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Workshop on Coherent Phenomena in Disordered Optical Systems

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Unconventional Quantum Phases in Kicked Rotors

Chushun TIAN
Institute for Advanced Study, Tsinghua University
Beijing
People's Republic of China

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Institute for Advanced Study, Tsinghua University, Beijing

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Collaborators:

Alexander Altland (Cologne), Markus Garst (Cologne), and Jiao Wang (Xiamen)

References:

[1] C. Tian and A. Altland, New. J. Phys. 12, 043043 (2010).

[2] C. Tian, A. Altland, and M. Garst, Phys. Rev. Lett. 107, 074101 (2011).

[3] J. Wang, C. Tian, and A. Altland, Phys. Rev. B 89, 195105 (2014).

[4] C. Tian and A. Altland, in preparation

Acknowledgements:

S. Fishman and I. Guarneri for discussions over years.

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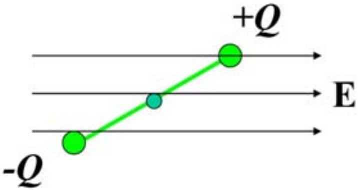
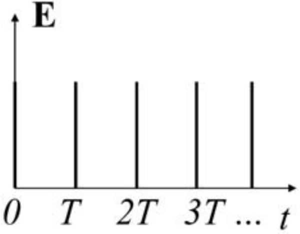




$$\hat{H} = \frac{\hat{J}^2}{2I} - k\cos\hat{\theta}\sum_n \delta(t - nT)$$

$$E(t) = \langle\psi(t)|(-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2})|\psi(t)\rangle?$$

Rescaling

$$\begin{aligned} t &\rightarrow t/T \equiv t \\ \hat{J} &\rightarrow T\hat{J}/I \equiv \hat{l} \\ k &\rightarrow Tk/I \equiv K \\ \hbar &\rightarrow T\hbar/I \equiv \tilde{h} \end{aligned}$$

$$\hat{H} \rightarrow \frac{\hat{l}^2}{2} - K\cos\hat{\theta}\sum_n \delta(t-n) \equiv \hat{H}, [\hat{\theta}, \hat{l}] = i\tilde{h}$$

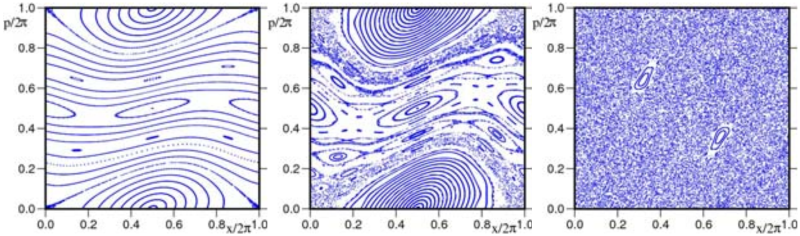
A quantum kicked rotor (QKR) is controlled by two parameters:

1. classical stochastic parameter: K
2. effective Plank's constant: $\tilde{h}$

Classical limit($\tilde{h} \to 0$): transition to global stochasticity

Chirikov-Taylor mapping

$$
\begin{aligned}
l_{n+1} &= l_n + K \sin\theta_{n+1} \\
\theta_{n+1} &= \theta_n + l_n
\end{aligned}
$$



$K = 0.5$ $\quad$ $K = K_c = 0.9716$ $\quad$ $K = 5$

Diffusion as a manifestation of global stochasticity

Joul's heating ⇔ random walk in angular momentum space

$$l_{t+1} = l_t + \xi_t \qquad \langle \xi_t \xi_{t'} \rangle \propto K^2 \delta_{tt'}$$

