Stochastic Processes and Applications — **Physics of Complex Systems**

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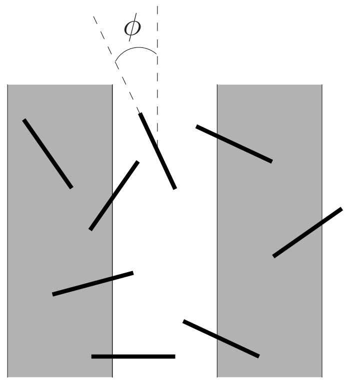
> *This is a collection of problems related to some of the topics we have discussed during the lectures. You are warmly encouraged to solve them. Those marked with (*) require some additional effort.*

1. Buffon's needle

Consider a floor consisting of equal, parallel strips of width b and drop at random on it some needles (8 in the sketch on the right) of length $a < b$. Some of these needles (6 in the situation sketched on the right) will likely cross the line separating two adjacent strips. What is the probability $p(a, b)$ for a single needle to cross one of such lines?



This problem was first formulated in 1777 by the French naturalist Georges-Louis Leclerc, Comte de Buffon (7 September 1707 —16 April 1788).

(a) Determine $p(a, b)$, assuming that the probability distributions of the angle ϕ and of the position x of the center of the needle are uniform.

(b) Drop N needles onto the floor. What is the probability $p_N(n)$ that n of them will cross a line? Calculate $\langle n \rangle$ and the variance $\sigma^2 = \langle n^2 \rangle - \langle n \rangle^2$.

(c) Explain how to use the result of point (b) in order to estimate numerically the value of π. Determine the number N of trials required in order to reduce below 1% the relative fluctuations in the estimate of π.

2. Random sums

Assume that $\{X_r, r \geq 1\}$ is a collection of independent and identically distributed (continuous) random variables with moment generating function $M_X(t) = \langle \mathrm{e}^{itX} \rangle$ and variance σ_X^2. Let $N \in \mathbb{N}$ be a positive *random* number, independent of $\{X_r, r \geq 1\}$, with probability generating function $G_N(t) = \langle t^N \rangle$ and variance σ_N^2. Consider the random sum

$$S = \sum_{r=1}^{N} X_r \ . \tag{1}$$

Prove (a) that the moment generating function $M_S(t)$ of S is given by $M_S(t) = G_N(M_X(t))$ and (b) that $\sigma_S^2 = \langle N \rangle \sigma_X^2 + \sigma_N^2 \langle X_r \rangle^2$.

3. Decimal and binary random walks

Consider a random walker in one dimension which starts from $S_0 = 0$ and takes random steps according to the rule that the amplitude d_r of the r-th step ($r \geq 1$)

is given by $x_r/10^r$, where $x_r \in \{0, 1, 2 \ldots, 9\}$ is an integer and uniformly distributed random variable. The distance traveled by the walker after n steps is $S_n = \sum_{r=1}^{n} d_r$:

(a) Calculate the average and the variance of S_n for finite n and in the limit $n \to \infty$.

(b) Determine the smallest and the largest distance that the walker can ever travel.

(c) Does the central limit theorem apply to this random walk? Why?

(d) Determine the probability distribution function for $n \to \infty$.
[Hint: do not attempt a direct calculation.]

Assume now that the steps of the walker are given by $b_r = y_r/2^r$ where $y_r \in \{0, 1\}$ is a uniformly distributed random variable.

(e) Highlight the analogies with the random walk considered above.

(f) Calculate the asymptotic probability density function of the random walk which starts from $S_0 = -1/2$, i.e., of $S_n = -1/2 + \sum_{r=1}^{n} b_r$.
[Hint: Use the fact that $\prod_{r=1}^{\infty} \cos(t/2^r) = (\sin t)/t$]

4. Poisson processes

In continuous time, consider the Markovian counting process in which the number $n \in \mathbb{N}$ of counts increases by 1 with a time-dependent rate $\lambda(t)$, with $t \geq 0$.

(a) Write down the master equation for the evolution of the probability $P_n(t)$ of this process.

(b) Introduce the characteristic (or probability generating) function $g(x, t) = \langle x^n \rangle_t$, where $\langle \cdot \rangle_t$ is calculated at time t. Solve the evolution equation for $g(x, t)$ assuming that $n = 0$ at time $t = 0$. Calculate $P_n(t)$ and $\langle n \rangle_t$.

(c) Determine $P_{1|1}(n, t|n', t')$ of this process (assuming $t > t'$) and calculate the correlation function $\langle n(t)n(t') \rangle$. Can this $P_{1|1}$ be stationary?

