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Preparatory School to the Winter College on Optics: Fundamentals of Photonics - Theory, Devices and

3 - 7 February 2014

Fundamentals of signal theory
(Continuous and discrete signals. Sampling theorem. Aliasing. White and coloured noise)

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Phase and Group Velocities

Helmholtz equation in a medium with refractive index n:

$$\left(\nabla^2 + k^2 n^2\right) \mathbf{E}(\vec{r},\omega) = 0$$

Forward propagating solution $\mathcal{E}(z,\omega) = \mathcal{E}_0 e^{iknz}$, so in time domain, monochromatic wave:

$$E = \mathcal{E}_0(\omega)\, e^{i(knz-\omega t)} = \mathcal{E}_0(\omega)\, e^{i(Kz-\omega t)}, \quad K = kn$$

If $n = n_r + i n_i$, $K = K_r + i K_i$

$$E = \mathcal{E}_0(\omega)\, \underbrace{e^{-K_i z}}_{\text{exponential decay}}\, e^{i(K_r z - \omega t)} = \mathcal{E}_0(\omega)\, e^{-K_i z}\, e^{iK_r\left(z - \frac{\omega}{K_r} t\right)}$$

The oscillations move at the Phase Velocity $V_{ph} = \frac{\omega}{K_r} = \frac{c}{n}$

Consider now a superposition of several frequencies very near ω.

For simplicity use only two, ω & $\omega + \Delta\omega$:

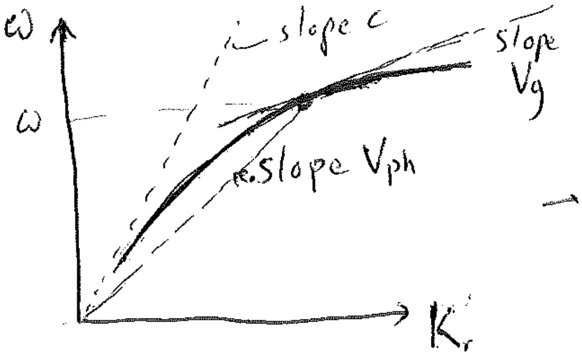
$$E = \mathcal{E}_0(\omega)\, e^{i(K(\omega)z - \omega t)} + \underbrace{\mathcal{E}_0(\omega+\Delta\omega)}_{\approx \mathcal{E}_0(\omega) + \cancel{\dots}}\, e^{i\left(\underbrace{K(\omega+\Delta\omega)}_{\approx K(\omega) + K'(\omega)\Delta\omega + \cancel{\dots}} z - (\omega+\Delta\omega)t\right)}$$

$$\approx \mathcal{E}_0(\omega)\, e^{i(K(\omega)z - \omega t)} \left(1 + e^{i[K'(\omega)\Delta\omega z - \Delta\omega t]}\right)$$

so $$I \propto |E|^2 \approx \left|1 + e^{i\Delta\omega K'(\omega)\left[z - \frac{1}{K'(\omega)} t\right]}\right|$$

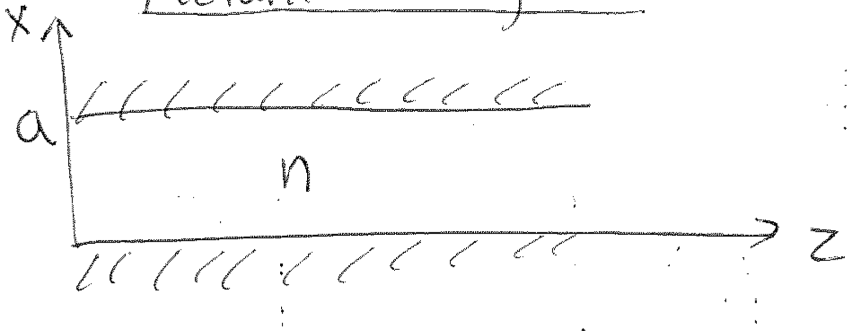
So, (neglecting K_i) the intensity features move at the group velocity

