tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_o e^{i(kz-\omega t)}$$

Due to a monochromatic EM plane wave incident on an atom- the above relation is incorrect !

Lorentz Model

A more correct way to treat this situation is to consider the bound atomic electrons as classical, damped, forced harmonic oscillators -described by inhomogeneous 2nd-order differential equation

$$\boxed{m_e \ddot{\tilde{\vec{r}}} + m_e \gamma \dot{\tilde{\vec{r}}} + k_e \tilde{\vec{r}} = -e\tilde{\vec{E}}(\vec{r})}$$

$$\boxed{\underbrace{m_e \frac{\partial^2 \tilde{\vec{r}}(t)}{\partial t^2}}_{m_e \vec{a}} + m_e \gamma \frac{\partial \tilde{\vec{r}}(t)}{\partial t} + k_e \tilde{\vec{r}}(t) = -e\tilde{\vec{E}}(\vec{r},t)}$$

n.b. we have neglected the $e\tilde{\vec{v}} \times \tilde{\vec{B}} \left(\ll e\tilde{\vec{E}} \right)$ term here...

Lorentz Model

$$m_e \gamma \frac{\partial \tilde{\vec{r}}(t)}{\partial t}$$ → Velocity-dependent damping term

$\gamma \equiv$ damping constant

$$k_e \tilde{\vec{r}}(t)$$ → Potential Force (binding of atomic electrons to atom)

$$-e\tilde{\vec{E}}(\vec{r},t)$$ → Driving Force

$m_e =$ electron mass $= 9.1\times 10^{-31} kg$

Lorentz Model

Suppose the driving term varies periodically in time with angular frequency ω

$$\tilde{\vec{F}}_e(\vec{r},t) = -e\tilde{\vec{E}}(\vec{r},t) = -e\tilde{E}_o e^{-i\omega t}\hat{r}$$

because $\tilde{\vec{E}}(\vec{r},t) = \tilde{E}_o e^{-i\omega t}\hat{r}$

n.b. The electric field $\vec{E}$ is now complex $\tilde{\vec{E}}$.

Then the inhomogeneous force equation becomes:

$$m_e\ddot{\tilde{\vec{r}}} + m_e\gamma\dot{\tilde{\vec{r}}} + k_e\tilde{\vec{r}} = -e\tilde{E}_o e^{-i\omega t}\hat{r}$$ with $$\tilde{\vec{r}}(t) = \tilde{r}(t)\hat{r}$$

Lorentz Model

In the steady- state, we have:

$$\boxed{m_e\ddot{\tilde{\vec{r}}} + m_e\gamma\dot{\tilde{\vec{r}}} + k_e\tilde{\vec{r}} = -e\tilde{E}_o e^{-i\omega t}\hat{r}}$$

Since $\tilde{\vec{r}}$ physically represents the {vector} spatial displacement of each atomic electron from its equilibrium $\{\vec{E}=0\}$ position, then:

$$\tilde{\vec{r}}(t) = \tilde{r}_o(\omega)e^{-i\omega t}\hat{r} \quad \{n.b.\ \tilde{\vec{r}}(t) \text{ is now complex}\}$$

Lorentz Model

Thus:

$$\boxed{m_e \ddot{\tilde{\vec{r}}} + m_e \gamma \dot{\tilde{\vec{r}}} + k_e \tilde{\vec{r}} = -e\tilde{E}_o e^{-i\omega t} \hat{r}}$$

$$\boxed{m_e \frac{\partial^2 \tilde{\vec{r}}(t)}{\partial t^2} + m_e \gamma \frac{\partial \tilde{\vec{r}}(t)}{\partial t} + k_e \tilde{\vec{r}}(t) = -e\tilde{\vec{E}}(\vec{r}, t)}$$

$$\boxed{-m_e \omega^2 \tilde{r}_o \cancel{e^{-i\omega t}} - i\omega m_e \gamma \tilde{r}_o \cancel{e^{-i\omega t}} + k_e \tilde{r}_o \cancel{e^{-i\omega t}} = -e\tilde{E}_o \cancel{e^{-i\omega t}}}$$

$$\boxed{\left(m_e \omega^2 - k_e + i\omega m_e \gamma\right) \tilde{r}_o = e\tilde{E}_o}$$

Lorentz Model

Divide this equation through out by m_e :

$$\boxed{\left(\omega^2 - \left(\frac{k_e}{m_e}\right) + i\omega\gamma\right)\tilde{r}_o = \frac{e}{m_e}\tilde{E}_o}$$

Define:

$$\boxed{\omega_0^2 \equiv \left(\frac{k_e}{m_e}\right)} \quad \underline{\text{or}}: \quad \boxed{\omega_0 \equiv \sqrt{\frac{k_e}{m_e}}}$$

characteristic/natural resonance frequency.