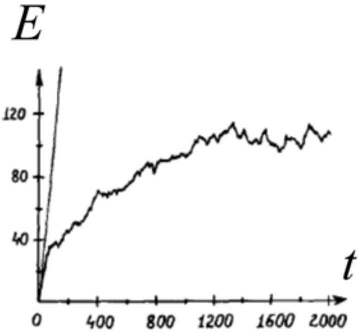
$$\langle E(t) \rangle \sim D_{cl} t$$

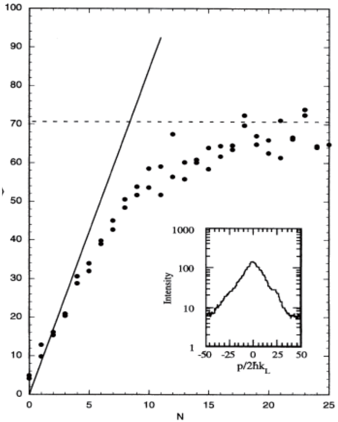
classical diffusion coefficient

$$D_{cl} = \begin{cases} (K - K_c)^3 & K_c \leq K < 4.5 \\ \frac{K^2}{4}\left[1 - 2J_2(K) - 2J_1^2(K) + 2J_3^2(K)\right] & K > 4.5 \end{cases}$$

Dynamical localization at irrational values of $\tilde{h}/(4\pi)$

Saturation of rotor's energy observed in cold-atom experiments





Casati, Chirikov, Ford, and Izrailev '79

Raizen et. al. '95

Analogy to Anderson localization (Fishman, Grempel, and Prange '82, '84)

$$\bar{\phi}_\alpha(n) = \tfrac{1}{2}(\langle n\tilde{h}|\phi_\alpha^+\rangle + \langle n\tilde{h}|\phi_\alpha^-\rangle)$$

$$|\phi_\alpha^+\rangle = e^{iK\cos\hat{\theta}/\tilde{h}}|\phi_\alpha^-\rangle$$

$$\tan(\omega - \tilde{h}n^2/2)\bar{\phi}_\alpha(n) + \textstyle\sum_r W_{n-r}\bar{\phi}_\alpha(r) = 0$$

$$\hat{W} = -\tan(K\cos\hat{\theta}/2\tilde{h})$$

W_n rapidly decays away at $|n| > K/\tilde{h}$

pseudo-randomness at irrational $\tilde{h}/(4\pi)$

Exponential localization of eigenfunctions

Bloch-Floquet theory:

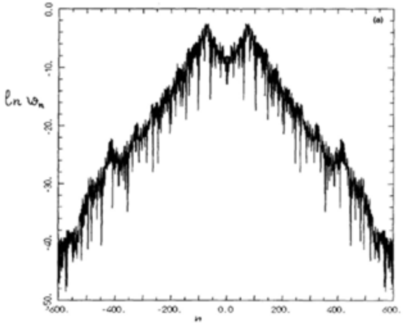
One-step evolution operator

$\hat{U} = e^{i\tilde{h}\hat{n}^2/2} e^{iK\cos\hat{\theta}/\tilde{h}}$
$|\psi(t)\rangle = \hat{U}^t|\psi(0)\rangle$

Quasi eigenenergy and eigenstate

$\hat{U}|\phi_\alpha\rangle = e^{i\omega_\alpha}|\phi_\alpha\rangle$
$\hat{U}|\phi_\alpha(t)\rangle = e^{i\omega_\alpha t}|\phi_\alpha(t)\rangle$
$|\phi_\alpha(t+1)\rangle = |\phi_\alpha(t)\rangle$
$\phi_\alpha(n) = \langle n|\phi_\alpha\rangle \quad w_n = |\phi_\alpha(n)|$

Localized quasienergy eigenstates



Shepelyansky '86; Casati et. al. '90

Izrailev and Shepelyansky '79 '80

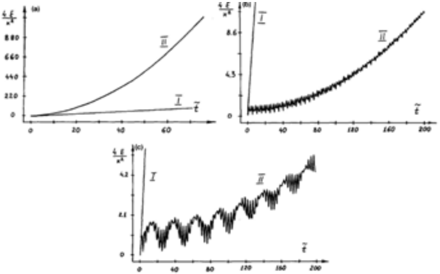


Fig. 9. The time dependence of the rotator energy E in the case of quantum resonance; the straight line (curve I) corresponds to classical diffusion, $E = k^2\tilde{t}/4$, here $k = 1$; (a) $\tau = 4\pi/17$, $k = 19$, $K = 14.0$; (b) $\tau = 8\pi/5$, $k = 0.5$, $K = 2.5$; (c) $\tau = 4\pi/3$, $k = 0.25$, $K = 1.0$ (after [IS79, IS80]).