$$V_g = \frac{1}{K'(\omega)} = \frac{c}{\frac{d(\omega n)}{d\omega}}$$



Waveguides in 2D

Metallic waveguide



Use Helmholtz equation

$$\left(\nabla^2 + k^2 n^2\right) E(x,z) = 0$$

with boundary conditions

$$E(0,z) = E(a,z) = 0$$

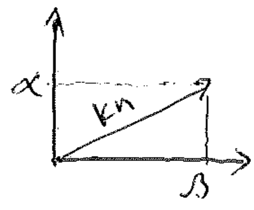
Propose separation of variables

$$E(x,z) = f(x)\, e^{i\beta z}$$

This leads to $f'' + \left(k^2 n^2 - \beta^2\right) f = 0$, $f(0) = f(a) = 0$

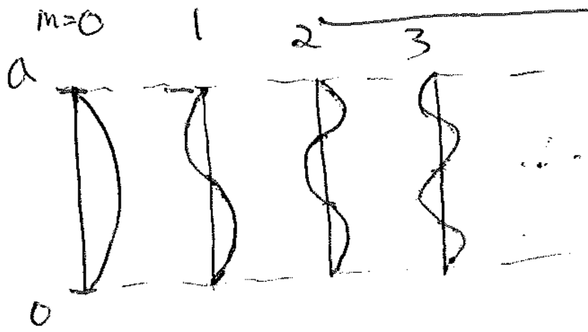
propose $\underline{f(x) = \sin \alpha_m x}$, $\alpha_m = \frac{m+1}{a}\pi$ (so it meets B.C.)

Then $\alpha_m^2 + \beta^2 = k^2 n^2$



$$\beta_m(\omega) = \sqrt{k^2 n^2(\omega) - \alpha_m^2}$$

$$= \sqrt{\frac{\omega^2 n^2(\omega)}{c^2} - \frac{(m+1)^2 \pi^2}{a^2}}$$



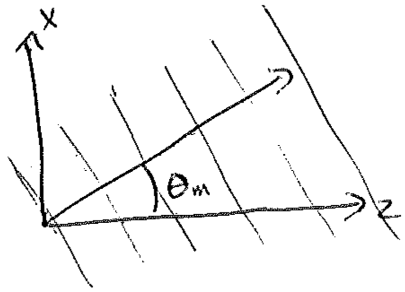
Phase velocity: $V_{ph} = \dfrac{\omega}{\beta_m}$ increases with m

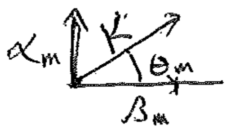
Group velocity: $V_g = \dfrac{1}{\beta_m'(\omega)} = \underbrace{\dfrac{1}{\frac{\partial \beta_m}{\partial K}}}_{\text{waveguide contribution}} \underbrace{\dfrac{1}{\frac{\partial K}{\partial \omega}}}_{\text{material contribution}} = \dfrac{V_g^{(wg)}\, V_g^{(mat)}}{V_{ph}^{(mat)}}$

In this case, $\beta_m = \sqrt{K^2 - \alpha_m^2}$, $\dfrac{V_g^{(wg)}}{V_{ph}^{(mat)}} = \left(\dfrac{K}{\beta_m}\right)^{-1} = \dfrac{\beta_m}{K}$

$V_g^{(wg)} = \dfrac{\beta_m}{\omega}$ decreases with m.