Lorentz Model

Then:

$$\left(\omega^2 - \omega_0^2 + i\gamma\omega\right)\tilde{r}_o = \left(\frac{e}{m_e}\right)\tilde{E}_o$$

or:

$$\tilde{r}_o(\omega) = \frac{\left(e/m_e\right)\tilde{E}_o}{\left[\omega^2 - \omega_0^2 + i\gamma\omega\right]} = \boxed{\begin{array}{c}\text{Atomic electron spatial}\\ \text{displacement amplitude}\\ \{\boldsymbol{n.b.}\ \text{complex!}\}\end{array}}$$

Now:

$$\vec{\mathrm{P}}(\vec{r},t) = -n_e e\tilde{\vec{r}}(t) = -n_e e\tilde{r}_o(\omega)e^{-i\omega t}\hat{r}$$

P is now complex and frequency-dependent!!!

Frequency Dependence of Polarization

Thus

$$\vec{\tilde{\mathrm{P}}}(\vec{r},t)=\frac{-n_e\left(e^2/m_e\right)\left(\tilde{E}_o e^{-i\omega t}\hat{r}\right)}{\left[\omega^2-\omega_0^2+i\gamma\omega\right]}=+\frac{n_e\left(e^2/m_e\right)}{\left[\omega_0^2-\omega^2-i\gamma\omega\right]}\vec{\tilde{E}}(\vec{r},t)$$

For ω = 0

$$\vec{\tilde{\mathrm{P}}}(\omega=0)=\frac{n_e\left(e^2/m_e\right)}{\omega_0^2}\tilde{E}_o\hat{r}=\frac{n_e\left(e^2/m_e\right)}{\left(k_e/m_e\right)}\tilde{E}_o\hat{r}=\left(\frac{n_e e^2}{k_e}\right)\tilde{E}_o\hat{r}$$

Static polarization $\vec{\tilde{\mathrm{P}}}(\omega=0)$ is in-phase with $\vec{\tilde{E}}$

Frequency Dependence of Polarization

Note also that the phase of (complex) $\tilde{\vec{\mathrm{P}}}(\omega)$ depends on the frequency ω – *i.e.* $\tilde{\vec{\mathrm{P}}}(\omega)$ lags behind $\tilde{\vec{E}}(\omega)$ by a phase angle

$$\phi_{\mathrm{P}}(\omega) = \tan^{-1}\left[\frac{\Im m\left(\vec{\mathrm{P}}(\vec{r},t)\right)}{\Re e\left(\vec{\mathrm{P}}(\vec{r},t)\right)}\right] = \tan^{-1}\left[\frac{\gamma\omega}{\left(\omega_0^2-\omega^2\right)}\right] \Leftarrow$$

n.b. The damping constant γ has the same units as ω: radians/sec

When: $\omega < \omega_0 = \sqrt{\dfrac{k_e}{m_e}}$, $\phi_{\mathrm{P}} > 0 \Rightarrow \tilde{\vec{\mathrm{P}}}$ lags $\tilde{\vec{E}}$

When: $\omega > \omega_0 = \sqrt{\dfrac{k_e}{m_e}}$, $\phi_{\mathrm{P}} < 0 \Rightarrow \tilde{\vec{\mathrm{P}}}$ leads $\tilde{\vec{E}}$

Frequency Dependence of Polarization

- From the above formula- note that if the damping constant, γ =0 then ϕ_P =0, the polarization $\tilde{\vec{\mathrm{P}}}(\omega)$ is then always in-phase with $\tilde{\vec{\mathrm{E}}}(\omega)$ because if $\gamma = 0$, then $\Im m\left(\tilde{\vec{\mathrm{P}}}(\vec{r},t)\right)=0$

- The electric polarization $\vec{\mathrm{P}}(\omega)$ is purely real! Physically - a damping constant of $\gamma = 0$ means that the width $\Gamma = \gamma/ 2\pi$ of the atomic/molecular resonance is infinitely narrow,

- And there are no dissipative processes present at the microscopic atomic/molecular level in this macroscopic medium - γ has SI units of radians/second.

Frequency Dependence of Polarization

Note further that $\tilde{\vec{E}}$ in the above expression is actually the internal macroscopic electric field of the dielectric: $\vec{E}_{int}$

$$\tilde{\vec{E}} = \tilde{\vec{E}}_{int} = \tilde{\vec{E}}_{ext} + \tilde{\vec{E}}_{P}$$

The sum of the macroscopic external applied electric field and the macroscopic electric field due to the polarization of the dielectric medium.