$$E(t) \sim t^2, \quad \tilde{h}/(4\pi) = p/q,\ (p,q) = 1$$

A case study: $\tilde{h} = 4\pi, \quad p = q = 1$

initial states: $|\psi(0)\rangle = |0\rangle$

$E(t) = \frac{1}{2}\sum_n n^2 \langle n|\hat{U}^t|0\rangle\langle 0|\hat{U}^{\dagger t}|n\rangle$

matrix elements of $\hat{U}$

$\langle n|e^{\frac{i\tilde{h}}{2}\hat{n}^2}|n'\rangle = \delta_{nn'}$

$\Rightarrow$ 'Disorders' (pseudo-randomness) do not play any roles!

$\langle n|e^{\frac{iK}{\tilde{h}}\cos\hat{\theta}}|n'\rangle = J_{n-n'}(K/\tilde{h})$

$\langle n|\hat{U}|n'\rangle = J_{n-n'}(K/\tilde{h})$, $\langle n|\hat{U}^t|n'\rangle = J_{n-n'}(Kt/\tilde{h})$

$$E(t) = \frac{1}{2}\sum_n n^2 J_n^2(Kt/\tilde{h}) = \left(\frac{K}{2\tilde{h}}\right)^2 t^2$$

Challenge: analytic studies beyond low-q values

Quantum resonance at generic q (Tian and Altland '10)

a universal law for the diffusive-ballistic crossover for $\ell \sim K/\tilde{h} \ll q \ll \xi$

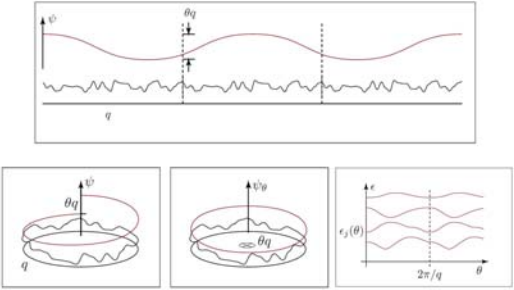
$$E(t)/(D_q q) = F(t/q),$$

$$F(x) = \begin{cases} x + x^3/3, & x < 1 \\ x^2 + 1/3, & x > 1 \end{cases}$$

• This result coincides with a conjecture (by Wójcik and Dorfman '04) on the mean-square displacement of one-dimensional periodic quantum maps.

• Relation to transport in periodic disordered systems (Taniguchi and Altshuler '93)

Origin of quantum resonance: translational symmetry in angular momentum space



- $[\hat{U}, \hat{T}_q] = 0 \Rightarrow$ ($\hat{T}_q$: translational operator) 'one-dimensional crystal' of lattice constant q
- Bloch band $\epsilon_j(\theta)$
- twisted boundary condition: $\psi(0) = \psi(q)e^{i\theta q}$

$$E(t) \sim \int d\theta d\theta' e^{i(\epsilon_j(\theta)-\epsilon_j(\theta'))t - in(\theta-\theta')} n^2 \sim \int d\theta_+ (\partial_{\theta_+}\epsilon_j)^2 t^2 \propto t^2$$

What is quasi-periodic KR?

