The Abdus Salam
International Centre
for Theoretical Physics
50th Anniversary 1964-2014

2570-5

Preparatory School to the Winter College on Optics: Fundamentals of Photonics - Theory, Devices and

3 - 7 February 2014

Basics of quantum optics and electrodynamics (photons, coherent states, number states)

Imrana Ashraf Zahid
Quaid-i-Azam University
Islamabad
Pakistan

1 REVIEW OF QUANTUM MECHANICS

1.1 CLASSICAL MECHANICS

Classical mechanics is based on the assumption that any physically interesting variables connected with a system/particle, such as its position, velocity or its energy can be measured with arbitrary precision and without mutual interference from any other such measurement. Laws of classical mechanics can be expressed in various mathematical forms,

1. Newtonian mechanics
2. Hamiltonian mechanics

Hamiltonian of a physical system gives its total energy

$$H = T + V$$

Quantum mechanics is based on the realization that the measuring process may affect the physical system. It is therefore impossible to measure simultaneously certain pairs of variables with precision. In quantum mechanics physical system is described by a state vector or wave function, and variables are represented by operators.

Quantum mechanics can be expressed by,

1. Wave mechanics
2. Dirac's notation

1.2 WAVE MECHANICS:

A quantum mechanical system (such as atoms, molecules, ions etc.) are given by its wave function $\psi(r,t)$. Itself $\psi(r,t)$ has no physical meaning but it allows to calculate the expectation values of all observables of interest.

Measurable quantities are called observables and are represented by hermition operators$\hat{O}$. Expectation value is given by

$$\left\langle \hat{O} \right\rangle = \int d^3r \psi^*(r,t)\hat{O}\psi(r,t).$$

1.3 Probability:

The probability of finding the system in the volume element d^3r is

$$\psi^*(r,t)\hat{O}\psi(r,t).$$

As the system exist, its probability of being somewhere has to equal 1.

$$\begin{aligned} \int \psi^*(r,t)\psi(r,t)d^3r &= 1 \\ \int \psi_n^*(r,t)\psi_m(r,t)d^3r &= \delta_{nm}. \end{aligned}$$

The time development of wave function is determined by schrodinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi(r,t) = H\psi(r,t),$$

where H is the Hamiltonian of the system. The energy of the unperturbed system for instance an atom not interacting with light is the sum of its potential and kinetic energies

$$H = \frac{p^2}{2m} + V(r).$$

1.4 Stationary state:

Stationary states of schrodinger equation are those for which space and time dependence are separated,

$$\psi_n(r,t) = U_n(\overrightarrow{r})e^{-i\omega t}.$$

Time independent equation,

$$HU_n(\overrightarrow{r}) = E_n U_n(\overrightarrow{r}) = \hbar\omega_n U_n(\overrightarrow{r}),$$

where $U_n(\overrightarrow{r})$ is an eigen function of H with eigen values $E_n = \hbar\omega_n$. The eigen values of hermition operators are real numbers. The eigen functions of hermitian operators belonging to different eigen values are orthogonal, and eigen functions having same eigen values are normal.

$$\int U_n^*(r)U_m(r)d\overrightarrow{r} = \delta_{nm}$$

and complete

$$\sum_n U_n^*(r)U_n(r) = 1.$$

The completeness relation means that any function can be written as a linear combination of the $U_n(r,t)$. The wave function

$$\psi(r,t) = \sum_n \psi_n(r,t) = \sum_n C_n(t)U_n(\overrightarrow{r})e^{-i\omega_n t},$$

here $C_n(t)$ are the expansion coefficients.

$C_n(t)$ - constant for problems related to free part of Hamiltonian

$C_n(t)$ - change in time if we include the interaction part of Hamiltonian.

Putting the values of $\psi(r,t)$ in the normalization condition we get

$$\sum_n |C_n|^2 = 1,$$

gives the probability of finding the system in state n. The expectation value in terms of C_n

$$\left\langle \hat{O} \right\rangle = \sum_{n,m} C_n C_m^* \hat{O}_{nm}\ e^{-i\omega_{nm}t},$$

Where

$$\hat{O}_{nm} = \int d^3r \; U_m^*(r)\hat{O} \; U_n(r),$$

and

$$\omega_{nm} = \omega_n - \omega_m.$$

1.5 DIRAC NOTATION:

The wave function of wave mechanics corresponds to the state vector in Dirac's formulation of quantum mechanics. The relation between state vector and wave function is analogous to using vectors instead of coordinates. A vector $\overrightarrow{V}$ can be expanded as,

$$\overrightarrow{V} = V_x \hat{x} + V_y \hat{y}.$$

In Dirac's notation

$$|V\rangle = V_x |x\rangle + V_y |y\rangle .$$

x-component of a vector is obtained by

$$\overrightarrow{V} . \hat{x} = V_x,$$

in Dirac's notation

$$\langle x| V\rangle = V_x \quad \text{and} \quad \langle \; y| V\rangle = V_y.$$

Using these Eqn's. we can write

$$\begin{aligned} |V\rangle &= |x\rangle \langle x| V\rangle + |y\rangle \; \langle \; y| V\rangle \\ &= (|x\rangle \langle x| + |y\rangle \; \langle \; y|) |V\rangle \end{aligned}$$

The identity diadic (outer product of two vectors)

$$|x\rangle \langle x| + |y\rangle \langle y| = 1$$

for n dimensions

$$\begin{aligned} |V\rangle &= \sum_n |n\rangle \langle n| V\rangle \\ \sum_n |n\rangle \langle n| &= I \end{aligned}$$

where $\{ \; |n\rangle\}$ are complete set of vectors, i.e. a basis. The inner products $\langle n| V\rangle$ are the expansion coefficients of the vector$|V\rangle$ in this basis. Expansion coefficients are in general complex.

$$\langle k| V\rangle = \langle V| k\rangle^*$$

For continuous basis $\{ \; |r\rangle\}$

$$I = \int d^3r \, |r\rangle \langle r|$$

The wave vector

$$|\psi(t)\rangle = \int d^3r\, |r\rangle \langle r|\, \psi\rangle$$

Where the wave function

$$\begin{aligned}\psi(r) &= \langle r|\, \psi\rangle \\ \psi(x) &= \langle x|\, \psi\rangle\end{aligned}$$

The expectation value of the operator $\hat{O}$ is given by,

$$\left\langle \hat{O} \right\rangle = \langle\psi(t)|\, \hat{O}\, |\psi(t)\rangle$$

Hermitian

$$\begin{aligned}\langle\psi(t)|\, \hat{O}\, |\psi(t)\rangle &= \left[\langle\psi(t)|\, \hat{O}^{\dagger}\, |\psi(t)\rangle\right]^{*} = \langle\psi(t)|\, \hat{O}\, |\psi(t)\rangle^{*} \\ \hat{O}^{\dagger} &= \hat{O}\end{aligned}$$

The set of eigen vectors of a hermitian operator is complete. This means that any arbitrary vector $|\psi(t)\rangle$ can be expressed as a sum of orthogonal eigen vectors.

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n\, |x_n\rangle$$

Eigen vectors are orthonormal

$$\begin{aligned}\langle X_n|\, X_m\rangle &= \delta_{nm} \\ \delta_{nm} &= 1\ for\ n = m \\ &= 0\ for\ n \neq m\end{aligned}$$

Completeness relation for discrete case is

$$\sum_n |X_n\rangle \langle X_n| = I$$

The state vector $|\psi\rangle$ in terms of position eigen states (which are continuous).

$$\begin{aligned}|\psi(t)\rangle &= \int d\overrightarrow{x}\, |x\rangle \langle x|\, \psi\rangle \\ \int d\overrightarrow{x}\, |x\rangle \langle x| &= I\end{aligned}$$

The normalization of eigen vectors with a continuous set of eigen values must be normalized with the help of dirac delta function having properties,

$$\begin{aligned}\delta(x - x^{'}) &= 0 \quad\quad if\ \ x \neq x^{'} \\ \delta(x - x^{'}) &= \infty \quad\quad\quad x = x^{'} \\ \langle x|\, x^{'}\rangle &= \delta(x - x^{'})\end{aligned}$$

State vectors obey the Schrodinger's equation

$$\begin{aligned} i\hbar \left|\dot{\psi}\right\rangle &= H\left|\psi\right\rangle, \\ \left|\psi\right\rangle &= \sum_n C_n e^{-i\omega_n t}\left|n\right\rangle \end{aligned}$$

Expectation value can be written as

$$\left|\psi\right\rangle = \sum_{n,m} C_n^* C_m e^{-i(\omega_n - \omega m)t}\left|n\right\rangle \hat{O}_{nm}$$

Where

$$\hat{O}_{nm} = \left\langle m\right|\hat{O}\left|n\right\rangle$$

1.6 Two level system:

Wave function for two level system is

$$\psi(r,t) = C_a U_a(\overrightarrow{r})e^{-i\omega_a t} + C_b U_b(\overrightarrow{r})e^{-i\omega_b t}$$

State vector

$$\left|\psi\right\rangle = C_a e^{-i\omega_a t}\left|a\right\rangle + C_b e^{-i\omega_b t}\left|b\right\rangle$$

1.7 SCHRODINGER, INTERACTION AND HEISENBERG PICTURES:

1.8 SCHRODINGER PICTURE:

The interaction of radiation with matter involves a hamiltonian.

$$H = H_\circ + V$$

$H_\circ$ - unperturbed energy
V - Interaction energy
The corresponding Schrodinger equation

$$\begin{aligned} \left|\dot{\psi}(t)\right\rangle &= \frac{-i}{\hbar}H\left|\psi(t)\right\rangle \\ \left|\dot{\psi}(t)\right\rangle &= \frac{-i}{\hbar}(H_\circ + V)\left|\psi(t)\right\rangle, \\ \left|\psi(t)\right\rangle &= e^{\frac{-iHt}{\hbar}}\left|\psi(0)\right\rangle \end{aligned}$$

Expectation value of an operator $\hat{O}$ which represents the observables.

$$\left\langle \hat{O}\right\rangle = \left\langle\psi(t)\right|\hat{O}(0)\left|\psi(t)\right\rangle$$

Operator $\hat{O}$ is independent of time, but $\left|\psi(t)\right\rangle$ is a function of time. This is the schrodinger picture way of writing the expectation value of an operator.

1.9 HEISENBERG PICTURE:

In Heisenberg picture total time dependence goes into operator, so state vector is independent of time, the expectation value of an operator in Schrodinger picture is,

$$\left\langle \hat{O} \right\rangle = \langle\psi(t)|\, \hat{O}(0)\, |\psi(t)\rangle$$

It can also be written as,

$$\left\langle \hat{O}\,(t) \right\rangle = \langle\psi(t)|\, e^{\frac{-iHt}{\hbar}} e^{\frac{+iHt}{\hbar}} \hat{O}(0) e^{\frac{-iHt}{\hbar}} e^{\frac{+iHt}{\hbar}} \,|\psi(t)\rangle$$

Where H is the total Hamiltonian. According to Schrodinger equation,

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H\, |\psi(t)\rangle$$

Integrating

$$\begin{aligned} |\psi(t)\rangle &= e^{\frac{-iHt}{\hbar}}\, |\psi(0)\rangle \\ e^{\frac{+iHt}{\hbar}}\, |\psi(t)\rangle &= |\psi(0)\rangle \end{aligned}$$

Using this we can write

$$\left\langle \hat{O}\,(t) \right\rangle = \langle\psi(0)|\, e^{\frac{+iHt}{\hbar}} \hat{O}(0) e^{\frac{-iHt}{\hbar}} \,|\psi(0)\rangle$$

Define

$$\hat{O}(t) = e^{\frac{+iHt}{\hbar}} \hat{O}(0) e^{\frac{-iHt}{\hbar}}$$

Then

$$\left\langle \hat{O}\,(t) \right\rangle = \langle\psi(0)|\, \hat{O}(t)\, |\psi(0)\rangle$$

Which is called Heisenberg picture. In this all time dependence lies in operator.while wave function is independent of time.

