Spring College on the Physics of Complex Systems
Advanced methods in non-equilibrium statistical mechanics
13th of June of 2014
Exam

1 Questions

Answers should be concise and to the point.

1. Which model did we use to mimic the bath in the derivation of the Langevin equation?

2. Which is the main reason for having the *same* function $\Gamma(t-t')$ in the noise-noise correlation and the friction kernel in the generalised Langevin equation that we derived in the lectures?

3. Mention at least two reasons why a particle set in contact with an equilibrium environment (at a given time called $t=0$) could remain out of equilibrium forever.

4. What do you expect for the dynamics of a particle moving in one dimension according to the stochastic equation

$$m\dot{v}(t) = \int_0^t dt' \; \Gamma_1(t-t') \; v(t') = -V'(x(t)) + \xi(t) \qquad (1)$$

with m the mass, $v(t)$ the instantaneous velocity, and $x(t)$ the instantaneous position, $V(x) = kx^2/2$ the potential energy. $\xi(t)$ is a time-dependent Gaussian noise with zero mean and correlation

$$\langle \xi(t)\xi(t')\rangle = k_B T \; \Gamma_2(t-t') \qquad (2)$$

5. Give the name and the mathematical statement of the relation between the linear response of an observable $A(t)$ to an infinitesimal perturbation applied at time t' that modifies the Hamiltonian as $H \mapsto H - hB$, $R_{AB}(t,t')$, and the two-time correlation, $C_{AB}(t,t')$, of the same observables A and B, measured at times t and t', respectively, in a system evolving in thermal equilibrium.

6. Explain – in no more than five lines of text – the formalism and the technique with which this property was shown in the lectures.

7. Imagine that a experimental colleague of yours shows you some two-time self-correlation data that are well-described by the function

$$C(t,t') = a\ e^{-[(t-t')/t_d]^b} \tag{3}$$

with a, b and t_d three fitting parameters.
a) With just this information, would you say that the system evolves in equilibrium? Give a reason to your claim.

Next, the experimentalist tells you that he/she constructed the integrated linear response function

$$\chi(t,t') \equiv \int_{t'}^{t} dt''\ R(t,t'') \tag{4}$$

with $R(t,t')$ the linear response function of the same observable, and that a function of $\tau \equiv t - t'$ fitted well the data.
b) Could you now assert that the system is in equilibrium?

He/she next shows you the parametric plot $\chi(\tau)$ against $C(\tau)$, where $C(\tau)$ is the same correlation as in Eq. (3) and $\chi(\tau)$, the integrated linear response in (4), both written just in terms of the time-difference, $\tau = t - t'$. We reproduce this plot in Fig. 1, read the caption for more details.

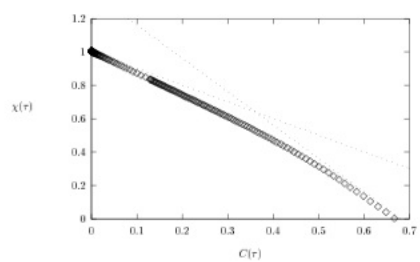

Figure 1: The integrated linear response $\chi(\tau)$, see Eq. (4), against the self-correlation function $C(\tau)$ in a plot constructed by using τ as a parameter that varies from 0 to ∞. For instance, for $\tau = 0$ one has $C(0) = a$ and $\chi(0) = 0$, while for $\tau \to \infty$ one has $C(\tau) \to 0$ and $\chi(\tau) \to 1$.

c) What would you say now about the dynamics of this system?

8. If you had to explain the domain growth process to a colleague who does not know this phenomenon: how would you do it by using – at most – 10 lines.

9. What is the main difference in the behaviour of the two-time correlation function of a macroscopic system quenched from $T_0 \to \infty$
– to the critical point, $T = T_c$ and
– into the ordered low temperature phase $T < T_c$.

10. Can one identify "out of equilibrium" universality classes in domain growth phenomena? If so, which is the function used to quantify the dynamics of the system that serves the purpose of qualifying them?

2 Problem

Take a point-like particle with mass m moving in a two dimensional space. The position of this particle is $\vec{r} = (x, y) = x\hat{i} + y\hat{j}$ in a Cartesian coordinate system. The particle feels a potential $V(x, y) = kx^2$.

1. The particle is in contact with a *generic* environment in thermal equilibrium at temperature T. Propose a generic equation for the time evolution of the particle? Explain the origin of each term in this equation.

2. We will discuss the behaviour of the phase space variables $(\vec{p}, \vec{r})$. For simplicity, we will use a bath with additive white noise (with uncorrelated components, e.g. $\langle \xi_x(t)\xi_y(t')\rangle = 0$ for all t and t'). We will call γ_0 the friction coefficient. What do you expect to happen with the statistical properties of the momentum $\vec{p}$?

3. What do you expect to happen with the statistical properties of the position $\vec{r}$?

4. Take the over-damped (Smoluchowski) limit in which the inertia term is neglected from the dynamic equation. Compute the four correlation functions $C_{xx}(t, t') = \langle x(t)x(t')\rangle$, $C_{xy}(t, t') = \langle x(t)y(t')\rangle$, $C_{yx}(t, t') = \langle y(t)x(t')\rangle$, and $C_{yy}(t, t') = \langle y(t)y(t')\rangle$.

5. Apply a small perturbation to the particle that modifies the potential V according to $V \to V - \vec{h} \cdot \vec{r}$. Compute the four linear response functions $R_{xx}(t,t') = \delta\langle x(t)\rangle_{\vec{h}}/\delta h_x(t')|_{\vec{h}=\vec{0}}$, $R_{xy}(t,t') = \delta\langle x(t)\rangle_{\vec{h}}/\delta h_y(t')|_{\vec{h}=\vec{0}}$, $R_{yx}(t,t') = \delta\langle y(t)\rangle_{\vec{h}}/\delta h_x(t')|_{\vec{h}=\vec{0}}$, and $R_{yy}(t,t') = \delta\langle y(t)\rangle_{\vec{h}}/\delta h_y(t')|_{\vec{h}=\vec{0}}$.

6. Compare the results obtained for the linear response with the correlation functions and conclude about the validity, or not, of the fluctuation-dissipation theorem.