Casati, Guarneri, and Shepelyansky '89

$K \to K(t) =$
$K\, f(\cos(\theta_1 + \omega_1 t), \cdots, \cos(\theta_{d-1} + \omega_{d-1} t))$

$\hat{H} = \frac{\hat{l}^2}{2} - K(t) \cos \hat{\theta} \sum_n \delta(t-n)$
$[\hat{\theta}, \hat{l}] = i\tilde{h}$

Basic phenomena of d-frequency driven quasiperiodic KR: irrational values of $\tilde{h}/4\pi$

Garreau, Delande *et. al.*; '08, '09; Tian, Altland, and Garst '11

d-frequency driven quasiperiodic KR $\Leftrightarrow$ d-dimensional disordered quantum systems

$d \leq 2$: Anderson insulator

$d > 2$: Anderson transition

- $K/\tilde{h} \gg 1$: $E(t) \sim t \Rightarrow$ metallic;
- $K/\tilde{h} = \mathcal{O}(1)$: Diffusion constant D_ω is strongly renormalized $D_\omega \xrightarrow{\omega \to 0} i\omega \to E(t) \to const. \Rightarrow$ insulating;
- At critical point: $D_\omega \sim (-i\omega)^{(d-2)/d} \Rightarrow E(t) \sim t^{2/d}$

Basic phenomena of d-frequency driven quasiperiodic KR: rational values of $\tilde{h}/4\pi$

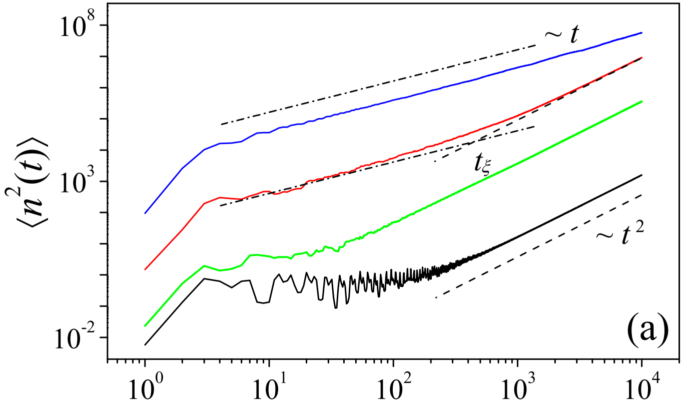
Tian, Altland, and Garst '11; Wang, Tian, and Altland '14

metal-supermetal transition

- $d = 2, 3$: $E(t) \sim t^2$ supermetallic;
- $d \geq 4$: if $K/\tilde{h}$ is larger than some critical value, then a metallic phase is formed $\Rightarrow E(t) \sim t$;
- $d \geq 4$: if $K/\tilde{h}$ is smaller than some critical value, then a supermetallic phase is formed $\Rightarrow E(t) \sim t^2$;
- $d \geq 4$: at the critical point $K/\tilde{h} = \mathcal{O}(1)$ a metal-supermetal transition occurs.

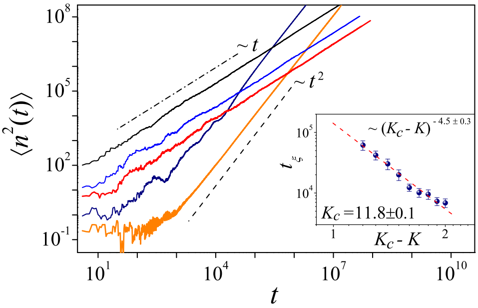
Numerical evidence of supermetal at $p = 1, q = 3, d = 2$ (Wang, Tian, and Altland '14)

From bottom to top: $K = 4$, 8, 64, and 512.



Numerical evidence of metal-supermetal transition $p = 1, q = 3, d = 4$ (Wang, Tian, and Altland '14)

From bottom to top at the left side, $K = 4, 8, 20, 30$, and 80.



$$t_\xi \sim (K_c - K)^{-\alpha}, \quad K_c = 11.8 \pm 0.1, \quad \alpha = 4.5 \pm 0.3$$

Quantum phase structures for low q

TABLE I: Summary of main results.