1.9.1 Why called Heisenberg picture?

$$\begin{aligned} \hat{O}(t) &= e^{\frac{+iHt}{\hbar}} \hat{O}(0) e^{\frac{-iHt}{\hbar}} \\ \dot{O}(t) &= \frac{i}{\hbar} H\hat{O} + \frac{-i}{\hbar} \hat{O} H \\ \dot{O}(t) &= \frac{i}{\hbar} \left[H, \hat{O} \right] \end{aligned}$$

Which is Heisenberg equation of motion.That is why we call it Heisenberg picture. In between two extremes of Schrodinger and Heisenberg picture. There is an intermediate picture called interaction picture.

1.10 INTERACTION PICTURE:

Consider again the equation,

$$\left\langle \hat{O}\left(t\right)\right\rangle = \left\langle \psi(0)\right| e^{\frac{+iHt}{\hbar}} \hat{O}(0) e^{\frac{-iHt}{\hbar}} \left|\psi(0)\right\rangle$$

As
$H = H_\circ + V$
$H_\circ$- free Hamiltonian,
V- interaction part of Hamiltonian,

If the time dependence created by the interaction energy is only assigned to the state vector and rest of time dependence goes to the operator, then expectation value is written as,

$$\begin{aligned} \left\langle \hat{O}\left(t\right)\right\rangle &= \left\langle \psi(0) e^{\frac{+ivt}{\hbar}} \right| e^{\frac{+iH_\circ t}{\hbar}} \hat{O}(0) e^{\frac{-iH_\circ t}{\hbar}} \left| e^{\frac{-ivt}{\hbar}} \psi(0) \right\rangle \\ \left\langle \hat{O}\left(t\right)\right\rangle &= \left\langle \psi^I(t)\right| \hat{O}(t) \left|\psi^I(t)\right\rangle \end{aligned}$$

Interaction picture state vector

$$\left|\psi^I(t)\right\rangle = e^{\frac{-ivt}{\hbar}} \left|\psi(0)\right\rangle$$

Equation of motion

$$\left|\dot{\psi}^I(t)\right\rangle = \frac{-iV}{\hbar} \left|\psi^I(t)\right\rangle$$

This equation is simpler than ordinary schrodinger equation, but requires the calculation of $\hat{O}^I(t)$, where

$$\hat{O}^I\left(t\right) = e^{\frac{+iH_\circ t}{\hbar}} \hat{O}(0) e^{\frac{-iH_\circ t}{\hbar}}$$

In interaction picture both state vector and operator are time dependent. The interaction picture state vector,

$$\left|\psi^I(t)\right\rangle = \sum_n C_n(t) \left|n\right\rangle$$

The schrodinger picture state vector is

$$\left|\psi(t)\right\rangle = \sum_n c_n(t) \left|n\right\rangle = \sum_n C_n(t) e^{\frac{-i\omega_n t}{\hbar}} \left|n\right\rangle$$

Where

$$c_n(t) = C_n(t) e^{\frac{-i\omega_n t}{\hbar}}$$

The complete time dependence is given by $c_n(t)$, but due to interaction energy is given by $C_n(t)$.

1.11 PAULI SPIN MATRIX:

Another method to describe two- level atom is a use of 2×2 matrix notation. The eigen function U_a and U_b or eigen vectors $|a\rangle$ and $|b\rangle$ can be represented by the column vectors

$$\begin{aligned} |a\rangle &= U_a \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |b\rangle &= U_b \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

And the wave function and wave vector by the column vectors

$$\psi(r,t) = \begin{bmatrix} C_a \\ C_b \end{bmatrix}$$

And

$$|\psi(r,t)\rangle = \begin{pmatrix} C_a \\ C_b \end{pmatrix}$$

The energy and electric- dipole operators are written in terms of the Pauli spin matrices as

$$\begin{aligned} \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

These matrices are hermition, but the spin- flip operators

$$\begin{aligned} \sigma_+ &= \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \text{and} \qquad \sigma_- &= \frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

are not hermitian, σ_- flips the system from upper- level to a lower- level

$$\sigma_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

While, σ_+ flips the system from lower- level to the upper- level

$$\sigma_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

1.12 TWO LEVEL ATOMIC SYSTEM AND HAMILTONIAN IN TERMS OF MATRICES:

State vector of a system can be written as,

$$|\psi(r,t)\rangle = C_a e^{-i\omega_a t} |a\rangle + C_b e^{-i\omega_b t} |b\rangle \,,$$

Which corresponds to the wave function

$$\psi(r,t) = C_a U_a(\overrightarrow{r}) e^{-i\omega_a t} + C_b U_b(\overrightarrow{r}) e^{-i\omega_b t}$$

The matrix form from for the unit vectors (eigen vectors) $|a\rangle$ and $|b\rangle$are

$$\psi(r,t) \leftrightarrow C_a e^{-i\omega_a t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_b e^{-i\omega_b t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_a e^{-i\omega_a t} \\ C_b e^{-i\omega_b t} \end{pmatrix}$$

The two- level Hamiltonian of the semi-classical treatment is given as,

$$H = \hbar\omega_a |a\rangle \langle a| + V_{ab} |a\rangle \langle b| + V_{ba} |b\rangle \langle a| + \hbar\omega_b |b\rangle \langle b|$$

In matrix notation

$$H = \begin{pmatrix} H_{aa} & H_{ab} \\ H_{ba} & H_{bb} \end{pmatrix} = \begin{pmatrix} \hbar\omega_a & V_{ab} \\ V_{ba} & \hbar\omega_b \end{pmatrix}$$

Thus the matrix form of the schrodinger equation is

$$i\hbar \frac{d}{dt} \begin{pmatrix} C_a e^{-i\omega_a t} \\ C_b e^{-i\omega_b t} \end{pmatrix} = \begin{pmatrix} \hbar\omega_a & V_{ab} \\ V_{ba} & \hbar\omega_b \end{pmatrix} \begin{pmatrix} C_a e^{-i\omega_a t} \\ C_b e^{-i\omega_b t} \end{pmatrix}$$

1.14 DENSITY OPERATOR

For a given physical system there exist a state vector $|\psi\rangle$ which contain all possible information about the system. If we want to extract a piece of information about the system we must calculate the expectation value of the corresponding operator $\hat{O}$

$$\left\langle \hat{O} \right\rangle_{Q\ M} = \langle\psi| \hat{O} |\psi\rangle$$

In many situations we do not know $|\psi\rangle$ but we know P_ψ, probability of finding the system in $|\psi\rangle$. For such a situation we not only need to take quantum mechanical average but also the ensemble average over many identical systems that have been similarly prepared

$$\left\langle \left\langle \hat{O} \right\rangle_{Q\ M} \right\rangle_{ensemble} = \sum_\psi P_\psi \langle\psi|\hat{O}|\psi\rangle$$

It is called a quantum statistical system.

$$\begin{aligned}
\sum_n |n\rangle \langle n| &= 1 \\
\left\langle \left\langle \hat{O} \right\rangle_{Q\ M} \right\rangle_{ensemble} &= \sum_n \sum_\psi P_\psi \langle\psi|n\rangle \langle n| \hat{O} |\psi\rangle \\
&= \sum_n \sum_\psi P_\psi \langle n| \hat{O} |\psi\rangle \langle\psi|n\rangle \\
&= \sum_n \langle n| \hat{O} \left[\sum_\psi P_\psi |\psi\rangle \langle\psi| \right] |n\rangle \\
&= \sum_n \langle n| \hat{O}\rho |n\rangle = \sum_n (\hat{O}\ \rho)_{nn}
\end{aligned}$$

where

$$\rho = \sum_\psi P_\psi |\psi\rangle \langle\psi|$$

The sum of diagonal elements gives

$$\left\langle \left\langle \hat{O} \right\rangle_{Q\ M} \right\rangle_{enemble} = Tr(\hat{O}\rho) = Tr(\rho\hat{O})$$

where

$$\rho = \sum_\psi P_\psi |\psi\rangle \langle\psi|$$

is called density operator. In a particular case where all P_ψ are zero except the one for a state $|\psi_\circ\rangle$ then

$$\rho = |\psi_\circ\rangle \langle\psi_\circ|$$

and state is called a Pure state. It follows from the conservation of probability that $Tr(\rho) = 1$. For a pure state

$$Tr(\rho^2) = 1$$

For mixed case

$$Tr(\rho^2) \prec 1$$

1.15 EQUATION OF MOTION FOR DENSITY OPERATOR

$$\begin{aligned}\rho &= \sum_\psi P_\psi |\psi\rangle \langle\psi| \\ \dot{\rho} &= \sum_\psi P_\psi [\left|\dot{\psi}\right\rangle \langle\psi| + |\psi\rangle \left\langle\dot{\psi}\right|]\end{aligned}$$

From Schrodinger equation, we know that

$$\left|\dot{\psi}\right\rangle = \frac{1}{i\hbar} H|\psi\rangle$$

Also

$$\langle\dot{\psi}| = -\frac{1}{i\hbar}\langle\psi|H$$

Using these values, we get

$$\begin{aligned}\dot{\rho} &= \frac{1}{i\hbar}\sum_\psi P_\psi [H|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| H] \\ &= \frac{1}{i\hbar}[H\rho - \rho H]\end{aligned}$$

where ρ and H are operators.

$$\dot{\rho} = \frac{1}{i\hbar}[H, \rho]$$

It is called "Liouville equation" or "von Neumann equation" and it is equivalent to Schrodinger , but more general (because it has quantum mechanical as well as statistical aspect). In the above equation we have not included the decay of atomic levels due spontaneous emission. The excited atomic levels can also decay because of colllision and other phenomena. The finite life time of the atomic level can be dicribed by adding phenominalogical decay terms to the density operator.

The decay rates can be incorporated in equation by a relaxation matrix Γ, which is defined by the equation

$$\Gamma_{nm} = \langle n|\Gamma|m\rangle = \gamma_n \delta_{nm}$$

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] - \frac{1}{2}\{\Gamma,\rho\}; \qquad \{\Gamma\rho + \rho\Gamma\} = \{\Gamma,\rho\}$$

Equation of motion for the density matrix elements is

$$\dot{\rho}_{ij} = \frac{1}{i\hbar}\sum_k [H_{ik}\rho_{kj} - \rho_{ik}H_{kj}] - \frac{1}{2}\sum_k [\Gamma_{ik}\rho_{kj} + \rho_{ik}\Gamma_{kj}]$$

1.16 TWO LEVEL ATOM

$$\begin{aligned}
|\psi\rangle &= c_a(t)\,|a\rangle + c_b(t)\,|b\rangle \\
\rho &= |\psi\rangle\,\langle\psi| \\
\langle\psi| &= c_a^*(t)\,\langle a| + c_b^*(t)\,\langle b| \\
\rho &= |\psi\rangle\,\langle\psi| = |c_a(t)|^2\,|a\rangle\,\langle a| + c_a(t)c_b^*(t)\,|a\rangle\,\langle b| + c_a^*(t)c_b(t)\,|b\rangle\,\langle a| + |c_b(t)|^2\,|b\rangle\,\langle b| \\
\rho_{aa} &= \langle a|\rho|a\rangle = |c_a|^2 \qquad \text{probability of upper state} \\
\rho_{bb} &= \langle b|\rho|b\rangle = |c_b|^2 \qquad \text{probability of lower state} \\
\rho_{ab} &= c_a c_b^* \qquad \text{is propotional to dipole moment} \\
\rho_{ba} &= c_b c_a^* = \ \rho_{ab}^*
\end{aligned}$$

In matrix form

$$\rho = \begin{pmatrix} |c_a|^2 & c_a c_b^* \\ c_a^* c_b & |c_b|^2 \end{pmatrix}$$

$$P(z,t) = c_a c_b^* p_{ba} + c.c = \rho_{ab}(z,t)p_{ba} + c.c$$

In spiner notation,

$$|\psi\rangle = \begin{pmatrix} c_a \\ c_b \end{pmatrix} \qquad \langle\psi| = \begin{pmatrix} c_a^* & c_b^* \end{pmatrix}$$

$$\rho = \begin{pmatrix} c_a \\ c_b \end{pmatrix}\begin{pmatrix} c_a^* & c_b^* \end{pmatrix} = \begin{pmatrix} |c_a|^2 & c_a c_b^* \\ c_a^* c_b & |c_b|^2 \end{pmatrix}$$