The electric field due to polarization of the medium is:

$$\boxed{\tilde{\vec{E}}_{P} = -\frac{1}{3\varepsilon_o}\tilde{\vec{P}}} \Rightarrow \boxed{\tilde{\vec{E}} = \tilde{\vec{E}}_{int} = \tilde{\vec{E}}_{ext} - \frac{1}{3\varepsilon_o}\tilde{\vec{P}}}$$

Frequency Dependence of Polarization

Therefore:

$$\boxed{\tilde{\vec{\mathrm{P}}} = \frac{n_e\left(e^2/m_e\right)}{\left[\omega_o^2 - \omega^2 - i\gamma\omega\right]}\left[\tilde{\vec{E}}_{ext} - \frac{1}{3\varepsilon_o}\tilde{\vec{\mathrm{P}}}\right]}$$

where:

$$\boxed{\omega_0 \equiv \sqrt{\frac{k_e}{m_e}}}$$

Frequency Dependence of Polarization

Now solve for $\tilde{\vec{\mathrm{P}}}$: Skipping writing out some of the complex algebra, we obtain:

$$\boxed{\tilde{\vec{\mathrm{P}}} = \frac{n_e\left(e^2/m_e\right)}{\left[\omega_1^2 - \omega^2 - i\gamma\omega\right]}\tilde{\vec{E}}_{ext}}$$

where:

$$\boxed{\omega_1 \equiv \sqrt{\omega_0^2 - \left(n_e e^2/3\varepsilon_o m_e\right)}} =$$

effective angular resonance frequency of bound atomic electrons

Relation Between Complex Polarization and Electric Field

This formula is identical *e.g.* to the {complex} displacement amplitude formula for a driven harmonic oscillator and for many other physical systems exhibiting a {damped} resonance-type behavior.

Now if $\tilde{\vec{E}}_{ext} = \tilde{\vec{E}}$ - field associated with a monochromatic plane *EM* wave propagating in a dielectric medium:

$$\tilde{\vec{E}}_{ext}(z,t) = \tilde{\vec{E}}_o e^{i(kz-\omega t)},$$

Because of the *linear relationship between the* polarization $\tilde{\vec{P}}$ and

$$\tilde{\vec{E}}_{ext}(z,t) = \underbrace{\tilde{E}_o e^{i(kz-\omega t)}\hat{x}}_{\text{for } EM \text{ plane wave}},$$

Relation Between Complex Polarization and Electric Field

Gauss' Law becomes since $\rho_{free}(\vec{r}) = 0$

$$\boxed{\vec{\nabla} \bullet \tilde{\vec{E}}_{ext} = -\frac{1}{\varepsilon_o} \vec{\nabla} \bullet \tilde{\vec{\mathrm{P}}} = \tilde{\rho}_{bound} = 0}$$

The wave equation for a dielectric medium with

$\tilde{\rho}_{free}(\vec{r}) = 0$ and $\tilde{\vec{J}}_{free} = 0$

$$\nabla^2 \tilde{\vec{E}}_{ext} - \frac{1}{c^2} \frac{\partial^2 \tilde{\vec{E}}_{ext}}{\partial t^2} = \mu_o \frac{\partial^2 \tilde{\vec{\mathrm{P}}}}{\partial t^2} = \frac{\mu_o n_e \left(e^2 / m_e \right)}{\left[\omega_1^2 - \omega^2 - i\gamma\omega \right]} \frac{\partial^2 \tilde{\vec{E}}_{ext}}{\partial t^2}$$

with: $\boxed{\frac{1}{c^2} = \varepsilon_o \mu_o}$

Relation Between Complex Polarization and Electric Field

Or:

$$\boxed{\nabla^2 \tilde{\vec{E}}_{ext} = \frac{1}{c^2}\left[1+\left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\frac{1}{\left[\omega_1^2-\omega^2-i\gamma\omega\right]}\right]\frac{\partial^2 \tilde{\vec{E}}_{ext}}{\partial t^2}}$$

with:

$$\boxed{\omega_1 \equiv \sqrt{\omega_0^2 - \frac{n_e e^2}{3\varepsilon_o m}}}$$

The general solution to this **dispersive wave equation** is of the form:

$$\boxed{\tilde{\vec{E}}_{ext}(z,t) = \tilde{\vec{E}}_o e^{i\left(\tilde{k}z-\omega t\right)}}$$ with complex $$\boxed{\tilde{k} = k + i\kappa}$$

Frequency Dependent Complex Wave-number

and

$$\boxed{\tilde{k}^2 = \frac{\omega^2}{c^2}\left[1+\left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\frac{1}{\omega_1^2-\omega^2-i\gamma\omega}\right]}$$