parameter	$q = 1, 2$		$q = 3, 5, 6 \ldots$			$q = 4$		
	$\langle \hat{n}^2(t) \rangle$	phase	$\langle \hat{n}^2(t) \rangle$	phase	crossover time	$\langle \hat{n}^2(t) \rangle$	phase	crossover time
$d = 2$	quasiperiodic oscillation	insulator (non-Anderson)	$\sim t^2$	supermetal	$t_\xi \sim K^2$	$\sim t^2$	supermetal	$t_\xi \sim K$
$d = 3$					$\ln t_\xi \sim K^2$			
$d = 4$			$\sim t^2 \ (K < K_c)$	supermetal	$t_\xi \sim (K_c - K)^{-\alpha}$			
			$\sim t \ (K \geq K_c)$	metal	∞			

Quasiperiodic KR: probing high-dimensional physics in one dimension

Gauge transformation

$\hat{\Phi}(t) \equiv \exp\left(-it\sum_i \omega_i \hat{n}_i\right)$

$\hat{H}(t) \rightarrow \hat{\Phi}(t)\hat{H}(t)\hat{\Phi}^{-1}(t) \equiv \hat{H}(t)$

$\hat{H}(t) \equiv \frac{\tilde{h}^2\hat{n}^2}{2} + \sum_{i=1}^{d-1}\omega_i\hat{n}_i +$

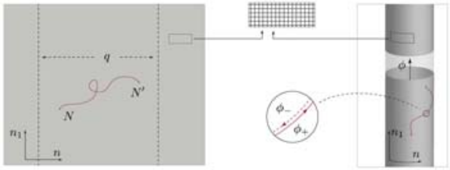
$K\cos\hat{\theta} f(\cos\hat{\theta}_1, \cdots, \cos\hat{\theta}_{d-1}) \sum_m \delta(t-m)$

$[\hat{\theta}_i, \hat{n}_j] = i\delta_{ij}$

- The kicking term is time-periodic.
- The quasiperiodic KR is mapped onto a d-dimensional system.

Rational $\tilde{h}/(4\pi)$: analogy to Aharonov-Bohm physics

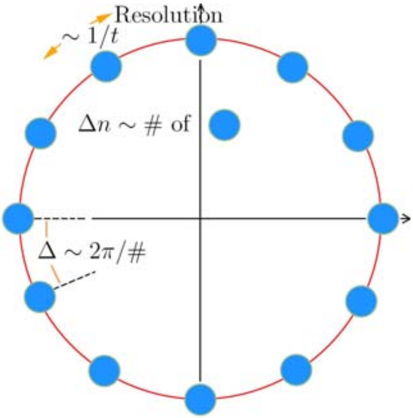
- q-periodicity in the real angular momentum space
$\Rightarrow$ Bloch momentum ϕ as AB flux, $\hat{\theta} \rightarrow \hat{\theta} + \phi_{\pm}$;
- non-periodic in the $(d-1)$-dimensional virtual angular momentum space;
- the dynamics reduced to the one in infinite d-dimensional unit cell.



special case of $d = 2$

Origin of supermetal ($d = 2$ as an example)