1.17 EQUATION OF MOTION FOR DENSITY MATIRX ELEMENTS

Consider $\quad i = a, j = a$

$$\dot{\rho}_{aa} = \frac{1}{i\hbar}\sum_{k=a,b}[H_{ak}\rho_{ka} - \rho_{ak}H_{ka}] - \frac{1}{2}\sum_{k=a,b}[\Gamma_{ak}\rho_{ka} + \rho_{ak}\Gamma_{ka}]$$

using this

$$
\begin{aligned}
\Gamma_{nm} &= \langle n|\Gamma|m\rangle = \gamma_n\delta_{nm} \qquad \text{and} \\
H &= H_\circ + \gamma \\
\dot{\rho}_{aa} &= \frac{1}{i\hbar}\sum_{k=a,b}[(H_\circ)_{ak}\rho_{ka} - \rho_{ak}(H_\circ)_{ka}] - \frac{1}{2}[\gamma_a\rho_{aa} + \rho_{aa}\gamma_a] \\
&+\frac{1}{i\hbar}\sum_{k=a,b}[V_{ak}\rho_{ka} - \rho_{ak}V_{ka}] \\
(H_\circ)_{ab} &= \langle a|H_\circ|b\rangle = E_b\langle a|b\rangle = 0 \\
(H_\circ)_{aa} &= \langle a|H_\circ|a\rangle = E_a \qquad \text{and} \qquad V_{aa=}V_{bb} = 0 \\
\dot{\rho}_{aa} &= -\gamma_a\delta_{aa} + \frac{1}{i\hbar}\sum_{k=a,b}[V_{ab}\rho_{ba} - \rho_{ab}V_{ba}] \\
\dot{\rho}_{bb} &= -\gamma_b\delta_{bb} + \frac{1}{i\hbar}[V_{ba}\rho_{ab} - \rho_{ba}V_{ab}] \\
\dot{\rho}_{ab} &= -(i\omega_\circ + \gamma_{ab})\delta_{ab} + \frac{1}{i\hbar}[V_{ab}\rho_{bb} - \rho_{aa}V_{ab}] \\
&\text{where} \\
\gamma_{ab} &= \frac{1}{2}(\gamma_a + \gamma_b) \qquad \text{and} \qquad \omega_\circ = \frac{1}{\hbar}(E_a - E_b)
\end{aligned}
$$

The population of excited level decays in time because of spontaneous emission. In soome cases the upper level decays to ground state lower level then

$$\dot{\rho}_{aa} = -\Gamma\rho_{aa} + \frac{1}{i\hbar}[V_{ab}\rho_{ba} - \rho_{ab}V_{ba}]$$

and

$$\dot{\rho}_{bb} = \Gamma\rho_{bb} + \frac{1}{i\hbar}[V_{ba}\rho_{ab} - \rho_{ba}V_{ab}]$$

Where Γ is the upper-to-lower level decay constant.

1 Quantization of the Free Electromagnetic Field

Dirac combined the wave and particle like aspects of light. Wave nature shows all the interference phenomena. Particle nature shows the exitation of a specific atom absorbing one photon of energy.

Classical field fails to explain

1. Spontaneous emission
2. Atomic decay
3. Lamb shift
4. Photon statistics

An interesting consequences of the quantization of the radiation is the fluctuations associated with the zero-point energy or so called vacuum fluctuations. These fluctuations have no classical analog and are responsible for many interacting phenomena in quantum optics.

1.1 Spontaneous Emission and Atomic Decay

A phenomena which we described phenomenologically in our treatment of semi-classical theory requires a quantum field. Spontaneous emission is often said to be the result of stimulating the atom by vacuum fluctuations.

1.2 Lamb Shift

According to the classical description of the field (*Dirac theory*) the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states in the hydrogen atom should have equal energies. Experimentally the two levels differ by approximately 1057 MHz. a fully quantized treatment of the field and atomic systems gives impressive agreement with the experimentally observed shift, because of the radiative correction due to the interaction between the atomic electron and the vacuum shift the $2S_{\frac{1}{2}}$ level higher in energy by around 1057 MHz relative to the $2P_{\frac{1}{2}}$ level.

1.3 Photon Statistics

In order to explain the photon statistics the concept of a particle, the photon is either necessary or convenient. For the quantization of the electromagnetic field in free space, it is convenient to begin with the classical description of the field based on Maxwell's equations. In MKS system

$$\begin{aligned} \nabla . D &= 0 \\ \nabla . B &= 0 \\ \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \times H &= \frac{\partial D}{\partial t} \end{aligned}$$

where in free space

$$\begin{aligned} D &= \epsilon_0 E \\ B &= \mu_0 H \end{aligned}$$

here ϵ_0 and μ_0 are the free space permitivity and permeability respectively and

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

where c is the speed of light in vacuum. Using these Maxwell,s equations we know that $E(r,t)$ and also $B(r,t)$ satisfies the wave equation

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Following the Dirac approach we associate each mode of the radiation field with a quantized simple Harmonic oscillator. Energy of the Harmonic oscillator (classically) is given by the hamiltonian

$$H = \frac{1}{2} m\omega^2 x^2 + \frac{p^2}{2m}$$

and quantum mechanically it is written as

$$H = \frac{1}{2} m\omega^2 \hat{x}^2 + \frac{\hat{p}^2}{2m}$$

2 Mode Expansion of The Field

2.1 Quantization of Field Inside the Cavity of Length L

Electric field is linearly polarized in the x direction. Expanding the field in the normal modes of the cavity

$$E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z)$$

j corresponds to different modes such that

$$L = j\frac{\lambda}{2};\ \lambda = \frac{2\pi}{k} \text{ and } L = \frac{j\pi}{k_j}, \text{where } j = 1, 2, 3, ...$$

Where q_j is the normal mode amplitude with the dimensions of length (position) and

$$\begin{aligned} A_j &= \left(\frac{2\omega_j^2 m_j}{V\epsilon_0}\right)^{1/2} \\ \text{where } \omega_j &= ck_j = \frac{j\pi c}{L} \end{aligned}$$

is the cavity eigen frequency. $V = LA$ is the volume (A is the transverse area of the optical resonator) m_j is a constant with the dimensions of mass, included to make an anology with SHO nothing to do with mass of photon. The E.M.F is assumed to be transverse with electric field polarized in the x-direction. Such field satisfies

$$\nabla .E = 0$$

The nonvanishing component of the magnetic field in the cavity is obtained by using Maxwell's 4rth equation i.e,

$$\nabla \times H = \frac{\partial D}{\partial t} = \epsilon_0 \frac{\partial E}{\partial t}$$

$$\begin{aligned} \nabla \times H &= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{pmatrix} \\ &= \hat{i}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{j}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{k}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \end{aligned}$$

x-component of $(\nabla \times H)_x$ is written as

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)_x = \epsilon_0 \frac{\partial E_x}{\partial t}$$

$H_z = 0$, z is the direction of propagation

$$\begin{aligned} -\frac{\partial H_y}{\partial z} &= \epsilon_0 \frac{\partial}{\partial t}\left(\sum_j A_j q_j(t) \sin(k_j z)\right) \\ &= \epsilon_0 \sum_j A_j \dot{q_j}(t) \sin(k_j z). \end{aligned}$$

As

$$\sin(k_j z) = -\frac{1}{k_j}\frac{\partial}{\partial z}\cos(k_j z),$$

putting the value of $\sin(k_j z)$ in the above equation we can write

$$\begin{aligned} -\frac{\partial H_y}{\partial z} &= -\frac{\partial}{\partial z}\left(\sum_j A_j \left(\frac{\dot{q_j}(t)\,\epsilon_0}{k_j}\right)\cos(k_j z)\right), \\ H_y &= \sum_j A_j \left(\frac{\dot{q_j}(t)\,\epsilon_0}{k_j}\right)\cos(k_j z). \end{aligned}$$

The classical Hamiltonian of the field i.e, the (total energy of the field) is

$$H = \frac{1}{2}\int_v d\tau \left(\epsilon_0 E_x^2 + \mu_0 H_y^2\right)$$

where the integration is over the volume of the cavity. Substituting the values of E_x and H_y in the above equation and performing the integration we get,

$$
\begin{aligned}
H &= \frac{1}{2}\sum_j \left(m_j \omega_j^2 q_j^2 + m_j \dot{q}_j^2 \right) \\
&= \frac{1}{2}\sum_j \left(m_j \omega_j^2 q_j^2 + \frac{p_j^2}{m_j} \right)
\end{aligned}
$$

where $p_j = m_j \dot{q}_j$ is the canonical momentum of the jth mode. The above equation expresses the hamiltonian of the radiation field as a sum of independent oscillator energies. Each mode of the field is dynamically equivalent to a mecahanical harmonic oscillator.

3 Quantization

The present dynamical problem can be quantized by identifying q_j and p_j as operators, which obey the commutation relations

$$
\begin{aligned}
\left[\hat{q}_j, \hat{p}_{j'}\right] &= i\hbar\delta_{jj'}. \\
\left[\hat{q}_j, \hat{q}_{j'}\right] &= \left[\hat{p}_j, \hat{p}_{j'}\right] = 0.
\end{aligned}
$$

It can be transformed as

$$
\widehat{a}_j = \frac{1}{\sqrt{2m_j\hbar\omega_j}} \left(m_j\omega_j\widehat{q}_j + i\widehat{p}_j \right) \exp\left(i\omega_j t \right)
$$

and

$$
\widehat{a}_j^{\dagger} = \frac{1}{\sqrt{2m_j\hbar\omega_j}} \left(m_j\omega_j\widehat{q}_j - i\widehat{p}_j \right) \exp\left(-i\omega_j t \right)
$$

$$
\begin{aligned}
\widehat{q}_j &= \left(\widehat{a}_j \exp\left(-i\omega_j t\right) + \widehat{a}^{\dagger}{}_j \exp\left(i\omega_j t\right) \right) \sqrt{\frac{\hbar}{2m_j\omega_j}} \\
\widehat{p}_j &= -i\sqrt{\frac{m_j\omega_j\hbar}{2}} \left(\widehat{a}_j \exp\left(-i\omega_j t\right) - \widehat{a}^{\dagger}{}_j \exp\left(i\omega_j t\right) \right)
\end{aligned}
$$

The commutation relations between $\widehat{a}_j$ and $\widehat{a}^{\dagger}$ follow from those between $\widehat{q}_j$ and $\widehat{p}_j$,

$$
\begin{aligned}
\left[\hat{a}_j, \hat{a}_j^{\dagger}\right] &= \frac{1}{2m_j\hbar\omega_j} \left[-im_j\omega_j \left[\hat{q}_j, \hat{p}_j\right] + im_j\omega_j \left[\hat{p}_j, \hat{q}_j\right] \right] \\
&= \frac{1}{2m_j\hbar\omega_j} \left[-im_j\omega_j \left(i\hbar\right) + im_j\omega_j \left(-i\hbar\right) \right] \\
&= 1.
\end{aligned}
$$

Similarly

$$\left[\hat{a}_j, \hat{a}_{j'}\right] = \left[\hat{a}^\dagger{}_j, \hat{a}^\dagger{}_{j'}\right] = 0$$

$$\left[\hat{a}_j, \hat{a}^\dagger{}_{j'}\right] = \delta_{jj'}$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ are referred to as the destruction and creation operators, they are not hermitian. Substituting the value of $\widehat{q}_j$ and $\widehat{p}_j$ in the equation for Hamiltonian we get

$$\begin{aligned} H &= \sum_j \frac{1}{2} m_j \omega_j^2 \left(\frac{\hbar}{2m_j\omega_j}\right)\left(a_j^2 \exp\left(-2i\omega_j t\right) + a^{\dagger 2}{}_j \exp\left(2i\omega_j t\right) + \hat{a}_j \hat{a}_j^\dagger + \hat{a}_j^\dagger \hat{a}_j\right) \\ &\quad + \frac{1}{2m_j}\left(-\frac{m_j \hbar\omega_j}{2}\right)\left(a_j^2 \exp\left(-2i\omega_j t\right) + a^{\dagger 2}{}_j \exp\left(2i\omega_j t\right) - \hat{a}_j \hat{a}_j^\dagger - \hat{a}_j^\dagger \hat{a}_j\right) \\ &= \sum_j \left(\frac{\hbar\omega_j}{2}\right)\left(\hat{a}_j \hat{a}_j^\dagger + \hat{a}_j^\dagger \hat{a}_j\right) \end{aligned}$$

As

$$\begin{aligned} \left[\hat{a}_j, \hat{a}_j^\dagger\right] &= 1 \\ \left(\hat{a}_j \hat{a}_j^\dagger - \hat{a}_j^\dagger \hat{a}_j\right) &= 1 \\ \hat{a}_j \hat{a}_j^\dagger &= \hat{a}_j^\dagger \hat{a}_j + 1 \end{aligned}$$

thus we can write

$$H = \sum_j \hbar\omega_j \left(\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2}\right)$$

In terms of $\hat{a}_j$and $\hat{a}_j^\dagger$, the electric and magnetic fields can be written as

$$\begin{aligned} E_x(z,t) &= \sum_j \varepsilon_j \left(\widehat{a}_j \exp\left(-i\omega_j t\right) + \widehat{a}^\dagger{}_j \exp\left(i\omega_j t\right)\right)\sin\left(k_j z\right) \\ H_y(z,t) &= -i\epsilon_0 c \sum_j \varepsilon_j \left(\widehat{a}_j \exp\left(-i\omega_j t\right) - \widehat{a}^\dagger{}_j \exp\left(i\omega_j t\right)\right)\cos\left(k_j z\right) \end{aligned}$$

where the quantity

$$\varepsilon_j = \left(\frac{\hbar\omega_j}{\epsilon_0 V}\right)^{1/2}$$

has the dimensions of an electric field.