It shows that the complex wave-number is explicitly dependent on the angular frequency ω , *i.e.*

$$\tilde{k}(\omega)=k(\omega)+i\kappa(\omega)$$

Thus a monochromatic plane EM waves propagating in a dispersive dielectric medium are exponentially attenuated, because of complex wave-number

$$\tilde{\vec{E}}_{ext}(z,t)=\tilde{\vec{E}}_o e^{i(\tilde{k}z-\omega t)}=\tilde{\vec{E}}_o e^{-\kappa z}e^{i(kz-\omega t)} \quad ,$$

Frequency Dependent Susceptibility

That is the $\kappa(\omega) = \Im m\left(\tilde{k}(\omega)\right)$

term corresponds to absorption/dissipation in the macroscopic dielectric, and is proportional to the damping constant γ .

Note that we can also write:

$$\tilde{\vec{\mathrm{P}}}(z,t,\omega) \equiv \varepsilon_o \tilde{\chi}_e(\omega) \tilde{\vec{E}}_{ext}(z,t,\omega)$$

The macroscopic electric susceptibility $\tilde{\chi}_e(\omega)$

is also now complex and is also frequency-dependent, *e.*

$$\tilde{\chi}_e(\omega) \equiv \chi_e(\omega) + i\zeta_e(\omega)$$

Frequency Dependent Susceptibility

Where $\zeta_e(\omega) = \Im m\left(\tilde{\chi}_e(\omega)\right)$

term corresponds to absorption/dissipation in the macroscopic dielectric and is physically related to the damping constant γ . The corresponding dissipative energy losses at the microscopic, atomic/molecular level in the macroscopic dielectric ultimately wind up as heat!

$$\tilde{\vec{\mathrm{P}}}(z,t) = \frac{n_e\left(e^2/m_e\right)}{\left[\omega_1^2 - \omega^2 - i\gamma\omega\right]}\tilde{\vec{E}}_{ext}(z,t) = \left(\frac{\varepsilon_o}{\varepsilon_o}\right)\frac{n_e\left(e^2/m_e\right)}{\left[\omega_1^2 - \omega^2 - i\gamma\omega\right]}\tilde{\vec{E}}_{ext}(z,t)$$

with $\omega_1 = \sqrt{\omega_0^2 - \dfrac{n_e e^2}{3\varepsilon_o m_e}}$

Frequency Dependent Susceptibility

$\therefore$ The complex electric susceptibility is:

$$\tilde{\chi}_e(\omega) = \left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\frac{1}{\left[\omega_1^2 - \omega^2 - i\gamma\omega\right]} \equiv \chi_e(\omega) + i\zeta_e(\omega)$$

Before proceeding this, we need to discuss another aspect of our model – namely that in most linear dielectric materials, the atoms comprising the material are multi-electron atoms, and consequently there are many different binding energies – the outer shell atomic electrons are weakly bound- hence have small k_e and thus small $\omega_0 = \sqrt{k_e / m_e}$,

Dispersion In Complex Dielectric Media

Whereas the inner-shell electrons are much more tightly bound- hence have larger k_e - larger $\omega_0 = \sqrt{k_e / m_e}$

In complex media- dielectrics with more than one kind of atom- electrons can be shared between atoms – i.e. they are bound to molecules - which can be weakly bound in some molecules.

There can be also be molecular resonances e.g. in the microwave and infra-red regions of the EM spectrum – atomic resonances are typically in the optical and UV regions (for the outer-most shell electrons), as well as in the far UV and x-ray regions (for the inner-shell electrons)!

Dispersion In Complex Dielectric Media

Allowing for all such resonances - we can write the complex electric polarization $\tilde{\vec{P}}$ as a summation over all of the resonances present in the linear dielectric as follows:

$$\tilde{\vec{P}}(z,t) = \frac{n_e e^2}{m_e}\left(\sum_{j=1}^{n} \frac{f_j^{osc}}{\left[\omega_{1j}^2 - \omega^2 - i\gamma_j \omega\right]}\right)\tilde{\vec{E}}_{ext}(z,t)$$

where: $\omega_{1j} \equiv \sqrt{\omega_{0j}^2 - \frac{n_e e^2}{3\varepsilon_o m_e}}$ and: $\omega_{0j} \equiv \sqrt{\frac{k_{ej}}{m_e}}$

$f_j^{osc} \equiv$ oscillator strength of *jth resonance, defined such that*

Dispersion Complex Dielectric Media

$$\sum_{j=1}^{n} f_j^{osc} = 1$$

Physically: $f_j^{osc} =$ fractional strength of the jth resonance
and γ_j = 2π*width of the jth resonance.