5. Brownian motion and Ornstein-Uhlenbeck process

In one spatial dimension, consider a Brownian particle with velocity v. Due to the surrounding fluid, the particle is subject to a friction which causes its velocity v to decrease according to

$$\dot{v} = -\gamma v \tag{2}$$

where γ is the friction coefficient. For a given velocity v at time t, the velocity $v(t+\tau)$ at time $t + \tau$ is a random variable such that the conditional average $\langle \ldots \rangle_{v(t)=v}$ of the increment $v(t + \tau) - v$ satisfies Eq. (2): $\langle v(t + \tau) - v \rangle_{v(t)=v} = -\gamma v \tau + O(\tau^2)$.

(a) Write down the Fokker-Planck equation for the probability $P_1(v,t)$ of the process $v(t)$, assuming that

$$\alpha_2(v) = \lim_{\tau\to 0} \frac{\langle [v(t+\tau) - v]^2 \rangle_{v(t)=v}}{\tau} = A_2 + O(v^2).$$

Accordingly, at small velocities v, $\alpha_2(v)$ is well approximated by the constant A_2.

(b) Determine the *stationary* solution $P_1^{(\mathrm{s})}(v)$ of the Fokker-Planck equation derived at point (a). [Hint: Use the fact that $\lim_{v\to\infty} vP_1^{(\mathrm{s})}(v) = 0$]

(c) Write down the canonical Maxwell distribution $P_M(v)$ of the velocity v of a free particle of mass m and kinetic energy $K(v) = mv^2/2$ in *equilibrium* at temperature T.

(d) By requiring that the stationary solution $P_1^{(\mathrm{s})}(v)$ of the Fokker-Planck equation determined at point (b) equals the Maxwell distribution $P_M(v)$ determined at point (c), show that the coefficient A_2 can be expressed in terms of the temperature T, Boltzmann's constant k_{B}, the friction coefficient γ and the mass m.

(e) Assuming that the process $v(t)$ is Markovian, write down and solve the Fokker-Planck equation for the time dependence of $P_{1|1}(v,t|v_0,t_0)$ (for simplicity, assume γ, $A_2 = 1$) and show that this solution coincides with the transition probability of the Ornstein-Uhlenbeck process mentioned in the lectures.

6. Branching and decay process with lethal competition

Consider a population consisting of individuals A, each of which might undergo the following processes, with the specified rates:

(*i*)	$A \xrightarrow{\sigma} A + A$	branching,
(*ii*)	$A \xrightarrow{\mu} 0$	annihilation,
(*iii*)	$A + A \xrightarrow{\lambda} A$	lethal competition.

In (*ii*), 0 indicates that the individual A dies, whereas the lethal competition introduced in (*iii*) occurs among all possible *pairs* of individuals of the population.

(a) Describe qualitatively the expected behavior of a population which is ruled by the elementary processes (*i*), (*ii*) and (*iii*), depending on the values of the corresponding transition rates σ, μ, and λ.

(b) Write down the transition rates $W(n \to n+1)$ and $W(n \to n-1)$ for the number n of individuals in the population, which are associated to each of the processes listed above. Is there any absorbing state for the dynamics?

(c) Write down the master equation for the probability $P_n(t)$ of having a population with n individuals at time t.

(d) On the basis of this master equation, determine the evolution equation of the average population $\langle n\rangle_t \equiv \sum_{n=0}^{\infty} nP_n(t)$ at time t.

[Hint: Calculate $\mathrm{d}\langle n\rangle_t/\mathrm{d}t$ and express it as a function of $\langle n\rangle_t$, $\langle n^2\rangle_t$, etc.]

(e) Focus here on the case $\lambda = 0$. Solve the evolution equation for $\langle n\rangle_t$ and discuss the qualitative features of the result as a function of σ/μ. Which is the asymptotic behavior of $\langle n\rangle_t$ for large times t in the absorbing and in the active phase? [For the definition of these phases, see the lectures.] What happens to $\langle n\rangle_t$ exactly at the transition point $\sigma = \mu$?

(f) Consider now the master equation for $\lambda = 1$ and introduce the *mean-field approximation* $\langle n^2\rangle_t \simeq \langle n\rangle_t^2$. Within this approximation calculate $\langle n\rangle$ in the stationary state, as a function of the transition rates σ, μ. Show that, depending on these parameters, there is a phase transition between the stationary states with $\langle n\rangle_{t=\infty} = 0$ and the one with $\langle n\rangle_{t=\infty} > 0$. What is the value of σ at this critical point? Compare it with the result of point (e).

(g) Within the assumptions of point (f), solve explicitly the evolution equation for $\langle n\rangle_t$ and discuss its qualitative behavior (increasing/decreasing, asymptotic behavior etc.) as a function of time for different values of the initial number of particles. Determine $\langle n\rangle_t$ exactly at the critical point.

7. Path integral for the Ornstein-Uhlenbeck process

Consider the Ornstein-Uhlenbeck process for a variable v. On the basis of the corresponding $P_{1|1}(v_2, t_2|v_1, t_1)$:

(a) construct the analogous of the Wiener measure for this process and derive the representation of $P_{1|1}$ as a path integral.