3.1 Quantization of Field Inside a Large Cavity of Finite Length L

Consider the field in a large but finite cubic cavity of side L. We consider the running wave solutions instead of the standing wave solutions. The classical electric and magnetic field can be expanded in terms of plane waves.

$$E(r,t) = \sum_k \hat{\epsilon}_k \varepsilon_k \alpha_k \exp(-i\omega_k t + ik.r) + c.c$$

using Maxwell's equation i.e,

$$\nabla \times H = \frac{\partial D}{\partial t}$$

we get

$$H(r,t) = \frac{1}{\mu_0} \sum_k \frac{k \times \hat{\epsilon}_k}{\omega_k} \varepsilon_k \alpha_k \exp(-i\omega_k t + ik.r) + c.c$$

where the summation is taken over an infinite discrete set of values of wave vector $k = (k_x, k_y, k_z)$, ϵ_k is a unit polarization vector, α_k is a dimensionless amplitude and

$$\varepsilon_k = \left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}$$

Periodic boundary conditions require that

$$k_x = \frac{2\pi n_x}{L}, k_y = \frac{2\pi n_y}{L}, k_z = \frac{2\pi n_z}{L}$$

where n_x, n_y, n_z are integers $0, \pm 1, \pm 2, \cdots$. A set of numbers (n_x, n_y, n_z) defines a mode of electromagnetic field. For transverse field

$$\nabla.D = 0$$

which requires

$$\vec{k}.\hat{\epsilon}_k = 0$$

There are two independent polarization directions of $\hat{\epsilon}_k$ for each k. Changing from a discrete distribution of modes to a continous distribution i.e,

$$\sum_k \Longrightarrow 2\left(\frac{L}{2\pi}\right)^3 \int d^3k$$

where factor of 2 accounts for two possible states of polarization. The number of modes available in a cavity is infinite, however the number of modes whose frequency lies between ω and $\omega + d\omega$ is finite. This is the same number of field modes having the magnitude of k, between k and $k + dk$. Making transformation from rectangular coordinates (k_x, k_y, k_z) to the polar coordinates $(k\sin\theta\cos\phi, k\sin\theta\sin\phi, k\cos\theta)$, the volume element in k-space is written as

$$\begin{aligned} d^3k &= k^2 dk \sin\theta d\theta d\phi \\ &= \frac{\omega^2}{c^3} d\omega \sin\theta d\theta d\phi. \end{aligned}$$

The total number of modes in the volume L^3 in the range between ω and $\omega + d\omega$ is given by

$$\begin{aligned} dN &= 2\left(\frac{L}{2\pi}\right)^3 \frac{\omega^2 d\omega}{c^3} \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \\ &= \frac{L^3\omega^2}{\pi^2 c^3} d\omega \end{aligned}$$

Radiation field is quantized by identifying α_k and α_k^* by the harmonic oscillator operators $\hat{a}_k$and $\hat{a}_k^\dagger$ respectively, which satisfiy the commutation relation

$$\left[\hat{a}_k, \hat{a}_k^\dagger\right] = 1$$

The quantized electric and magnetic fields takes the form

$$\begin{aligned} E(r,t) &= \sum_k \hat{\epsilon}_k \varepsilon_k \hat{a}_k \exp(-i\omega_k t + ik.r) + H.C \\ H(r,t) &= \frac{1}{\mu_0} \sum_k \frac{k \times \hat{\epsilon}_k}{\omega_k} \varepsilon_k \hat{a}_k \exp(-i\omega_k t + ik.r) + H.C \end{aligned}$$

where H.C is Hermetian conjugate. Seperating positive and negative frequency parts of these field operators

$$\begin{aligned} E^+(r,t) &= \sum_k \hat{\epsilon}_k \varepsilon_k \hat{a}_k \exp(-i\omega_k t + ik.r) \\ E^-(r,t) &= \sum_k \hat{\epsilon}_k \varepsilon_k \hat{a}_k^\dagger \exp(i\omega_k t - ik.r) \end{aligned}$$

where $E^+(r,t)$ is the annihilation operator and $E^-(r,t)$ is the creation operator.

4 Fock or Number States of Radiation Field

Consider a single mode of the field of frequency ω having creation and annihilation operators $\widehat{a}^{\dagger}$ and $\widehat{a}$ respectively. Let $|n\rangle$ be the energy eigen state corresponding to the energy eigen value E_n, i,e.

$$\begin{aligned} H\,|n\rangle &= \hbar\omega\left(a^{\dagger}a+\frac{1}{2}\right)|n\rangle \\ &= E_n\,|n\rangle \end{aligned} \tag{1}$$

applying operator $\hat{a}$ from the left of the eigenstates we have

$$Ha\,|n\rangle = \hbar\omega\left(a^{\dagger}a+\frac{1}{2}\right)a\,|n\rangle \tag{2}$$

$$\begin{aligned} \left[a,a^{\dagger}\right] &= aa^{\dagger}-a^{\dagger}a=1 \\ \Longrightarrow & \; aa^{\dagger}-1=a^{\dagger}a \end{aligned}$$

Putting in Eq. (2) we get

$$\begin{aligned} Ha\,|n\rangle &= \hbar\omega\left(aa^{\dagger}-1+\frac{1}{2}\right)a\,|n\rangle \\ &= \hbar\omega\left(aa^{\dagger}a-a+\frac{a}{2}\right)|n\rangle \\ &= a\hbar\omega\left(a^{\dagger}a+\frac{1}{2}-1\right)|n\rangle \\ &= a(\hbar\omega(a^{\dagger}a+\frac{1}{2})-\hbar\omega)\,|n\rangle \\ &= a(E_n-\hbar\omega)\,|n\rangle \\ &= (E_n-\hbar\omega)a\,|n\rangle \end{aligned}$$

where $a\,|n\rangle$ is an energy eigen state with eigen value $(E_n-\hbar\omega)$. Operator a lowers the energy and therefore it is called annihilation, destruction or absorption operator.

$$\Longrightarrow |n-1\rangle = \frac{a}{\alpha_n}\,|n\rangle\,,$$

is an energy eigen state but with the reduced eigen value i,e.

$$\begin{aligned} E_{n-1} &= (E_n-\hbar\omega) \\ H\,|n-1\rangle &= E_{n-1}\,|n-1\rangle\,, \end{aligned}$$

and α_n is a constant which will be determined from the normalization condition,

$$\langle n-1 \mid n-1\rangle = 1.$$

If we repeat this procedure n times we move down the energy ladder in steps of $\hbar\omega$ until we obtain

$$Ha\,|0\rangle = (E_0-\hbar\omega)\,a\,|0\rangle$$

E_0 is the ground state energy . $E_n - \hbar\omega$ is smaller than E_0 i,e, $E_n - \hbar\omega$ is negative. Since energy eigen value cannot be negative

$$a\,|0\rangle = 0$$

The state $|0\rangle$ is called the vaccum state. (in which no photon is excited).

$$\begin{aligned} H\,|0\rangle &= \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right)|0\rangle \\ &= \frac{1}{2}\hbar\omega\,|0\rangle \\ \Longrightarrow & \quad E_0 = \frac{1}{2}\hbar\omega \end{aligned}$$

is the energy of the ground state. Now we go step by step up as

$$\begin{aligned} E_{n-1} &= E_n - \hbar\omega \\ E_n &= E_{n-1} + \hbar\omega \end{aligned}$$

For $n = 1$ we can write

$$\begin{aligned} E_1 &= E_0 + \hbar\omega \\ &= \frac{1}{2}\hbar\omega + \hbar\omega = \frac{3}{2}\hbar\omega \end{aligned}$$

Similarly

$$\begin{aligned} E_2 &= E_1 + \hbar\omega \\ &= \frac{3}{2}\hbar\omega + \hbar\omega \\ &= \frac{5}{2}\hbar\omega \end{aligned}$$

It can also be written as

$$\begin{aligned} E_2 &= (2 + \frac{1}{2})\hbar\omega, \\ &\vdots \\ E_n &= \left(n + \frac{1}{2}\right)\hbar\omega \end{aligned}$$

$$\Longrightarrow H \mid n\rangle = E_n \mid n\rangle$$

$$\begin{aligned} \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right)|n\rangle &= \left(n + \frac{1}{2}\right)\hbar\omega\,|n\rangle \\ \Longrightarrow & \quad a^{\dagger}a\,|n\rangle = n\,|n\rangle \end{aligned}$$

$|n\rangle$ is also an energy eigen state of the number operator

$$n = a^\dagger a$$

The normalization constant can be now calculated

$$\langle n-1 \mid n-1\rangle = 1$$

as

$$\begin{aligned}
|n-1\rangle &= \frac{\hat{a}}{\alpha_n}|n\rangle \\
\langle n-1| &= \frac{\langle n| a^\dagger}{\alpha_n^*} \\
\langle n-1 \quad | \quad n-1\rangle &= \frac{\langle n| a^\dagger a |n\rangle}{\alpha_n^* \alpha_n} \\
&= \frac{1}{|\alpha_n|^2}\langle n| a^\dagger a |n\rangle \\
1 &= \frac{n}{|\alpha_n|^2}\langle n \mid n\rangle = |\alpha_n|^2 = n \\
\alpha &= \sqrt{n} e^{i\phi}
\end{aligned}$$

If we take the phase of the normalization constant α_n to be zero then $\alpha_n = \sqrt{n}$

$$\begin{aligned}
a|n\rangle &= \alpha_n \mid n-1\rangle \\
&= \sqrt{n} \mid n-1\rangle
\end{aligned}$$

now for operator $a^\dagger$

$$\begin{aligned}
Ha^\dagger |n\rangle &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) a^\dagger |n\rangle \\
&= \hbar\omega\left(a^\dagger a a^\dagger + \frac{a^\dagger}{2}\right)|n\rangle
\end{aligned}$$

using $aa^\dagger = a^\dagger a + 1$,

$$\begin{aligned}
Ha^\dagger \quad | \quad n\rangle = \hbar\omega\left(a^\dagger a^\dagger a + a^\dagger + \frac{a^\dagger}{2}\right)|n\rangle \\
&= a^\dagger\left(E_n + \hbar\omega\right)|n\rangle \\
(\hbar\omega\left(a^\dagger a + \frac{1}{2}\right) a^\dagger |n\rangle) &= \left(E_n + \hbar\omega\right) a^\dagger |n\rangle
\end{aligned}$$