Thus, we see that the complex electric susceptibility

$$\tilde{\chi}_e(\omega) \equiv \chi_e(\omega) + i\zeta_e(\omega)$$

is: $$\tilde{\chi}_e(\omega) = \left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\left(\sum_{j=1}^{n} \frac{f_j^{osc}}{\left[\omega_{1j}^2 - \omega^2 - i\gamma_j \omega\right]}\right) \equiv \chi_e(\omega) + i\zeta_e(\omega)$$

Frequency Dependent Complex Electric Permittivity

The complex electric permittivity

$$\tilde{\varepsilon}(\omega) = \varepsilon_o\left(1 + \tilde{\chi}_e(\omega)\right) \equiv \varepsilon(\omega) + i\varsigma(\omega)$$

of a dispersive, linear dielectric medium is:

$$\tilde{\varepsilon}(\omega) = \varepsilon_o\left(1 + \tilde{\chi}_e(\omega)\right) = \varepsilon_o\left(1 + \left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\left(\sum_{j=1}^{n} \frac{f_j^{osc}}{\left[\omega_{1j}^2 - \omega^2 - i\gamma_j\omega\right]}\right)\right) \equiv \varepsilon(\omega) + i\varsigma(\omega)$$

with the relations: $\varepsilon(\omega) = \Re e\left(\tilde{\varepsilon}(\omega)\right) = \varepsilon_o\left(1 + \chi_e(\omega)\right)$

and $\varsigma(\omega) = \Im m\left(\tilde{\varepsilon}(\omega)\right) = \varepsilon_o \zeta_e(\omega)$

Dispersive Wave Solution

Thus, monochromatic plane EM wave solutions to the dispersive wave equation are of the form:

$$\tilde{\vec{E}}(z,t,\omega) = \tilde{\vec{E}}_o e^{i(\tilde{k}z-\omega t)}$$

With complex wave-number

$$\tilde{k}(\omega) = k(\omega) + i\kappa(\omega) \equiv \sqrt{\tilde{\varepsilon}(\omega)\mu_o}\,\omega$$

Thus:

$$\boxed{\tilde{\vec{E}}_{ext}(z,t,\omega) = \tilde{\vec{E}}_o e^{i(\tilde{k}z-\omega t)} = \tilde{\vec{E}}_o \underbrace{e^{-\kappa z}}_{\substack{\text{exponential}\\ \text{damping}\\ \text{of } EM \text{ wave}}} e^{i(kz-\omega t)}}$$

Frequency Dependent Complex Refractive Index

Introducing a frequency-dependent complex wave-number

$$\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$$

is equivalent to introducing a

frequency-dependent complex index of refraction

$$\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$$

For a linear, dispersive dielectric, the complex index of refraction and complex wave-number are related to each other by:

$$\tilde{k}(\omega) = \left(\frac{\omega}{c}\right)\tilde{n}(\omega)$$

Frequency Dependent Complex Refractive Index

$$\boxed{\left(k(\omega)+i\kappa(\omega)\right)=\left(\frac{\omega}{c}\right)\left(n(\omega)+i\eta(\omega)\right)=\left(\frac{\omega}{c}\right)n(\omega)+i\left(\frac{\omega}{c}\right)\eta(\omega)}$$

$$\boxed{k(\omega)=\left(\frac{\omega}{c}\right)n(\omega)} \quad \text{and} \quad \boxed{\kappa(\omega)=\left(\frac{\omega}{c}\right)\eta(\omega)}$$

The complex index of refraction is related to the complex electric permittivity $\tilde{\varepsilon}(\omega)=1+\tilde{\chi}_e(\omega)$

and thus the complex electric susceptibility via the relation

$$\tilde{n}(\omega)=\sqrt{\tilde{\varepsilon}(\omega)/\varepsilon_o}=\sqrt{1+\tilde{\chi}_e(\omega)}$$

Frequency Dependent Complex Refractive Index

Squaring both sides:

$$\tilde{n}^2(\omega) = \frac{\tilde{\varepsilon}(\omega)}{\varepsilon_o} = 1 + \tilde{\chi}_e(\omega) = 1 + \left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\left(\sum_{j=1}^{n} \frac{f_j^{osc}}{\left[\omega_{1j}^2 - \omega^2 - i\gamma_j \omega\right]}\right)$$

But:

$$\tilde{k}^2(\omega) = \left(\frac{\omega}{c}\right)^2 \left(1 + \left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\left[\sum_{j=1}^{n} \frac{f_j^{osc}}{\left[\omega_{1j}^2 - \omega^2 - i\gamma_j \omega\right]}\right]\right)$$

$$= \left(k(\omega) + i\kappa(\omega)\right)^2 = k^2(\omega) + 2ik(\omega)\kappa(\omega) + \kappa^2(\omega)$$