Geometric interpretation in terms of rays/plane waves

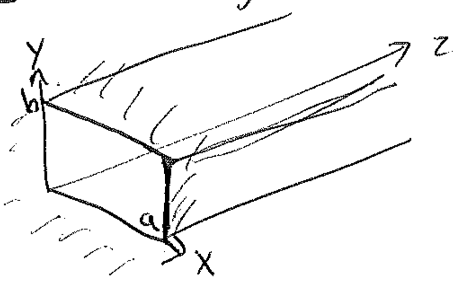




, $\beta_m = K\cos\theta_m$

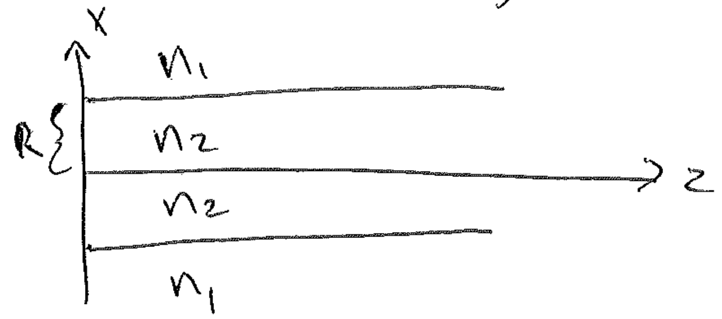
$V_{ph} = \dfrac{V_{ph}^{(mat)}}{\cos\theta_m}$, $V_g = V_g^{(mat)}\cos\theta_m$

<u>Excercise:</u> Rectangular metallic waveguide in 3D



Find modes, V_g, V_{ph}.

2D dielectric waveguide



$n_2 > n_1$

$$\left(\nabla^2 + k^2 n^2\right) E(x,z) = 0 \quad , \quad n(x) = \begin{cases} n_1, & |x| > R \\ n_2, & |x| < R \end{cases}$$

again, propose $E(x,z) = f(x) e^{i\beta z}$

$$f'' + (k^2 n^2 - \beta^2) f = 0 \quad , \quad f \,\&\, f' \text{ continuous.}$$

For $|x| < R$: $\quad f'' + (k^2 n_2^2 - \beta^2) f = 0, \quad f(x) = f_2 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \alpha x,$

$$\alpha = \sqrt{k^2 n_2^2 - \beta^2}$$

For $|x| > R$: $\quad f'' - \underbrace{(\beta^2 - k^2 n_1^2)}_{\text{need this to be positive to have exp. decay}} f = 0 \qquad f(x) = f_1 e^{-\gamma |x|}$

$$\gamma = \sqrt{\beta^2 - k^2 n_1^2}$$

There are even & odd modes.

<u>Even modes:</u> $\quad f(x) = f_2 \cos \alpha x \;, \quad |x| < R$

Boundary conditions

$$f(R_-) = f(R_+)$$
$$f'(R_-) = f'(R_+)$$

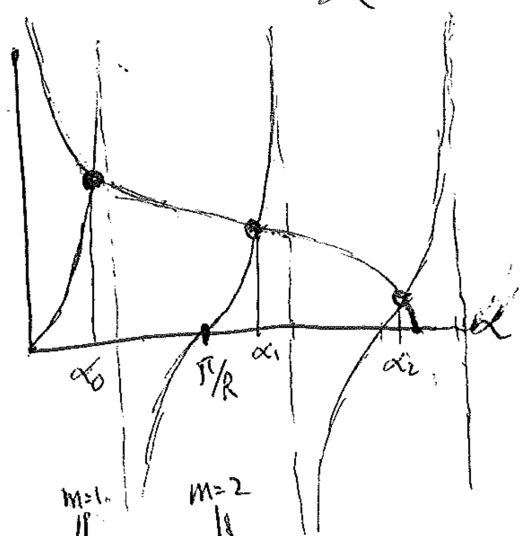
$$\left.\begin{aligned} f_2 \cos \alpha R &= f_1 e^{-\gamma R} \\ -\alpha f_2 \sin \alpha R &= -\gamma f_1 e^{-\gamma R} \end{aligned}\right\} \quad \text{divide second by (-)first:}$$