Thus $a^\dagger|n\rangle$ is also an energy eigen state of the field with eigen value $E_n+\hbar\omega$. We define

$$\begin{aligned}
|n+1\rangle &= \frac{\hat{a}^\dagger}{\beta_n}|n\rangle \\
E_{n+1} &= E_n + \hbar\omega \\
&\Longrightarrow H|n+1\rangle = E_{n+1}|n+1\rangle
\end{aligned}$$

using the same procedure we get

$$a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

A repeated use of the above equation gives,

$$|n\rangle = \frac{(a^{\dagger})^{n}}{\sqrt{n!}}\mid 0\rangle$$

The energy eigen states $|n\rangle$ are called fock states or photon number states. They form a complete set of state i,e.

$$\sum_{n=0}^{\infty}|n\rangle\langle n| = 1$$

The energy eigen value are discrete in contrast to classical electromagnetic theory where energy can have any value. State vector is written as the superposition of energy eigen states. i,e

$$|\Psi\rangle = \sum_{n} c_n |n\rangle$$

where c_n are complex coefficients. The energy $E_0 = \frac{1}{2}\hbar\omega$ is called zero-point energy. The energy levels for Q.M oscillations associated with the electromagnetic field are given as

$$\begin{aligned}
-----------E_{n+1} &= \left(n+\frac{3}{2}\right)\hbar\omega \\
a^{\dagger}\uparrow \qquad & \\
-----------E_{n} &= \left(n+\frac{1}{2}\right)\hbar\omega \\
a\downarrow \qquad & \\
-----------E_{n-1} &= \left(n-\frac{1}{2}\right)\hbar\omega \\
&\vdots \\
-----------E_{2} &= \frac{5}{2}\hbar\omega \\
-----------E_{1} &= \frac{3}{2}\hbar\omega \\
-----------E_{0} &= \frac{1}{2}\hbar\omega
\end{aligned}$$

The operators a and $a^{\dagger}$ are not hermitian but some of the combinations are Hermitian such as,

$$\begin{aligned}
a_1 &= \left(a+a^{\dagger}\right)/2, \\
a_2 &= (a-a^{\dagger})/2i.
\end{aligned}$$

Different energy eigen states of the field are orthagonal. The only non-vanishing matrix elements of a and $a^\dagger$are of the types;

$$\begin{aligned} \langle n-1 \quad \mid \quad a \mid n\rangle &= \sqrt{n} \\ \langle n+1 \quad \mid \quad a^\dagger \mid n\rangle &= \sqrt{n+1} \end{aligned}$$

An important property of $\mid n\rangle$ is that the expectation value of the single mode linearly polarized field operator vanishes. Using

$$E_x\left(z,t\right) = \varepsilon(ae^{-i\omega t} + a^\dagger e^{i\omega t})\sin kz,$$

or

$$E\left(r,t\right) = \varepsilon a e^{-i\omega t + ik.r} + \varepsilon^* a^\dagger e^{i\omega t - ik.r}$$

$$\langle n \mid E\left(r,t\right) \mid n\rangle = \varepsilon\langle n \mid a \mid n\rangle e^{-i\omega t+ik.r} + \varepsilon^*\langle n \mid a^\dagger \mid n\rangle e^{i\omega t-ik.r} = 0$$

Now in order to find the average value of $\langle E^2\rangle$, we write

$$\langle n \quad \mid \quad E^2 \mid n\rangle = \left|\varepsilon\right|^2 \langle n \mid aa^\dagger + a^\dagger a \mid n\rangle + \varepsilon^2 e^{-2i\omega t+2ik.r}\langle n \mid a^2 \mid n\rangle$$
$$+\varepsilon^2 e^{2i\omega t-2ik.r}\langle n \quad \mid \quad \overset{\dagger}{a}^2 \mid n\rangle.$$

$$\begin{aligned} \langle E^2\rangle &= (2n+1)\left|\varepsilon\right|^2 = 2\left(n+\frac{1}{2}\right)\left|\varepsilon\right|^2 \\ \Delta E^2 &= \langle E^2\rangle - \langle E\rangle^2 \\ &= 2\left(n+\frac{1}{2}\right)\left|\varepsilon\right|^2, \end{aligned}$$

as $\langle E\rangle^2 = 0$. For $n = 0$ i,e. in vaccum

$$\Delta E^2 \neq 0,$$

but is equal to

$$\Delta E^2 = \left|\varepsilon\right|^2$$

From these equations we conclude that the mean value is zero but fluctuations are present. These fluctuations are considered to be responsible for spontaneous emission, Lamb shift etc.

1 The Coherent Photon States

The single-mode states of physical importance are not the indivisual number states $\mid n\rangle$ (because the electromagnetic wave generated by practical light source do not have definite numbers of photons), but the linear superposition of states $\mid n\rangle$. There is a wide variety of possible superposition states.

A superposition state can be constructed for which uncertainties in the expectation values of the phase operators $\hat{\cos}\phi$ and $\hat{\sin}\phi$ are both equal to zero. Such states have $\Delta n = \infty$. They cannot be excited in any real experiment. Another kind is the coherent state. A coherent state has equal amount of uncertainities in amplitude and phase. A field in coherent state is in a minimum uncertainity state. For coherent state an electric field variation approaches that of classical wave of stable amplitude and fixed phase, in the limit of high excitation. They are improtant because, they are the closest quantum mecahnical approach to a classical electromagnetic wave. A single mode laser operated well above threshold generates a coherent state excitation.

The coherent state $\mid \alpha\rangle$ is the eigen state of the positive frequency part of the electric field operator or the eigen state of the destruction operator of the field.

$$\hat{a} \mid \alpha\rangle = \alpha \mid \alpha\rangle,$$

where α is complex, $\mid \alpha\rangle$ in terms of linear superposition of number state $\mid n\rangle$ is given by

$$\mid \alpha\rangle = \sum_{n=0}^{\infty} C_n \mid n\rangle \tag{1}$$

$$\begin{aligned} \hat{a} \mid \alpha\rangle &= \sum_{n=0}^{\infty} C_n \hat{a} \mid n\rangle \\ &= \sum_{n=0}^{\infty} C_n \sqrt{n} \mid n-1\rangle \\ &= 0 + C_1\sqrt{1} \mid 0\rangle + C_2\sqrt{2} \mid 1\rangle + \ldots \\ &= \sum_{n=0}^{\infty} C_{n+1}\sqrt{n+1} \mid n\rangle \end{aligned} \tag{a}$$

From eqn(1) multiplying with α we can write

$$\begin{aligned} \alpha \mid \alpha\rangle &= \sum_{n=0}^{\infty} C_n \alpha \mid n\rangle \\ \hat{a} \mid \alpha\rangle &= \sum_{n=0}^{\infty} C_n \alpha \mid n\rangle \end{aligned} \tag{b}$$

comparing eqn(a) and (b)

$$\begin{aligned} C_{n+1}\sqrt{n+1} &= C_n\alpha \\ C_n\sqrt{n} &= C_{n-1}\alpha \\ C_n &= \frac{\alpha}{\sqrt{n}}C_{n-1} \\ C_n &= \frac{\alpha}{\sqrt{n}}\frac{\alpha}{\sqrt{n-1}}\frac{\alpha}{\sqrt{n-2}},\ldots,\frac{\alpha}{\sqrt{1}}C_o \\ &= \frac{\alpha^n}{\sqrt{n!}}C_0 \end{aligned}$$

putting in Eqn(1) ,

$$\mid \alpha\rangle = C_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\mid n\rangle$$

the constant C_0 can be found by normalization

$$\begin{aligned} \langle\alpha \quad \mid \quad \alpha\rangle &= C_0^*C_0\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^m}{\sqrt{m!}}\langle n\mid m\rangle \\ \langle\alpha \quad \mid \quad \alpha\rangle &= |C_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} \end{aligned}$$

using

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$$

we can write

$$\begin{aligned} 1 &= |C_0|^2 e^{|\alpha|^2} \\ |C_0|^2 &= e^{-|\alpha|^2} \\ C_0 &= e^{-\frac{|\alpha|^2}{2}} \end{aligned}$$

Therefore

$$\mid \alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\mid n\rangle$$

Another way of proving the above relation which interpret $\mid \alpha\rangle$ as a superposition of number state is,

$$\mid \alpha\rangle = \sum_n \mid n\rangle\langle n \mid\mid \alpha\rangle \tag{1}$$

where $\sum_n \mid n\rangle\langle n \mid = 1$ is the completeness relation for number state. As

$$\mid n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\mid 0\rangle \tag{2}$$

$$\langle n \mid = \langle 0 \mid \frac{(a)^n}{\sqrt{n!}} \tag{3}$$

Putting in Eqn(1) we can write as

$$\mid \alpha\rangle = \sum_n \mid n\rangle\langle 0 \mid \frac{(a)^n}{\sqrt{n!}} \mid \alpha\rangle$$

we know that

$$\begin{aligned}
\hat{a} \quad &\mid \quad \alpha\rangle = \alpha \mid \alpha\rangle \\
(\hat{a})^n \quad &\mid \quad \alpha\rangle = \alpha^n \mid \alpha\rangle \\
&\Longrightarrow \quad \mid \alpha\rangle = \sum_n \mid n\rangle \frac{(\alpha)^n}{\sqrt{n!}} \langle 0 \mid \alpha\rangle \qquad (4)
\end{aligned}$$

The value of $\langle 0 \mid \alpha\rangle$ is obtained by normalization i,e

$$\langle \alpha \mid \alpha\rangle = 1$$

$$\begin{aligned}
\langle \alpha \quad &\mid \quad \alpha\rangle = \sum_n \sum_m \frac{(\alpha^*)^m}{\sqrt{m!}} \frac{(\alpha)^n}{\sqrt{n!}} \langle m \mid n\rangle \left|\langle 0 \mid \alpha\rangle\right|^2 \\
&= \quad \sum_n \frac{(\alpha^* \alpha)^n}{n!} \left|\langle 0 \mid \alpha\rangle\right|^2 \\
&= \quad \sum_n \frac{(|\alpha|^2)^n}{n!} \left|\langle 0 \mid \alpha\rangle\right|^2 \\
&\Longrightarrow \quad 1 = e^{|\alpha|^2} \left|\langle 0 \mid \alpha\rangle\right|^2 \\
\left|\langle 0 \mid \alpha\rangle\right|^2 \quad &= \quad e^{-|\alpha|^2} \\
\langle 0 \quad &\mid \quad \alpha\rangle = e^{-\frac{|\alpha|^2}{2}}
\end{aligned}$$

Putting in Eqn(4) we get

$$\mid \alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mid n\rangle$$

hence proved.

