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Spring College on the Physics of Complex Systems

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Stochastic processes and applications
Lectures 4 and 5

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(see p. 24)

We have solved: (coming from the continuum limit of the RW on the lattice...)

$$\frac{\partial}{\partial t} P_1 = -v \frac{\partial P_1}{\partial x} + D \frac{\partial^2 P_1}{\partial x^2}$$

with $P_1(x,0) = \delta(x)$ $\Rightarrow$ the particle is in $x=0$ at $t=0$

$\Rightarrow$ P_1 is nothing but $P_{1|1}(x,t|0,0)$

but the process, as def. <u>on the lattice</u>, is <u>STATIONARY</u>

$$\Rightarrow P_{1|1}(x_2 t_2 | x_1 t_1) = P_{1|1}(x_2 - x_1, t_2 - t_1 | 0, 0)$$

$$(t_2 > t_1) \qquad = \frac{1}{\sqrt{4\pi D (t_2 - t_1)}} \exp\left\{ - \frac{(x_2 - x_1)^2}{4D(t_2 - t_1)} \right\}$$

from now on we assume $v = 0$

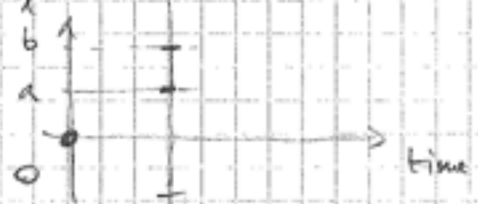
(<u>Ex.</u>: consider $v \neq 0$!)

— o — o —

Intermezzo: $\frac{\partial P}{\partial t} = D \frac{\partial^2}{\partial x^2} P$ ... Euclidean version of the Schrödinger eq for the free particle.

— o — o —

Consider now the process of diffusion in the continuum and we ask: what is the probab. to start in 0 at $t=0$ and to pass in gate 1 ?



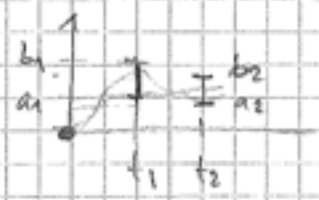
$$\text{Prob}\left(x(t_1) \in [a,b] \right) = \int_a^b dx \; \underbrace{P_{1|1}(x,t|0,0)}_{P_1(x,t) \text{ with initial cond.}}$$

$$\text{Prob}\left(x(t_1) \in [a_1,b_1],\; x(t_2) \in [a_2,b_2] \right) = \; ?$$

but:
- statistical indep. of subsequent jumps (MARKOV)
- translational invariance in space and time

$\Rightarrow$ $\mathrm{Prob}\left(x(t_1) \in [a_1, b_1],\ x(t_2) \in [a_2, b_2] \right) =$

$$= \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \quad P_2\left(x_2 t_2, x_1 t_1 \right) =$$



$$= \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \quad P_{1|1}\left(x_2 t_2 | x_1 t_1\right) P_{1|1}\left(x_1 t_1 | 0 0\right)$$

$\Rightarrow$ $\mathrm{Prob}\left(x(0) = 0,\ x(t_1) \in [a_1, b_1], \ \dots\ x(t_N) \in [a_N, b_N] \right) =$

$$= \int_{a_N}^{b_N} dx_N \int_{a_{N-1}}^{b_{N-1}} dx_{N-1} \cdots \int_{a_1}^{b_1} dx_1 \quad P_{1|1}(x_N t_N | x_{N-1} t_{N-1}) \times$$
$$P_{1|1}(x_{N-1} t_{N-1} | x_{N-2} t_{N-2}) \times$$
$$\vdots$$
$$\times P_{1|1}(x_1 t_1 | 0 0)$$

Question: which is the "probability" of a trajectory?
$\rightsquigarrow$ we identify the trajectory by
$N \to \infty$ with $\Delta t_i \equiv t_{i+1} - t_i \to 0$
$\Delta x_i \equiv b_i - a_i \to 0$

$\mathrm{Prob}\,(\text{traject}) =$

$$= \lim_{\substack{\Delta t_i \to 0 \\ N \to \infty}} dx_N\, dx_{N-1} \cdots dx_1 \ \frac{1}{\sqrt{4\pi D \Delta t_N}} \exp\left\{ - \frac{(x_N - x_{N-1})^2}{4D(\Delta t_N)} \right\} \times$$
$$\times \frac{1}{\sqrt{4\pi D \Delta t_{N-1}}} \exp\left\{ - \frac{(x_{N-1} - x_{N-2})^2}{4D(\Delta t_{N-1})} \right\} \times$$
$$\vdots$$
$$\times \frac{1}{\sqrt{4\pi D \Delta t_1}} \exp\left\{ - \frac{(x_2 - x_1)^2}{4D(\Delta t_1)} \right\}$$

$$= \lim_{\substack{\Delta t_i \to 0 \\ N \to \infty}} \exp\left\{ - \frac{1}{4D} \underbrace{\sum_{i=1}^{N} \left(\frac{x_i - x_{i-1}}{t_i - t_{i-1}} \right)^2 \Delta t_i}_{\substack{\downarrow\, \Delta t_i \to 0 \\ \int_0^{t_N \equiv t} d\tau\, (\dot{x}(\tau))^2}} \right\} \underbrace{\prod_{i=1}^{N} \frac{dx_i}{\sqrt{4D\, \Delta t_i}}}_{\prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{4D\, d\tau}}}$$

formal definition

$$\equiv \exp\left\{ - \frac{1}{4D} \int_0^t d\tau \, (\dot{x}(\tau))^2 \right\} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{4D\, d\tau}} \equiv d_W x(\tau)$$

"Wiener measure"