$t < t_\xi$: diffusion motion in virtual space



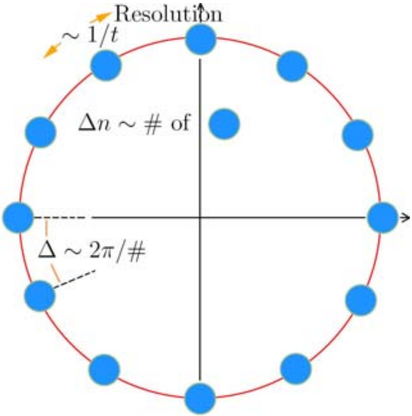
diffusion picture

$\langle \Delta n_1^2 \rangle \sim Dt$

spectrum picture

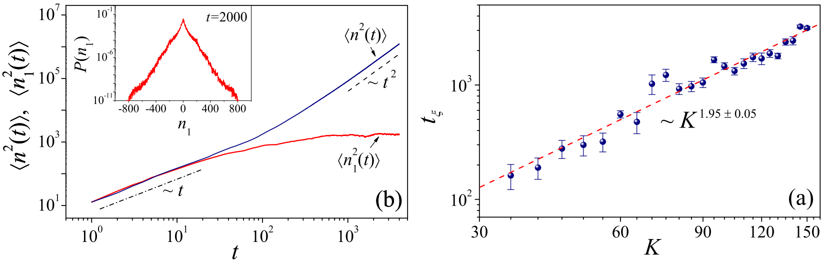
$\# \sim \Delta n_1 \sim \sqrt{Dt}$ unperturbed angular momentum states are excited;
$\# \sim \Delta n_1$ quasi eigenenergies uniformly distributed over the unit circle;
mean spacing $\Delta(t) \sim 2\pi/\#$;
energy resolution $1/t \gg \Delta(t)$, eigenenergies are not distinguishable $\rightarrow$ diffusion

$t \gtrsim t_\xi$: localization in virtual space



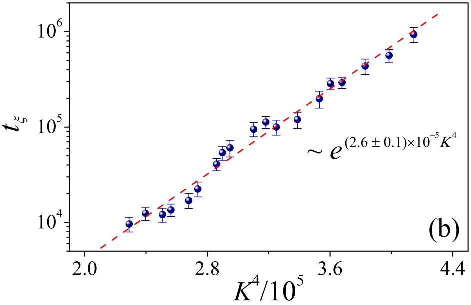
individual levels resolved (localization!) when $\Delta(t) \gtrsim 1/t \rightarrow t_\xi \sim D \sim K^2$

Origin of supermetal – numerical evidence (Wang, Tian, and Altland '14)



- The saturation of $\langle \hat{n}_1^2(t)\rangle$ and the supermetallic growth of $\langle \hat{n}^2(t)\rangle$ simultaneously occur.
- $t_\xi \sim D \sim K^2$

Supermetal in high dimensions



- $\ln t_\xi \sim D^2$
- Anderson localization in 2D virtual space $\Rightarrow$ $d = 3$ supermetal

$d \geq 4$

Anderson transition in $(d-1)$-dimensional virtual space $\Rightarrow$ metal-supermetal transition in d dimension

Energy growth for $\tilde{h}/(4\pi) = p/q$

unperturbed basis

real space $\otimes$ virtual space $\Rightarrow |N\rangle \equiv |n, n_1, \cdots, n_{d-1}\rangle$

One-step evolution operator of QQKR

$\hat{U} = e^{i\hat{T}/\tilde{h}} e^{i\hat{V}/\tilde{h}}$
$\hat{T} = \frac{1}{2}\tilde{h}^2\hat{n}^2 + \sum_i \omega_i \hat{n}_i, \quad \hat{V} = K\cos\hat{\theta} f(\cos\hat{\theta}_1, \cdots, \cos\hat{\theta}_{d-1})$
$[\hat{n}_i, \hat{\theta}_j] = i\delta_{ij}$

Main characteristic of transport

$E(t) = \frac{1}{2}\sum_N |\langle N|\hat{U}^t|0\rangle|^2 n^2$

Density correlation function

$E(t) = \frac{1}{2}\int \frac{d\omega}{2\pi} e^{-i\omega t} \int dN \int_0^{\frac{2\pi}{q}} d\phi\, \partial_{\phi_+}\partial_{\phi_-}\Big|_{\phi_\pm=\phi} K_\omega(N, 0)$

Supersymmetric field theory for $\tilde{h}/(4\pi) = p/q$

$$K_\omega(N,N') = \langle\langle N|\hat{G}^+_{\phi_+}(\omega_+)|N'\rangle\langle N'|\hat{G}^-_{\phi_-}(\omega_-)|N\rangle\rangle_{\omega_0}$$
$$\hat{G}^\pm_{\phi_\pm}(\omega_\pm) = \frac{1}{1-(e^{i\omega_\pm}\hat{U}_{\phi_\pm})^{\pm 1}},\ \omega_\pm = \omega_0 \pm \omega/2$$