Some other representations of the coherent state

$$\begin{aligned}
| \quad \alpha\rangle &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mid n\rangle \\
| \quad n\rangle &= \frac{(a^\dagger)^n}{\sqrt{n!}} \mid 0\rangle \\
| \quad \alpha\rangle &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(a^\dagger)^n}{\sqrt{n!}} \mid 0\rangle \\
| \quad \alpha\rangle &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} \mid 0\rangle
\end{aligned}$$

since we know that $e^{-\alpha^* a} \mid 0\rangle = \mid 0\rangle$

$$\begin{aligned}
&\Longrightarrow \quad \mid \alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a} \mid 0\rangle \\
D(\alpha) &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a}
\end{aligned}$$

where $D(\alpha)$ is called the displacement operator.

$$\mid \alpha\rangle = D(\alpha) \mid 0\rangle$$

1.1 Baker-Hausdorff identity

If

$$[[A, B], A] = [[A, B], B] = 0$$

then

$$e^{A+B} = e^{-[A,B]/2} e^A e^B$$

Let we have $A = \alpha a^\dagger$ and $B = -\alpha^* a$

$$\begin{aligned}
e^{\alpha a^\dagger - \alpha^* a} &= e^{-\frac{1}{2}\left[-\alpha a^\dagger \alpha^* a + \alpha^* a \alpha a^\dagger\right]} e^{\alpha a^\dagger} e^{-\alpha^* a} \\
&= e^{-\frac{1}{2}|\alpha|^2\left[-a^\dagger a + a a^\dagger\right]} e^{\alpha a^\dagger} e^{-\alpha^* a} \\
&= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger - \alpha^* a}
\end{aligned}$$

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

another definition

$$\Longrightarrow \mid \alpha\rangle = e^{\alpha a^\dagger - \alpha^* a} \mid 0\rangle$$

The other equivalent antinormal form of $D(\alpha)$ is obtained by using $A = -\alpha^* a$ and $B = \alpha a^\dagger$, then we get

$$D(\alpha) = e^{\frac{|\alpha|^2}{2}} e^{-\alpha^* a} e^{\alpha a^\dagger}$$

or by using

$$
\begin{aligned}
e^{A+B} &= e^{\frac{1}{2}[A,B]} e^{B} e^{A} \\
D(\alpha) &= e^{\frac{|\alpha|^2}{2}} e^{-\alpha^* a} e^{\alpha a^\dagger}
\end{aligned}
$$

The operator $D(\alpha)$ is a unitary operator. i,e.

$$D^\dagger(\alpha) = D(-\alpha) = D^{-1}(\alpha)$$

It acts as a displacement operator upon the amplitudes a and $a^\dagger$ i.e.

$$
\begin{aligned}
D^{-1}(\alpha)\,\hat{a} D(\alpha) &= a + \alpha \\
D^{-1}(\alpha)\,\hat{a}^\dagger D(\alpha) &= a^\dagger + \alpha^*
\end{aligned}
$$

This can be proved by writing

$$
\begin{aligned}
D(\alpha) &= e^{\alpha a^\dagger - \alpha^* a} \\
D^\dagger(\alpha) &= e^{\alpha^* a - \alpha a^\dagger} = D^{-1}(\alpha)
\end{aligned}
$$

using these equations we get

$$D^{-1}(\alpha)\,\hat{a} D(\alpha) = e^{\alpha^* a} e^{-\alpha a^\dagger} \hat{a} e^{\alpha a^\dagger} e^{-\alpha^* a}$$

For any operators A and B we have

$$e^{-\alpha A} B e^{\alpha A} = B - \alpha [A, B] + \frac{\alpha^2}{2!} [A, [A, B]] + \ldots$$

For $A = \hat{a}^\dagger$ and $B = \hat{a}$

$$
\begin{aligned}
e^{-\alpha a^\dagger} \hat{a} e^{\alpha a^\dagger} &= a + \alpha \\
D^{-1}(\alpha)\,\hat{a} D(\alpha) &= a + \alpha
\end{aligned}
$$

Similarly for

$$D^{-1}(\alpha)\,\hat{a}^\dagger D(\alpha) = e^{\alpha^* a} \hat{a}^\dagger e^{-\alpha^* a}$$

here $A = \widehat{a}$ and $B = \hat{a}^\dagger$

$$\Longrightarrow D^{-1}(\alpha)\,\hat{a}^\dagger D(\alpha) = a^\dagger + \alpha^*$$

Prove

$$
\begin{aligned}
& e^{-\alpha A} B e^{\alpha A} = B - \alpha [A, B] + \frac{\alpha^2}{2!} [A, [A, B]] + \ldots \\
& \left(1 - \alpha A + \frac{\alpha^2 A^2}{2!} + \ldots\ldots\right) B \left(1 + \alpha A + \frac{\alpha^2 A^2}{2!} + \ldots\right) \\
& = B - \alpha (AB - BA) + \frac{\alpha^2}{2!} (\ldots) \\
& e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} = a - \alpha \left[a^\dagger, a\right] + \frac{\alpha^2}{2!} \left[a^\dagger, \left[a^\dagger, a\right]\right] + \ldots \\
& = a^\dagger + \alpha
\end{aligned}
$$

1.2 Properties of coherent states

Properties of a cavity mode excited to a coherent state $\mid \alpha\rangle$ can be detrmined by the method applied to the number state $\mid n\rangle$.

1- The mean number of photon in the coherent state $\mid \alpha\rangle$ is given by

$$\begin{aligned}\langle n\rangle &= \langle\alpha \mid \hat{n} \mid \alpha\rangle = \langle\alpha \mid a^{\dagger}a \mid \alpha\rangle = |\alpha|^2 \\ \mid & \alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}} \mid n\rangle \\ \langle\alpha &\mid = e^{-\frac{|\alpha|^2}{2}}\sum_{m=0}^{\infty}\frac{(\alpha^*)^m}{\sqrt{m!}}\langle m \mid\end{aligned}$$

therefore

$$\begin{aligned}\langle n\rangle &= e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}\langle m \mid a^{\dagger}a \mid n\rangle \\ &= e^{-|\alpha|^2}\sum_n\frac{(\alpha^*\alpha)^n}{n!}n \\ &= e^{-|\alpha|^2}\sum_n\frac{(|\alpha|^2)^n}{n!}n\end{aligned}$$

Let $x = |\alpha|^2$, and also

$$\begin{aligned}x\frac{\partial}{\partial x}\sum_n\frac{x^n}{n!} &= \sum_n\frac{x^n}{n!}n \\ x\frac{\partial}{\partial x}e^x &= \sum_n\frac{x^n}{n!}n\end{aligned}$$

therefore we can write

$$\begin{aligned}\langle n\rangle &= e^{-|\alpha|^2}\,|\alpha|^2\frac{\partial}{\partial|\alpha|^2}e^{|\alpha|^2} \\ &= |\alpha|^2\end{aligned}$$

Find

$$\begin{aligned}\langle\alpha &\mid \hat{n}^2 \mid \alpha\rangle = e^{-|\alpha|^2}\sum_n\frac{(\alpha^*\alpha)^n}{n!}n^2 \\ &= e^{-|\alpha|^2}\sum_n\frac{|\alpha|^{2n}}{n!}\{n(n-1)+n\}\end{aligned}$$

Let again $x = |\alpha|^2$, by the definition

$$\begin{aligned}\sum_n\frac{x^n}{n!}n(n-1) &= x^2\frac{\partial^2}{\partial x^2}\sum_n\frac{x^n}{n!} \\ &= x^2\frac{\partial^2}{\partial x^2}e^x\end{aligned}$$

so we can write

$$\langle \alpha \mid \hat{n}^2 \mid \alpha \rangle = e^{-|\alpha|^2} e^{|\alpha|^2} \left(|\alpha|^4 + |\alpha|^2 \right)$$

Root-mean-square deviation is

$$\Delta n = \sqrt{\langle n^2 \rangle - \langle n \rangle^2} = \sqrt{|\alpha|^2}$$

Where $|\alpha|^2$ is the mean number of photons in the cavity mode and uncertainty spread about the mean is equal to the square root of the mean number of photons.

ii)- Photon statistics: photon distribution function:

The probability of finding n-photons in the field $\mid \alpha \rangle$ is

$$p(n) = |\langle n \mid \alpha \rangle|^2$$

where

$$\langle n \mid \alpha \rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{(\alpha)^m}{\sqrt{m!}} \langle n \mid m \rangle$$

$$\begin{aligned} p(n) &= \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{n!} \\ &= \frac{e^{-\langle n \rangle} \langle n \rangle^n}{n!} \end{aligned}$$

is a poisson distribution.

iii)- Coherent state is the minimum energy state: i,e

$$\Delta p \Delta q = \frac{\hbar}{2}$$

$\hat{a}$ and $\hat{a}^\dagger$ are not hermitian but their combinations are.

Let $m_j = 1$ at $t = 0$ $r = 0$

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2\hbar\omega}} \left(\omega \hat{q} + i \hat{p} \right) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar\omega}} \left(\omega \hat{q} - i \hat{p} \right) \end{aligned}$$

adding these two we get

$$\begin{aligned} \frac{1}{2} \left(\hat{a} + \hat{a}^\dagger \right) &= \sqrt{\frac{\omega}{2\hbar}} \hat{q} \\ \hat{a} + \hat{a}^\dagger &= \sqrt{\frac{2\omega}{\hbar}} \hat{q} \end{aligned}$$

and

$$\begin{aligned}\hat{a} - \hat{a}^\dagger &= i\sqrt{\frac{2}{\hbar\omega}}\hat{p} \\ \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) &= \sqrt{\frac{1}{2\hbar\omega}}\hat{p}\end{aligned}$$

$\hat{p}$ and $\hat{q}$ are hermitain and represent observable quantities,

$$\begin{aligned}\left[\hat{q}, \hat{p}\right] &= i\hbar \\ \Delta\hat{q}\Delta\hat{p} &\geq \frac{\hbar}{2}\end{aligned}$$

for a coherent state we have to prove that

$$\Delta p \Delta q = \frac{\hbar}{2}$$

$$\begin{aligned}\langle\hat{p}\rangle &= \langle\alpha \mid \hat{p} \mid \alpha\rangle = \frac{\sqrt{2\hbar\omega}}{2i}\left(\langle a\rangle - \langle a^\dagger\rangle\right) \\ \langle a\rangle &= \langle\alpha \mid a \mid \alpha\rangle = \alpha\langle\alpha \mid \alpha\rangle\end{aligned}$$

$$\begin{aligned}\langle p\rangle &= \frac{\sqrt{2\hbar\omega}}{2i}\left(\alpha - \alpha^*\right) \\ \langle p^2\rangle &= \left(\frac{\sqrt{2\hbar\omega}}{2i}\right)^2 \langle(a - a^\dagger)(a - a^\dagger)\rangle \\ &= -\frac{2\hbar\omega}{4}\langle\alpha \mid a^2 + a^{\dagger 2} - aa^\dagger - a^\dagger a \mid \alpha\rangle \\ &= -\frac{2\hbar\omega}{4}(\alpha^2 + \alpha^{*2} - 2\alpha^*\alpha - 1)\end{aligned}$$

$$\begin{aligned}\Delta p^2 &= \langle p^2\rangle - \langle p\rangle^2 \\ &= -\frac{\hbar\omega}{2}(\alpha^2 + \alpha^{*2} - 2\alpha^*\alpha - 1) + \frac{\hbar\omega}{2}(\alpha^2 + \alpha^{*2} - 2\alpha^*\alpha) \\ &= \frac{\hbar\omega}{2}\end{aligned}$$

Similarly

$$\Delta q^2 = \frac{\hbar}{2\omega}$$

$$\Delta q \Delta p = \frac{\hbar}{2}$$

iv)- Coherent states are not orthagonal, but are normalized. i,e.

$$\langle \alpha \mid \alpha \rangle = e^{-|\alpha|^2} \sum_n \frac{(\alpha^* \alpha)^n}{n!} = 1$$

For two different complex numbers α and β.

$$\begin{aligned} \mid \quad \alpha\rangle &= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} \mid n\rangle \\ \mid \quad \beta\rangle &= e^{-\frac{|\beta|^2}{2}} \sum_m \frac{\beta^m}{\sqrt{m!}} \mid m\rangle \end{aligned}$$

$$\begin{aligned} \langle \alpha \quad \mid \quad \beta\rangle &= e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} e^{\alpha \times \beta} \neq 0 \\ \Longrightarrow \quad |\langle \alpha \mid \beta\rangle|^2 &= e^{-|\alpha-\beta|^2}. \end{aligned}$$

The $\mid \alpha\rangle$ form an over complete set of states and lack of orthogonality is a consequence of this.i, j, k are orthogonal and independent of each other. If we divide space in 5 directions they would not be orthogonal and independent of each other. Therefore over complete. i,e. there are many more coherent states $\mid \alpha\rangle$ than there are states $\mid n\rangle$.