Frequency Dependent Complex Refractive Index

Since:

$$\tilde{n}(\omega)=\left(\frac{c}{\omega}\right)\tilde{k}(\omega)$$

then:

$$\tilde{n}^2(\omega)=\left(\frac{c}{\omega}\right)^2\tilde{k}^2(\omega)=1+\left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\left[\sum_{j=1}^{n}\frac{f_j^{osc}}{\left[\omega_{1j}^2-\omega^2-i\gamma_j\omega\right]}\right]$$

$$=\left(n(\omega)+i\eta(\omega)\right)^2=n^2(\omega)+2in(\omega)\eta(\omega)-\eta^2(\omega)$$

Using the “standard” trick:

$$\tilde{z}=\frac{1}{x-iy}=\frac{1}{x-iy}\frac{x+iy}{x+iy}=\frac{x+iy}{x^2+y^2},$$

Frequency Dependent Complex Refractive Index

$$\Re e\left(\tilde{z}\right) = \frac{x}{x^2 + y^2}$$ and $$\Im m\left(\tilde{z}\right) = \frac{y}{x^2 + y^2}$$

Then equating the real and imaginary parts of the LHS & RHS of the above equation, we obtain the real part as:

$$n^2(\omega) - \eta^2(\omega) = 1 + \frac{n_e e^2}{\varepsilon_o m_e}\left[\sum_{j=1}^{n} \frac{f_j^{osc}\left(\omega_{1j}^2 - \omega^2\right)}{\left[\left(\omega_{1j}^2 - \omega^2\right)^2 + \gamma_j^2 \omega^2\right]}\right]$$

Frequency Dependent Complex Refractive Index

And the imaginary part as

$$2n(\omega)\eta(\omega) = \frac{n_e e^2}{\varepsilon_o m_e}\left[\sum_{j=1}^{n} \frac{f_j^{osc}\gamma_j\omega}{\left[\left(\omega_{1j}^2-\omega^2\right)^2+\gamma_j^2\omega^2\right]}\right]$$

2 equations and 2 unknowns: {n(ω) & η (ω)}

→ solve for *n(ω)&η(ω)*

First define:

$$\alpha_x(\omega) \equiv \left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\left[\sum_{j=1}^{n} \frac{f_j^{osc}\left(\omega_{1j}^2-\omega^2\right)}{\left[\left(\omega_{1j}^2-\omega^2\right)^2+\gamma_j^2\omega^2\right]}\right]$$

Frequency Dependent Complex Refractive Index

$$\beta_x(\omega) \equiv \left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\left[\sum_{j=1}^{n} \frac{f_j^{osc}(\gamma_j \omega)}{\left[\left(\omega_{1j}^2 - \omega^2\right)^2 + \gamma_j^2 \omega^2\right]}\right]$$

(n.b. $\beta_x(\omega) > 0$ - is always positive)

Then:

$$n^2(\omega) - \eta^2(\omega) = 1 + \alpha_x(\omega)$$

and

$$2n(\omega)\eta(\omega) = \beta_x(\omega) \Rightarrow \eta(\omega) = \beta_x(\omega)/2n(\omega)$$

Frequency Dependent Complex Refractive Index

Thus:

$$n^2(\omega) - \left(\frac{\beta_x(\omega)}{2n(\omega)}\right)^2 = (1 + \alpha_x(\omega))$$

$\Leftarrow$ multiply equation through by $n^2(\omega)$

$$n^4(\omega) - \left(\frac{\beta_x(\omega)}{2}\right)^2 = (1 + \alpha_x(\omega))n^2(\omega)$$

Or:

$$n^4(\omega) - (1 + \alpha_x)n^2(\omega) - \left(\frac{\beta_x(\omega)}{2}\right)^2 = 0$$

Frequency Dependent Complex Refractive Index

Define: $x \equiv n^2(\omega)$

Then:

$$x^2 - (1+\alpha_x)x - \left(\frac{\beta_x}{2}\right)^2 = 0 \quad \longrightarrow \quad \boxed{ax^2 + bx + c = 0}$$

with

$$a = 1, \quad b = -(1+\alpha), \quad c = -\left(\frac{\beta}{2}\right)^2$$

Frequency Dependent Complex Refractive Index

Solutions or roots of this quadratic equation are of the form:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$\boxed{x = \frac{+(1+\alpha_x) \pm \sqrt{(1+\alpha_x)^2 + 4\left(\frac{\beta_x}{2}\right)^2}}{2} = \frac{1}{2}\left[(1+\alpha_x) \pm \sqrt{(1+\alpha_x)^2 + \beta_x^2}\right]}$$

i.e.