Functional integral formalism

$$K_\omega(N,0) = -\int D[Q]e^{-S[Q]}(Q(N))_{+b1,-b1}(Q(0))_{-b1,+b1}$$

Supermatrix σ-model action

$$S[Z] = -\tfrac{1}{2}\mathrm{str}\ln(1 - Z\tilde{Z}) + \tfrac{1}{2}\mathrm{str}\ln(1 - e^{i\omega}\hat{U}^\dagger_{\hat{\phi}} Z \hat{U}_{\hat{\phi}}\tilde{Z})$$
$$\Downarrow$$
$$S[Q] = -\frac{1}{16}\int dN \mathrm{str}\left[\sum_{i=1}^{d-1} D_i(\partial_{n_i}Q)^2 + D_0((\partial_n + i[\hat{\phi},])Q)^2 + 2i\omega Q\sigma_3^{AR}\right]$$

Mathematical structure of supermatrix $Q = \{Q_{\lambda\alpha t}\}$

- q-periodicity in the real space: $Q(n) = Q(n+q)$
- $\lambda = +/-$(analyticity of advanced/retarded Green function), $\alpha = b/f$(commuting/anti-commuting variables $\Rightarrow$ 'supersymmetry'), $t = 1, 2$(time-reversal symmetry)
- $$Q = \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} \sigma_3^{AR} \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}^{-1}$$
- Z satisfies: $\tilde{Z}_{b,b} = Z^\dagger_{b,b}$, $\tilde{Z}_{f,f} = -Z^\dagger_{f,f}$ and $|Z_{b,b}Z^\dagger_{b,b}| < 1$

Application: linear-quadratic energy growth crossover

$l < q < \xi$: dominated by zero-mode field configuration

$$S[Q] = \frac{\pi}{8\Delta} str(D_q[\hat{\phi}, Q]^2 - 2i\omega Q \sigma^3_{AR})$$

polar coordinate representation $Q = \mathcal{R} Q_0 \mathcal{R}^{-1}$, $[\mathcal{R}, \sigma_3^{\mathrm{AR}}]$

$$Q_0 = \begin{pmatrix} \cos\hat{\Theta} & i\sin\hat{\Theta} \\ -\sin\hat{\Theta} & -\cos\hat{\Theta} \end{pmatrix}^{\mathrm{AR}}, \hat{\Theta} = \begin{pmatrix} \hat{\theta}_{11} & 0 \\ 0 & \hat{\theta}_{22} \end{pmatrix}^{\mathrm{BF}},$$

$\hat{\theta}_{11} = \theta \mathbf{1}^{\mathrm{T}}$, $\hat{\theta}_{22} = i\theta_1 \mathbf{1}^{\mathrm{T}} + i\theta_2 \sigma_1^{\mathrm{T}}$ Q replaced by Q_0

$0 < \theta < \pi$ and $\theta_{1,2} > 0$

$\lambda \equiv \cos\theta$ and $\lambda_{1,2} \equiv \cosh\theta_{1,2}$

Effective action

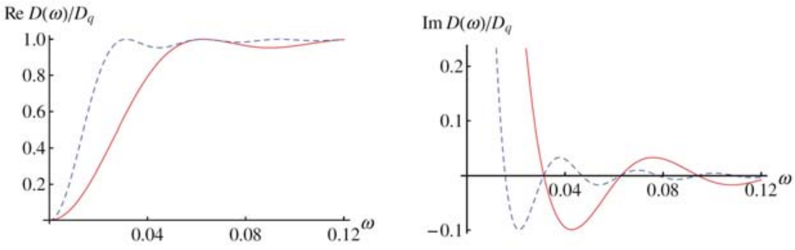
$S = \frac{\pi}{\Delta} \left(D_q[(\Delta\phi)^2(\lambda_1^2 - \lambda^2) + (\phi_+ + \phi_-)^2(\lambda_2^2 - 1)] + 2i\omega(\lambda - \lambda_1\lambda_2) \right)$

Dominant contributions come from $\lambda_2 = 1$!

$$S = \frac{\pi D_q (\Delta\phi)^2}{2\Delta}(\lambda_1^2 - \lambda^2) + i\frac{\pi\omega}{\Delta}(\lambda - \lambda_1)$$