Completeness relation:

For number states

$$\sum_n \mid n\rangle\langle n \mid = 1$$

Similarly the set of all coherent states $\mid \alpha\rangle$ is a complete set and satisfy the completeness relation.

$$\frac{1}{\pi} \int d^2\alpha \mid \alpha\rangle\langle \alpha \mid = 1$$

Let

$$\begin{aligned} \alpha &= re^{i\theta} \\ d^2\alpha &= rdrd\theta \end{aligned}$$

$$\begin{aligned} \int d^2\alpha \quad \mid \quad \alpha\rangle\langle\alpha \mid &= \int e^{-\frac{|\alpha|^2}{2}} \sum_m \frac{\alpha^m}{\sqrt{m!}} \mid m\rangle e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^{*n}}{\sqrt{n!}} \langle n \mid d^2\alpha \\ &= \int_0^\infty \int_0^{2\pi} e^{-|\alpha|^2} \sum_{n,m} \frac{|\alpha|^{n+m+1}}{\sqrt{n!m!}} e^{i(m-n)\theta} \mid m\rangle\langle n \mid d\,|\alpha|\, d\theta \end{aligned}$$

as

$$\int_0^{2\pi} d\theta e^{i(m-n)\theta} = 2\pi\delta_{nm}$$

therefore

$$\int d^2\alpha \quad | \quad \alpha\rangle\langle\alpha \mid = 2\pi \int_0^\infty \sum_n \frac{1}{n!} |\alpha|^{2n+1} e^{-|\alpha|^2} \mid n\rangle\langle n \mid d(|\alpha|)$$

$$= \quad \pi \sum_n \frac{1}{n!} \int_0^\infty 2\,|\alpha|\, d(|\alpha|)\, |\alpha|^{2n} e^{-|\alpha|^2} \mid n\rangle\langle n \mid$$

putting

$$x = |\alpha|^2$$

$$dx = 2\,|\alpha|\, d\,|\alpha|$$

$$\int d^2\alpha \mid \alpha\rangle\langle\alpha \mid = \pi \sum_n \frac{1}{n!} \int_0^\infty dx\ x^n e^{-x} \mid n\rangle\langle n \mid$$

$$\int_0^\infty dx\ x^n e^{-x} = n!$$

$$\int d^2\alpha \quad | \quad \alpha\rangle\langle\alpha \mid = \pi \sum_n \frac{1}{n!} n! \mid n\rangle\langle n \mid$$

$$= \pi \sum_n \mid n\rangle\langle n \mid$$

$$\frac{1}{\pi} \int d^2\alpha \quad | \quad \alpha\rangle\langle\alpha \mid = 1$$

The completeness property is essential for the utility of a set of states.$\mid \alpha\rangle$ is a complete but not orthogonal. As a result any coherent state can be expanded in terms of other states.

$$| \quad \alpha\rangle = \frac{1}{\pi} \int d^2\alpha' \mid \alpha'\rangle\langle\alpha' \mid \alpha\rangle$$

$$= \frac{1}{\pi} \int d^2\alpha \mid \alpha'\rangle \exp\left[-\frac{1}{2}|\alpha|^2 + \alpha'\alpha^* - \frac{1}{2}|\alpha'|^2\right]$$

This shows that the coherent states are overcomplete.

2 Squeezed states of the radiation field

It is possible to generate states in which fluctuations are reduced below the symmetric quantum limit in one quadrature component, at the expense of enhanced fluctuations in the canonically conjugate quadrature such that the Heisenberg uncertainty principle is not voilated. Such states of the radiation field are called squeezed states.

Consider two hermitian operators A and B which satisfy the commutation relation

$$[A, B] = iC$$

According to the Heisenberg uncertainty principle, the product of the uncertainties in determining the expectation values of two variables A and B is given by.

$$\Delta A \Delta B \geq \frac{1}{2} |\langle C \rangle|$$

A state of the system is called a squeezed state if the uncertainty in one of the observables (say A) satisfies the relation

$$(\Delta A)^2 < \frac{1}{2} |\langle C \rangle|$$

If in addition to the above condition the variances satisfy the minimum uncertainty relation i,e.

$$\Delta A \Delta B = \frac{1}{2} |\langle C \rangle|$$

then the state is called an ideal squeezed state.

In a squeezed state , therefore the quantum fluctuations in one variable are reduced below their value in a symmetric minimum uncertainty state.

$$(\Delta A)^2 = (\Delta A)^2 = \frac{1}{2} |\langle C \rangle|$$

at the expense of the corresponding increased fluctuations in the conjugate variable such that the uncertainty relation is not voilated.

2.1 Quadrature amplitude operators

Let us define Hermitian amplitude operators.

$$\begin{aligned} X_1 &= \frac{1}{2}\left(a + a^\dagger\right) \\ X_2 &= \frac{1}{2i}\left(a - a^\dagger\right) \end{aligned}$$

X_1 and X_2 are dimensionless position and momentum operators

$$\begin{aligned} q &= \frac{\sqrt{2\hbar / \omega m}}{2}\left(a + a^\dagger\right) \\ p &= \frac{\sqrt{2\hbar \omega m}}{2i}\left(a - a^\dagger\right) \end{aligned}$$

$$\hat{a} = X_1 + iX_2$$
$$\hat{a}^\dagger = X_1 - iX_2$$

The operators X_1 and X_2 are Hermitian and satisfy the commutation relation

$$[X_1, X_2] = \frac{i}{2}$$

These operators are also called quadrature operators. In terms of X_1 and X_2 quantized single mode field can be written as

$$E(t) = 2\varepsilon\hat{\epsilon}\left(X_1 \cos\omega t + X_2 \sin\omega t\right)$$

The Hermitian operators X_1 and X_2 are the amplitudes of the two quadratures of the field having a phase difference $\pi/2$.

From the commutation relation of X_1 and X_2, we get the uncertainty relation for the two amplitudes i,e.

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}$$

A squeezed state of the rdiation field is obtained if

$$(\Delta X_i)^2 < \frac{1}{4}, \quad for \;\; i = 1 \; or \; 2$$

An ideal squeezed state is obtained if in addition to the above equation, the relation

$$\Delta X_1 \Delta X_2 = \frac{1}{4}$$

also holds . Example of an ideal squeezed state is the two-photon coherent state. The quadrature operators allow us to represent a beam of light graphically. These so called phasor diagrams are very popular in quantum optics. Any state of light can be represented on a phasor diagram of the operators , i,e. a plot of X_1 versus X_2. Unlike classical vacuum, quantum mechanical vacuum is represented by a circle at the origin. $(\langle X_1\rangle, \langle X_2\rangle)$ every point in this circle will represent a wave. In quadrature representation each point represents a wave.

Now consider how does different states look like pictorially

i)- Vaccum

$$\Delta X_1 = \sqrt{\langle X_1^2\rangle - \langle X_1\rangle^2}$$

$$\begin{aligned}
\langle X_1\rangle &= \langle 0 \mid \frac{a + a^\dagger}{2} \mid 0\rangle = 0 \\
\langle X_1^2\rangle &= \langle 0 \mid \frac{(a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a)}{4} \mid 0\rangle \\
&= \langle 0 \mid \frac{(a^2 + a^{\dagger 2} + 2a^\dagger a + 1)}{4} \mid 0\rangle \\
&= \frac{1}{4}
\end{aligned}$$

$$\Longrightarrow \Delta X_1^2 = \frac{1}{4}$$

Similarly

$$\Delta X_2^2 = \frac{1}{4}$$

$$\begin{aligned}
\Delta X_1 &= \frac{1}{2}, \quad \Delta X_2 = \frac{1}{2} \\
\Delta X_1 \Delta X_2 &= \frac{1}{4}
\end{aligned}$$

ii)- Coherent States:

$$\begin{aligned}
\langle X_1 \rangle &= \frac{1}{2} \langle \alpha \mid a + a^\dagger \mid \alpha \rangle \\
&= \frac{1}{2} (\alpha + \alpha^*)
\end{aligned}$$

$$\alpha = |\alpha| \, e^{i\phi}$$

Therefore

$$\begin{aligned}
\langle X_1 \rangle &= |\alpha| \cos \phi \\
\langle X_2 \rangle &= \frac{1}{2i} \langle \alpha \mid a - a^\dagger \mid \alpha \rangle \\
&= \frac{1}{2i} (\alpha - \alpha^*) \\
&= |\alpha| \sin \phi
\end{aligned}$$

$$\begin{aligned}
\langle X_1^2 \rangle &= \frac{1}{4} \langle \alpha \mid a^2 + a^{\dagger 2} + a a^\dagger + a^\dagger a \mid \alpha \rangle \\
&= \frac{1}{4} \left(\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* + 1 \right) \\
\langle X_2^2 \rangle &= -\frac{1}{4} \left(\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* + 1 \right)
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad & \Delta X_1^2 = \langle X_1^2 \rangle - \langle X_1 \rangle^2 \\
= \quad & \frac{1}{4} \left(\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* + 1 \right) - \frac{1}{4} \left(\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* \right) \\
= \quad & \frac{1}{4}
\end{aligned}$$

Similarly

$$\Delta X_2^2 = \frac{1}{4}$$

which means that there is equal amount of uncertainty in both the quadratures. It represent a circle with displacement $|\alpha|$. Displacement operator displaced the vacuum by an amount α that creates a coherent state. Therefore coherent state is a displaced vacuum state.

$$| \alpha\rangle = D(\alpha) | 0\rangle$$

Coherent state has both amplitude and phase uncertainties.
Is Coherent and Fock state are squeezed states?
First consider the coherent state

$$\begin{aligned}
(\Delta \mathrm{X}_1)^2 &= \langle\alpha \mid X_1^2 \mid \alpha\rangle - (\langle\alpha \mid X_1 \mid \alpha\rangle)^2 \\
&= \frac{1}{4}\langle\alpha \mid a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a \mid \alpha\rangle - \frac{1}{4}\left(\langle\alpha \mid (a + a^\dagger) \mid \alpha\rangle\right)^2 \\
&= \frac{1}{4}
\end{aligned}$$

Similarly

$$(\Delta \mathrm{X}_2)^2 = \frac{1}{4}$$

$$\Delta \mathrm{X}_1 \Delta \mathrm{X}_2 = \frac{1}{4}$$

i,e. minimum uncertainty relation holds.
Coherent state is not a squeezed state.
Fock state:

$$\begin{aligned}
(\Delta \mathrm{X}_1)^2 &= \langle n \mid X_1^2 \mid n\rangle - (\langle n \mid X_1 \mid n\rangle)^2 \\
&= \frac{1}{4}(2n+1)
\end{aligned}$$

Simailarly

$$(\Delta \mathrm{X}_2)^2 = \frac{1}{4}(2n+1)$$

This implies that Fock states $| n\rangle$ are not squeezed states.

A coherent state with identical uncertainties in both X_1 and X_2 has a constant value for the variance of electric field. A squeezed state with reduced noise in X_1 has reduced uncertainty in the amplitude at the expense of large uncertainty in the phase of electric field.

A squeezed state with reduced noise in X_2 has reduced uncertainty in the phase at the expense of large uncertainty in the amplitude of the electric field. If we have a state such that ΔX_1^2 or $\Delta X_2^2 < \frac{1}{4}$ then the state is a squeezed state. According to heisenberg uncertainty principle, vacuum is a circle. Area of the circle should be the same, otherwise it will voilate Heisenberg uncertainty principle. In order to squeeze a coherent state we will define a squeezing operator.

2.1.1 Generation of squeezed state:

One example of generation of a squeezed state is the "Degenerate parametric process". The two photon Hamiltonian can be wirtten as

$$H = i\hbar(ga^{\dagger 2} - g^* a^2)$$

where g is the coupling constant . The state of the field is written as

$$\mid \Psi\,(t)\rangle = e^{(ga^{\dagger 2} - g^* a^2)} \mid 0\rangle$$

and this leads to define the unitary squeezed operator.