$$\boxed{x = \frac{1}{2}(1+\alpha_x)\left[1 \pm \sqrt{1 + \left(\frac{\beta_x}{(1+\alpha_x)}\right)^2}\right]}$$

n.b. the term: $\boxed{\left(\frac{\beta_x}{(1+\alpha_x)}\right)^2 > 0}$

Frequency Dependent Complex Refractive Index

→ Must select +ive root on physical grounds - since $x \equiv n^2 > 0$

$$x = n^2 = \frac{1}{2}(1+\alpha)\left[1+\sqrt{1+\left(\frac{\beta_x}{(1+\alpha_x)}\right)^2}\right]$$

Finally- we obtain:

$$n(\omega) \equiv \Re e(\tilde{n}(\omega)) = \sqrt{\left(\frac{1+\alpha_x(\omega)}{2}\right)\left[1+\sqrt{1+\left(\frac{\beta_x(\omega)}{(1+\alpha_x(\omega))}\right)^2}\right]}$$

Complex index of refraction: $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$

Frequency Dependent Complex Refractive Index

$$\eta(\omega) \equiv \Im m\left(\tilde{n}(\omega)\right) = \frac{\beta_x(\omega)}{2n(\omega)} = \frac{\beta_x(\omega)/2}{\sqrt{\left(\frac{1+\alpha_x(\omega)}{2}\right)\left[1+\sqrt{1+\left(\frac{\beta_x(\omega)}{(1+\alpha_x(\omega))}\right)^2}\right]}}$$

Where:

$$\alpha_x(\omega) \equiv \left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\left[\sum_{j-1}^{n} \frac{f_j^{osc}\left(\omega_{1j}^2 - \omega^2\right)}{\left[\left(\omega_{1j}^2 - \omega^2\right)^2 + \gamma_j^2 \omega^2\right]}\right]$$

$$\beta_x(\omega) \equiv \left(\frac{n_e e^2}{\varepsilon_o m_e}\right)\left[\sum_{j-1}^{n} \frac{f_j^{osc}\left(\gamma_j \omega\right)}{\left[\left(\omega_{1j}^2 - \omega^2\right)^2 + \gamma_j^2 \omega^2\right]}\right]$$

Frequency Dependent Complex Refractive Index

Explicit form of n(ω) and η (ω) is quite tedious – but these can be reasonably-easily coded up and plots of n(ω) vs. ω and η (ω) vs. ω can be obtained. We can also then obtain the following:

The complex relations: $\tilde{n}(\omega) \equiv n(\omega) + i\eta(\omega)$

and

$$\tilde{k}(\omega) \equiv k(\omega) + i\kappa(\omega) = \left(\frac{\omega}{c}\right)\tilde{n}(\omega)$$

and thus: $k(\omega) = \left(\frac{\omega}{c}\right)n(\omega)$ and $\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega)$

Frequency Dependent Intensity

The frequency-dependent intensity/irradiance

$$I(z,\omega)=\left\langle\left|\vec{S}(z,t,\omega)\right|\right\rangle$$

of a monochromatic plane EM wave propagating in a linear, dispersive dielectric is also exponentially decreased by a factor of $1/e = e^{-1}$ of its original value in going a characteristic distance of $z = 1/\alpha(\omega) = 1/2\kappa(\omega) \equiv \ell_{atten}(\omega)$. Defining: $\ell_{atten}(\omega) \equiv 1/\alpha(\omega) = 1/2\kappa(\omega)$ = intensity attenuation length – which is analogous to the skin depth, $\delta_{sc} \equiv 1/\kappa$ for conductors. However, note that $\delta_{sc} \equiv 1/\kappa$ is associated with the attenuation of the ***E*** and ***B***-fields, whereas attenuation effects in intensity/irradiance, *I* varies as the square of the **E**-field:

Frequency Dependent Intensity

$$\boxed{I(z) = \left\langle \left| \vec{S}(z,t) \right| \right\rangle \propto \left\langle E_{ext}^2(z,t) \right\rangle},$$

hence: $$\boxed{I(z) \propto E_o^2 e^{-2\kappa(\omega)z} = E_o^2 e^{-\alpha(\omega)z}}$$