$$\alpha \tan \alpha R = \gamma$$

but $\gamma = \sqrt{\beta^2 - k^2 n_1^2}$, $\beta^2 = k^2 n_2^2 - \alpha^2$, so

$$\boxed{\alpha \tan \alpha R = \sqrt{k^2 (n_2^2 - n_1^2) - \alpha^2}}$$

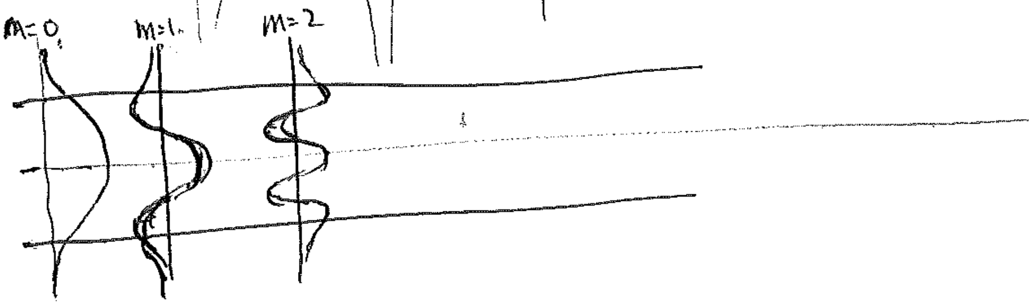
$$\tan \alpha R = \frac{\sqrt{k^2(n_2^2-n_1^2)-\alpha^2}}{\alpha}$$



$$\beta_m(\omega) = \sqrt{k^2 n_2^2 - \alpha_m^2(\omega)}$$

$$\gamma_m(\omega) = \sqrt{k^2(n_2^2-n_1^2) - \alpha_m^2(\omega)}$$

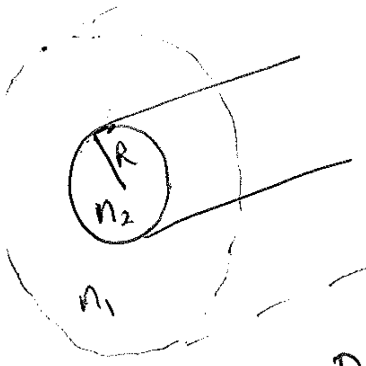
$$f_1 = f_2 \cos \alpha_m R \, e^{\gamma_m R}$$



Again $V_{ph} = \frac{\omega}{\beta_m(\omega)}$, $V_g = \frac{1}{\beta_m'(\omega)}$

Excercise: find odd modes.

3D Cylindrical waveguides (fibres)



$$\left(\nabla^2 + k^2 n^2\right) E(\rho, \phi, z) = 0 \qquad n(\rho) = \begin{cases} n_1, & \rho > R \\ n_2, & \rho < R \end{cases}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho} E\right) + \frac{1}{\rho^2}\frac{\partial^2 E}{\partial^2 \phi^2} + \frac{\partial^2 E}{\partial z^2} + k^2 n^2 E$$

Propose $E = f(\rho) e^{i l \phi} e^{i \beta z}$, $l = $ integer.

$$\frac{1}{\rho}(\rho f')' - \frac{l^2}{\rho^2} f - \beta^2 f + k^2 n^2 f = 0$$

or

$$\rho^2 f'' + \rho f' + \left[(k^2 n^2 - \beta^2)\rho^2 - l^2\right] f = 0$$

For $\rho < R$

$$\rho^2 f'' + \rho f' + (\alpha^2 \rho^2 - l^2) f = 0 \quad , \quad \alpha = \sqrt{k^2 n_2^2 - \beta^2}$$

Solution: $f(\rho) = f_2 J_l(\alpha \rho)$ Bessel function of the first kind

For $\rho > R$

$$\rho^2 f'' + \rho f' (-\gamma^2 \rho^2 - l^2) f = 0 \quad , \quad \gamma = \sqrt{\beta^2 - k^2 n_1^2}$$

Solution: $f(\rho) = f_1 I_l(\gamma \rho)$ modified Bessel function of the first kind

Boundary conditions

$$f_2 J_l(\alpha R) = f_1 I_l(\gamma R)$$

$$\alpha f_2 \frac{J_{l-1}(\alpha R) - J_{l+1}(\alpha R)}{2} = \gamma f_1 \frac{I_{l-1}(\gamma R) + I_{l+1}(\gamma R)}{2}$$

Their ratio (times 2) gives

$$\alpha \frac{J_{l-1}(\alpha R) - J_{l+1}(\alpha R)}{J_l(\alpha R)} = \gamma \frac{I_{l-1}(\gamma R) + I_{l+1}(\gamma R)}{I_l(\gamma R)}$$

solve for α, with $\gamma = \sqrt{k^2(n_2^2 - n_1^2) - \alpha^2}$

If $l=0$: $-\alpha \frac{J_1(\alpha R)}{J_0(\alpha R)} = \gamma \frac{I_1(\gamma R)}{I_0(\gamma R)}$.