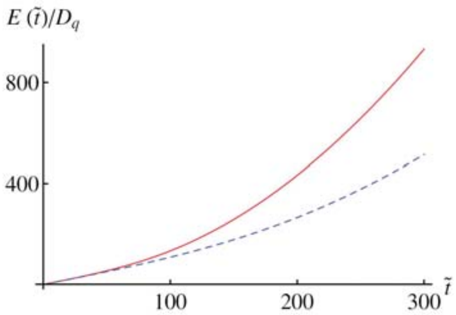
$$K_\omega(\Delta\phi) = \frac{q}{4}\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda \frac{\lambda_1 + \lambda}{\lambda_1 - \lambda} e^{-S}$$

$D(\omega) = D_q \left[1 - \frac{2}{q^2\omega^2}(1 - e^{iq\omega})\right]$ (Taniguchi and Altshuler '93)



$\tilde{h} = p/q$, redline for $p = 1$ and $q = 100$ while blueline for $p = 2$ and 201

$$E(t) = -\frac{1}{2}\int \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega^2}D(\omega)$$



$\tilde{h} = p/q$, redline for $p = 1$ and $q = 100$ while blueline for $p = 2$ and 201

Application: supermetallic energy growth at $d = 2$

$$S[Q] = \frac{\pi\nu}{8}\int dn_1\, \mathrm{str}(-D_0(\partial_{n_1}Q)^2 + D_0[\hat{\phi}, Q]^2 - 2i\omega Q\sigma^3_{\mathrm{AR}}),\ \nu = q/2\pi$$

Transfer matrix method (Efetov and Larkin '83)

$$K_\omega(\Delta\phi) = \frac{qD_0}{4}\int_1^\infty\int_{-1}^1 \frac{d\lambda_1 d\lambda}{(\lambda_1-\lambda)^2}[(\lambda_1^2-1)f_{1\varphi_1\to 0} + (1-\lambda^2)f_{\varphi_1\to 0}]\Psi$$

$$\hat{\mathcal{H}}_0\Psi = 0, \quad \Psi(\lambda_1 = \lambda = 1) = 1$$

$$2\pi\nu D_0\hat{\mathcal{H}}_0 f_{1\varphi_1\to 0} - 2\lambda_1\frac{\partial}{\partial\lambda_1}f_{1\varphi_1\to 0} + \frac{2(1-\lambda^2)}{(\lambda_1-\lambda)^2}(f_{1\varphi_1\to 0} - f_{\varphi_1\to 0}) = \Psi$$

$$2\pi\nu D_0\hat{\mathcal{H}}_0 f_{\varphi_1\to 0} + 2\lambda\frac{\partial}{\partial\lambda}f_{\varphi_1\to 0} + \frac{2(\lambda_1^2-1)}{(\lambda_1-\lambda)^2}(f_{\varphi_1\to 0} - f_{1\varphi_1\to 0}) = \Psi$$

$$\hat{\mathcal{H}}_0 = \hat{\mathcal{H}}_{EL} + \frac{\pi\nu D_0(\Delta\phi)^2}{2}(\lambda_1^2 - \lambda^2)$$

$$\hat{\mathcal{H}}_{EL} = -\frac{1}{2\pi\nu D_0}(\lambda_1-\lambda)^2\left[\frac{\partial}{\partial\lambda}\frac{1-\lambda^2}{(\lambda_1-\lambda)^2}\frac{\partial}{\partial\lambda} + \frac{\partial}{\partial\lambda_1}\frac{\lambda_1^2-1}{(\lambda_1-\lambda)^2}\frac{\partial}{\partial\lambda_1}\right] - i\pi\nu\omega(\lambda_1-\lambda)$$

$$E(t) \sim \left(\frac{t}{q}\right)^2, \text{ for } t \gtrsim q^2 D_0$$

Conclusion

- Quantum kicked rotors exhibit rich phase structures which are extremely sensitive the values of effective Planck's constant.
- A systematic first-principles (field-theoretic) approach has developed to study these quantum phases and their transition.