(b) Identify the quantum mechanical system which would have the same (euclidean) action.

8. Zeroes, maximum, and escape rate of the Wiener process

Consider the Wiener process starting at time $t = 0$ from $x = 0$.

(a) Calculate the probability $P(t, s)$ (with $t > s > 0$) that the process crosses zero *at least once* within the time interval $[s, t]$. This provides information on the distributions of the zeros of the random walk. Plot the resulting expression and discuss its qualitative features.

(b) Determine the distribution of the values of the minimum x_m of the Wiener process within the time interval $[0, T]$.

(c) Determine the probability $P_a(t)$ that the Wiener process has never left the strip $S_a = [-a/2, a/2]$ (with $a > 0$) up to time t. Determine the probability density function of the time T of first exit from S_a. Calculate the average value of this exit time. Is it finite?

(*)(d) Generalize the analysis of point (c) to the case of a strip $S_{ba} = [b, a]$ (with $b < 0 < a$). Calculate the average value of the first exit time as a function of a, b. What happens to this average for $b \to -\infty$? Why?

9. Maximum of the Brownian bridge

Consider the ensemble of trajectories of the Wiener process such that $x(0) = 0$ and $x(T) = b$, with fixed $T > 0$. For $b = 0$ these trajectories form closed loops, known as Brownian bridges.

(a) Consider, first, a standard Wiener process with $x(0) = 0$ (and arbitrary $x(T) = x_T$) and indicate with $\bar{x} \equiv \max\{x(t)|0 \le t \le T\}$ its maximum value. On the basis of the reflection principle show that, for $a > 0$,

$$\mathrm{Prob}(\bar{x} > a, x_T \in [b, b + \Delta b]) = \mathrm{Prob}(\bar{x} > a, x_T \in [2a - b - \Delta b, 2a - b]). \quad (3)$$

(b) From Eq. (3) infer the distribution of the maximum of a Brownian bridge, given by $\mathrm{Prob}(\bar{x} > a|x(T) = b)$. Determine the probability density function of $\bar{x}$ in the case $b = 0$ and discuss its qualitative features in comparison with the result of point (b) of Problem 8.

(c) Within the Wiener path-integral formalism, write down the expression for $\mathrm{Prob}(\bar{x} < a|x(T) = b)$ and re-derive the result of point (b).

10. Backward Fokker-Planck equation and first passage time

Consider a Markovian process with continuous sample paths.

(a) Following the approach we used in the lectures to derive the Kramers-Moyal expansion, write down the differential equation for the evolution equation of $P_{1|1}(x, t|x_0, t_0)$ as a function of the *initial time* t_0 and expand it in increasing moments of the "small" jumps from the *initial* point x_0. Which is the relation between the coefficients of this expansion and those of the Kramers-Moyal expansion discussed in the lectures?

Focus below on the case in which only the first two terms of the previous expansion do not vanish. The corresponding equation is known as *backward* Fokker-Planck equation:

$$\partial_{t_0} P_{1|1}(x, t|x_0, t_0) = -\alpha_1(x_0)\partial_{x_0} P_{1|1}(x, t|x_0, t_0) - \frac{1}{2}\alpha_2(x_0)\partial^2_{x_0} P_{1|1}(x, t|x_0, t_0). \quad (4)$$

This equation is useful for discussing the exit time from the strip $S_{ab} = [a, b]$ (see Problem 8). Indeed, assume that the random walker is removed whenever it reaches the boundaries of S_{ba}, i.e., the boundaries are *absorbing*.

(b) In terms of $P^{(s)}_{1|1}(x, t|x_0, 0)$ in the presence of these absorbing boundaries, express the probability $\pi(x_0, t)$ that the particle is still alive at time t, assuming that it started at time $t = 0$ from the point x_0.

(c) Under the assumption that the transition probability of the process is stationary, i.e., $P_{1|1}^{(s)}(x,t|x_0,0) = P_{1|1}^{(s)}(x,0|x_0,-t)$, use the backward Fokker-Planck equation satisfied by $P_{1|1}^{(s)}(x,t|x_0,0)$ in order to prove that

$$\partial_t \pi(x_0,t) = \alpha_1(x_0)\partial_{x_0}\pi(x_0,t) + \frac{1}{2}\alpha_2(x_0)\partial_{x_0}^2\pi(x_0,t). \tag{5}$$

[Note that $P_{1|1}$ in the presence (superscript (s)) or in the absence of the boundaries satisfy the same Fokker-Planck equation, the only difference being in the boundary conditions.]

(*)(d) Which are the proper boundary conditions for π? Solve Eq. (5) for the Wiener process without drift and with constant diffusion. Show that one recovers the results of Problem 8.