2.1.2 Squeeze operator:

The squeezed states can be generated by using the unitary squeezed operator.

$$S\,(\zeta) = \exp\left(\frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta a^{\dagger 2}\right) \tag{1}$$

this is a unitary operator i,e.

$$S^\dagger\,(\zeta) = S\,(-\zeta) = S^{-1}\,(\zeta)$$

where $\zeta = re^{i\theta}$. We want to find the values of ΔX_1^2 and ΔX_2^2 or (ΔY_1^2 and ΔY_2^2 for a rotated frame). For this we first find

$$S^\dagger\,(\zeta)\,\hat{a} S\,(\zeta) = e^{\frac{1}{2}\left(\zeta a^{\dagger 2} - \zeta^* a^2\right)} a e^{\frac{1}{2}\left(\zeta^* a^2 - \zeta a^{\dagger 2}\right)} \tag{2}$$

using

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2\perp}[A, [A, B]] + \frac{1}{3\perp}[A, [A, [A, B]]] + \ldots$$

Let we define

$$A = \frac{1}{2}\zeta a^{\dagger 2} - \frac{1}{2}\zeta^* a^2$$

and

$$B = \hat{a}$$

B will commute with the second term of A

$$[A, B] = \frac{1}{2}\zeta\left[a^{\dagger 2}, a\right] \tag{3}$$

using

$$\begin{aligned}\left[a, a^{\dagger n}\right] &= na^{\dagger(n-1)}\\ \left[a^\dagger, a^{n-1}\right] &= -na^{n-1}\end{aligned}$$

$$\implies \qquad [A,B]=\frac{1}{2}\zeta\left(-2a^{\dagger}\right)=-\zeta a^{\dagger}$$

Now we have to find

$$[A,[A,B]]=?$$

Ist term of A i,e. $\frac{1}{2}\zeta a^{\dagger 2}$ commutes with $[A,B]$

$$\begin{aligned}
[A,[A,B]] &= \left[-\frac{1}{2}\zeta^{*}a^{2},-\zeta a^{\dagger}\right] \\
&= \frac{1}{2}\zeta^{*}\zeta\left[a^{2},a^{\dagger}\right] \\
&= \frac{|\zeta|^{2}}{2}(2a) \\
&= |\zeta|^{2}a \qquad (4)
\end{aligned}$$

Now for

$$[A,[A,[A,B]]]=?$$

the second term of A will commute with $[A,[A,B]]$ therefore we have

$$\begin{aligned}
[A,[A,[A,B]]] &= \left[\frac{1}{2}\zeta a^{\dagger 2},|\zeta|^{2}a\right] \\
&= \frac{1}{2}\zeta|\zeta|^{2}\left[a^{\dagger 2},a\right] \\
&= \frac{1}{2}\zeta|\zeta|^{2}\left[-2a^{\dagger}\right] \\
&= -\zeta|\zeta|^{2}a^{\dagger} \qquad (5)
\end{aligned}$$

Using these relations we get

$$\begin{aligned}
S^{\dagger}(\zeta)\hat{a}S(\zeta) &= a-\zeta a^{\dagger}+\frac{1}{2!}|\zeta|^{2}a-\frac{\zeta|\zeta|^{2}}{3!}a^{\dagger}+\ldots \\
&= a\left[1+\frac{1}{2!}|\zeta|^{2}+\frac{1}{4!}|\zeta|^{4}\ldots\right]-a^{\dagger}\left[\zeta+\frac{\zeta|\zeta|^{2}}{3!}+\ldots\right]
\end{aligned}$$

as

$$\zeta=re^{i\theta} \qquad \implies |\zeta|=r$$

$$\begin{aligned}
S^{\dagger}(\zeta)\hat{a}S(\zeta) &= a\left[1+\frac{1}{2!}r^{2}+\frac{1}{4!}r^{4}+.....\right]-a^{\dagger}\left[re^{i\theta}+\frac{r^{3}e^{i\theta}}{3!}+......\right] \\
&= a\cosh r-a^{\dagger}e^{i\theta}\sinh r
\end{aligned}$$

Similarly

$$S^{\dagger}(\zeta)\hat{a}^{\dagger}S(\zeta)=a^{\dagger}\cosh r-e^{-i\theta}a\sinh r$$

As we defined

$$\begin{aligned} X_1 &= \frac{1}{2}\left(a + a^\dagger\right) \\ X_2 &= \frac{1}{2}\left(a - a^\dagger\right) \end{aligned}$$

$$\Longrightarrow \qquad a = X_1 + iX_2$$

$$a^\dagger = X_1 - iX_2$$

Rotate the axes by an amount $\theta/2$. Rotated axes $Y_1$and Y_2 in terms of $X_1$and X_2 are written as

$$\begin{aligned} Y_1 &= X_1 \cos\frac{\theta}{2} + X_2 \sin\frac{\theta}{2} \\ Y_2 &= -X_1 \sin\frac{\theta}{2} + X_2 \cos\frac{\theta}{2} \end{aligned}$$

$$\begin{aligned} Y_1 + iY_2 &= X_1\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) + X_2\left(\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right) \\ &= X_1 e^{-i\theta/2} + iX_2\left(-i\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right) \\ &= \left(X_1 + iX_2\right) e^{-i\theta/2} \end{aligned}$$

is the rotated complex amplitude at an angle $\theta/2$.

$$\begin{aligned} S^\dagger(\zeta)\left(X_1 + iX_2\right) S(\zeta) &= S^\dagger(\zeta)\, a S(\zeta) \\ S^\dagger(\zeta)\left(X_1 + iX_2\right) S(\zeta) &= a\cosh r - a^\dagger e^{i\theta}\sinh r \end{aligned}$$

For rotated frame we have

$$S^\dagger(\zeta)\left(X_1 + iX_2\right) e^{-i\theta/2} S(\zeta) = a\cosh r e^{-i\theta/2} - a^\dagger e^{i\theta/2}\sinh r$$

$$S^\dagger(\zeta)\left(Y_1 + iY_2\right) S(\zeta) = a e^{-i\theta/2}\left(\frac{e^r + e^{-r}}{2}\right) - a^\dagger e^{i\theta/2}\left(\frac{e^r - e^{-r}}{2}\right)$$

$$= \left(a e^{-i\theta/2} - a^\dagger e^{i\theta/2}\right)\frac{e^r}{2}$$

$$+ \left(a e^{-i\theta/2} + a^\dagger e^{i\theta/2}\right)\frac{e^{-r}}{2}$$

As

$$\begin{aligned} Y_1 &= X_1 \cos\frac{\theta}{2} + X_2 \sin\frac{\theta}{2} \\ &= \frac{1}{2}\left(a + a^\dagger\right)\cos\frac{\theta}{2} + \frac{1}{2i}\left(a - a^\dagger\right)\sin\frac{\theta}{2} \\ &= \frac{1}{2}a\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) + \frac{1}{2}a^\dagger\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \\ &= \frac{1}{2}\left(a e^{-i\theta/2} + a^\dagger e^{i\theta/2}\right) \qquad \text{(a)} \end{aligned}$$

$$\Longrightarrow S^{\dagger}(\zeta)(Y_1+iY_2)S(\zeta)=Y_1e^{-r}+iY_2e^{r}$$

where

$$iY_2=\frac{1}{2}\left(ae^{-i\theta/2}-a^{\dagger}e^{i\theta/2}\right) \tag{b}$$

The squeezed operatot attenuates one component of the (rotated) complex amplitude and it amplifies the other component. The degree of attenuation and amplification is determined by $r=|\zeta|$ which is called squeeze factor. The squeezed state $\mid\alpha,\zeta\rangle$ is obtained by first squeezing the vacuum and then displacing it.

$$\mid\alpha,\zeta\rangle=D(\alpha)S(\zeta)\mid 0\rangle$$

where α^2 is the intensity of the state, θ is the orientation of the squeezing axis and r the degree of squeezing. The reverse order of $D(\alpha)$ and $S(\zeta)$ in the above equation is also possible. This results in the so called two photon correlated state. A coherent state is generated by linear terms in a and $a^{\dagger}$ in the exponent

$$D(\alpha)\mid 0\rangle=\mid\alpha\rangle$$

where

$$D(\alpha)=e^{\alpha a^{\dagger}-\alpha^{*}a}$$

the squeezed coherent state requires quadratic terms.

2.1.3 Quadrature variance:

Now we will find $\Delta Y_1^2$and ΔY_2^2, in order to find the ΔY_1 we need to find $\langle a\rangle$, $\langle a^{\dagger}\rangle$, $\langle a^2\rangle$, $\langle a^{\dagger 2}\rangle$ and $\langle a^{\dagger}a\rangle$. We will find these one by one

$$\begin{aligned}
\langle a\rangle &= \langle\alpha,\zeta\mid a\mid\alpha,\zeta\rangle\\
&= \langle 0\mid D^{\dagger}(\alpha)S^{\dagger}(\zeta)aS(\zeta)D(\alpha)\mid 0\rangle\\
&= \langle\alpha\mid(a\cosh r-a^{\dagger}e^{i\theta}\sinh r)\mid\alpha\rangle\\
&= \langle\alpha\mid a\mid\alpha\rangle\cosh r-\langle\alpha\mid a^{\dagger}\mid\alpha\rangle e^{i\theta}\sinh r\\
&= \alpha\cosh r-\alpha^{*}e^{i\theta}\sinh r
\end{aligned}$$

$$\begin{aligned}
&\langle a^2\rangle=\langle 0\mid D^{\dagger}(\alpha)S^{\dagger}(\zeta)a^2S(\zeta)D(\alpha)\mid 0\rangle\\
&=\langle 0\mid D^{\dagger}(\alpha)S^{\dagger}(\zeta)aS(\zeta)S^{\dagger}(\zeta)aS(\zeta)D(\alpha)\mid 0\rangle\\
&=\langle\alpha\mid(a\cosh r-a^{\dagger}e^{i\theta}\sinh r)^2\mid\alpha\rangle\\
&=\langle\alpha\mid(a^2\cosh^2 r+e^{2i\theta}a^{\dagger 2}\sinh^2 r-aa^{\dagger}e^{i\theta}\cosh r\sinh r\\
&-a^{\dagger}ae^{i\theta}\cosh r\sinh r)\mid\alpha\rangle\\
&=\alpha^2\cosh^2 r+e^{2i\theta}\alpha^{*2}\sinh^2 r-2|\alpha|^2e^{i\theta}\cosh r\sinh r\\
&-e^{i\theta}\cosh r\sinh r\\
&=\langle(a^{\dagger})^2\rangle^{*}
\end{aligned}$$

Now

$$\begin{aligned}
\langle a^\dagger a\rangle &= \langle \alpha \mid S^\dagger(\zeta)\, a^\dagger S(\zeta)\, S^\dagger(\zeta)\, aS(\zeta) \mid \alpha\rangle \\
&= \langle \alpha \mid \left(a^\dagger \cosh r - e^{-i\theta} a \sinh r\right)\left(a \cosh r - a^\dagger e^{i\theta} \sinh r\right) \mid \alpha\rangle \\
&= |\alpha|^2 \left(\cosh^2 r + \sinh^2 r\right) - (\alpha^*)^2 e^{i\theta} \sinh r \cosh r \\
&\quad -\alpha^2 e^{-i\theta} \cosh r \sinh r + \sinh^2 r
\end{aligned}$$

As from Equ(a) and (b)

$$\begin{aligned}
Y_1 &= \frac{1}{2}\left(ae^{-i\theta/2} + a^\dagger e^{i\theta/2}\right) \\
Y_2 &= \frac{1}{2i}\left(ae^{-i\theta/2} - a^\dagger e^{i\theta/2}\right)
\end{aligned}$$

$$\begin{aligned}
\langle Y_1\rangle &= \frac{1}{2}\left[\langle a\rangle e^{-i\theta/2} + \langle a^\dagger\rangle e^{i\theta/2}\right] \\
&= \frac{1}{2}\left(\alpha \cosh r - e^{-i\theta}\alpha^* \sinh r\right) e^{-i\theta/2} + \frac{1}{2}\left(\alpha^* \cosh r - e^{-i\theta}\alpha \sinh r\right) e^{i\theta/2}
\end{aligned}$$

$$\langle Y_1^2\rangle = \frac{1}{4}\left[\langle a^2\rangle e^{-i\theta} + \langle a^{\dagger 2}\rangle e^{i\theta} + 2\langle a^\dagger a\rangle + 1\right]$$

Now

$$\begin{aligned}
(\Delta Y_1)^2 &= \langle Y_1^2\rangle - \langle Y_1\rangle^2 \\
&= \frac{1}{4}e^{-2r}
\end{aligned}$$

and

$$(\Delta Y_2)^2 = \frac{1}{4}e^{2r}$$

$$\Longrightarrow \qquad \Delta Y_1 \Delta Y_2 = \frac{1}{4}$$

A squeezed coherent state is therefore an ideal squeezed state. In the complex amplitude plane the coherent state error circle is squeezed into an error ellipse of the same area. The principle axes of the ellipse lie along the $\mathrm{Y}_1$and Y_2 axes, and the principle radii are ΔY_1 and ΔY_2.