In the exponential z-dependent term $e^{-2\kappa(\omega)z}$

since the energy densit(ies) $\left\langle u_{E,M}(z,t) \right\rangle$

And intensity $I(z) = \left\langle \left| \vec{S}(z,t) \right| \right\rangle$

are both proportional to E^2 i.e. both proportional to $e^{-2\kappa(\omega)z}$,

Frequency Dependent Absorption and Extinction coefficient

we define the frequency-dependent absorption coefficient

$$\boxed{\alpha(\omega) \equiv 2\kappa(\omega) = 1/\ell_{atten}(\omega)}.$$

Similarly, for the frequency-dependent complex index of refraction

$$\boxed{\tilde{n}(\omega) = n(\omega) + i\eta(\omega)}$$

we can also define the frequency-dependent extinction coefficient:

$$\boxed{\xi(\omega) \equiv 2\eta(\omega)}$$

Frequency Dependent Absorption coefficient

Since: $$\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega)$$ $\Rightarrow$ $$2\kappa(\omega) = \left(\frac{\omega}{c}\right)2\eta(\omega)$$

thus: $$\alpha(\omega) = \left(\frac{\omega}{c}\right)\xi(\omega) = 2\left(\frac{\omega}{c}\right)\eta(\omega)$$

The absorption coefficient:

$$\alpha(\omega) \equiv 2\kappa(\omega) = \left(\frac{\omega}{c}\right)2\eta(\omega) = \left(\frac{\omega}{c}\right)\xi(\omega) = 1/\ell_{atten}(\omega)$$

Frequency Dependent Extinction coefficient

The extinction coefficient: $$\boxed{\xi(\omega) \equiv 2\eta(\omega)}$$

Typical values of the (real) index of refraction $n(\omega)$ for solids and liquids are $n(\omega) \approx 1.3-1.7$ in the visible light region of *EM* spectrum, e.g. $n_{glass}(\omega) = 1.5$, $n_{H2O}(\omega) = 1.3$, $n_{plastic}(\omega) = 1.7$.

Then if index of refraction of glass in the visible light region:

$$n(\omega) = \sqrt{\left(\frac{1+\alpha_x(\omega)}{2}\right)\left[1+\sqrt{1+\left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}\right]} = 1.5$$

Frequency Dependent Refractive Index

Then:

$$n^2(\omega) = \left(\frac{1+\alpha_x(\omega)}{2}\right)\left[1+\sqrt{1+\left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}\,\right] = (1.5)^2 = 2.25$$

Thus:

$$\left(1+\alpha_x(\omega)\right)\left[1+\sqrt{1+\left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}\,\right] = 4.50$$

One equation & two unknowns: $\alpha_x(\omega)$ and $\beta_x(\omega)$

$\rightarrow$ Need another relation / independent constraint!!

Frequency Dependent Refractive Index

Note that glass doesn't have significant absorption in the visible light region –but such solid / liquid materials have absorption coefficients for visible light in the range of:

$$\alpha(\omega) \equiv 2k(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega) \approx 10^{-2} - 10^{-1} m^{-1}$$

Iintensity I falls off to $1/e = e^{-1} = 0.3679$ of initial ($z = 0$) value after light travels a distance ~ 10 – 100 m

So suppose:

$$\alpha(\omega) \equiv 2\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega) \simeq 10^{-1}\,\mathrm{m}^{-1}$$

in glass for visible light, $\omega_{vis} = 10^{16}$ radians / sec

Frequency Dependent Refractive Index

$$\eta(\omega) = \left(\frac{c}{\omega}\right)\alpha(\omega) \simeq \left(\frac{3\times10^{8}}{10^{16}}\right)10^{-1} = 3\times10^{-9} \ll 1$$

Now:

$$\eta(\omega) = \left(\frac{\beta_x(\omega)}{2n(\omega)}\right)$$

$n(\omega) \simeq 1.5$ for glass in visible light range of EM spectrum.

$$\eta(\omega) = \frac{1}{3}\beta_x(\omega)$$

or:

$\beta_x(\omega) = 3\eta(\omega) \simeq 9\times10^{-9}$ <<1 in the visible light range for glass

Frequency Dependent Refractive Index

Then: $$\left(1+\alpha_x(\omega)\right)\left[1+\sqrt{1+\left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}\right]=4.50$$

Now solve for $\alpha_x(\omega)$

$$\boxed{1+\sqrt{1+\left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}=4.50\Big/\left(1+\alpha_x(\omega)\right)}$$

Has a solution when: $\boxed{\alpha_x(\omega)\simeq 1.25}$ for $\boxed{\beta_x(\omega)=9\times 10^{-9}\ll\alpha_x(\omega)}$

Refractive Index of Glass

Thus, for $n(\omega)=1.5$ glass in the light region of the EM spectrum

with $\alpha_x(\omega)\simeq 1.25$ and $\beta_x(\omega)=9\times 10^{-9}$ as an explicit check we see that

$$n(\omega)=1.5$$

That is the refractive index of a glass in the visible light region of EM spectrum

References

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THANK YOU