Once we have formally defined the "integration measure"
(it is possible to prove that this is a mathematically sound measure)
one has :

this is intuitively clear taking into account the relation between measure and probability.

$$\mathbb{P}[\, x(0)=0,\; x(t)\in[a,b]\,] =$$

$$= \int_{x(0)=0}^{x(t)\in[a,b]} \prod_{t=0} \frac{dx(\tau)}{\sqrt{4\pi D d\tau}} \exp\left\{-\frac{1}{4D}\int_0^t d\tau\; \dot{x}^2(\tau)\right\}$$

Integration over the set of trajectories $x(\tau) \;\middle|\; \begin{matrix} x(0)=0 \\ x(t)\in[a,b] \end{matrix}$ $\longrightarrow$ the summation over this continuous set is the Wiener path integral

See (**) previous page

$$= \int_a^b dx \frac{1}{\sqrt{4\pi D t}} \exp\left\{-\frac{x^2}{4Dt}\right\}$$

Compare it with imaginary-time QM! $\longrightarrow$

From the probabilistic interpretation follows that

$$\int_{\substack{x(0)=0 \\ x(t)\ \text{arbitrary}}} d_W x(\tau) = 1$$

i.e. the probability of ending somewhere starting from $x(0)=0$ is 1.

(the measure over this set of function is called unconditional or full or absolute Wiener measure)

Brownian Motion & Wiener Integral

time evolution, $H = H_0 + V$

$$\mathcal{U}(t) = \exp\left\{-i\frac{H}{\hbar}t\right\} \xrightarrow{t = -i\tau} \mathcal{U}(\tau) = \exp\left\{-\frac{H}{\hbar}\tau\right\}$$

- H with positive spectrum
- $\tau > 0$

real:

$\mathcal{U}(t) = \exp\left\{-i\frac{H}{\hbar}t\right\}$ → unitary group

Feynman-Kac formula

$$\langle q'|\mathcal{U}(t)|q\rangle \equiv K(t,q',q) = \int_{q(0)=q}^{q(t)=q'} \mathcal{D}q \; e^{i\frac{S}{\hbar}}$$

$$S = \int_0^t dt' \left[\frac{1}{2}m\dot{q}^2 - V(q)\right]$$

$\int \mathcal{D}q$ = "sum over paths" in heuristic sense

imaginary

$\mathcal{U}(\tau) = \exp\left\{-\frac{H}{\hbar}\tau\right\}$ → Hermitian semigroup

$$\equiv \langle q'|\mathcal{U}(\tau)|q\rangle \equiv K(\tau,q',q) = \int_{q(0)=q}^{q(\tau)=q'} \mathcal{D}q \; e^{-S_E/\hbar}$$

$$S_E = \int_0^\tau d\tau' \left[\frac{1}{2}m\dot{q}^2 + V(q)\right]$$

Euclidean action (name coming from QFT where the metric changes from Minkowski → Euclid.

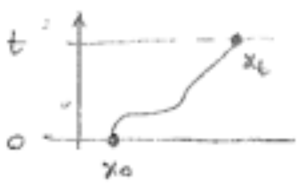
$\int \mathcal{D}q$ = sum over paths.,
mathematically founded.
Probabilistic interpretation,
Brownian motion.

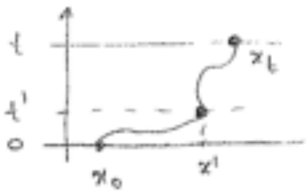
The conditional W.m. is defined on the set of paths with fixed endpoints:

$$\underbrace{\int\limits_{\substack{x(0)=0 \\ x(t)\ \text{arbitrary}}} d_W x(\tau)}_{\text{Unconditional}} = \int_{-\infty}^{\infty} dx_t \underbrace{\int\limits_{\substack{x(0)=0 \\ x(t)=x_t}} d_W x(\tau)}_{\text{conditional}}$$

Moreover, the obvious property:

$$\int\limits_{\substack{x(0)=x_0 \\ x(t)=x_t}} d_W x(\tau) = \int_{-\infty}^{\infty} dx' \int\limits_{\substack{x(0)=x_0 \\ x(t')=x'}} d_W x(\tau) \int\limits_{\substack{x(t')=x' \\ x(t)=x_t}} d_W x(\tau) \qquad 0<t'<t$$





Theorem (Wiener): The set of discontinuous as well as the set of differentiable functions has zero Wiener measure

Heuristic: $\langle x^2\rangle = \int_{-\infty}^{\infty} dx\ x^2\, \rho(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} dx\ x^2\, e^{-x^2/(4Dt)}$

$$= 4Dt\ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dy\ y^2 e^{-y^2}$$

$\Rightarrow$ the shift during a period of time t is typically:

$\bar{x} \equiv \sqrt{\langle x^2\rangle} \sim \sqrt{t}$

$\Rightarrow$ the speed of the Brownian particle : $v = \frac{\bar{x}}{t} \sim \frac{\sqrt{t}}{t} \to \infty$ for $t \to 0$

On the other hand: $\bar{x}(t) \to 0$ for $t \to 0$ $\Rightarrow$ the typical paths are continuous.

As a consequence: $\dot{x}$ in d_W has to be properly understood via the limiting procedure

(Something about fractal dimension! $\longrightarrow$ Drift!!

So far: Wiener path integral $\longleftrightarrow$ Euclidean path integral for the free particle.

? External potential?

Consider a Brownian motion in a medium where the particle can be annihilated with a time and space dependent rate $V(x,t) > 0$ per unit time.

$\Rightarrow$ in the case of a fixed particle with time-dependent annihilation rate $V(t)$, the survival probability at time t satisfies:

$$P_S(t+\Delta t) = P_S(t) - \Delta t\, V(t)\, P_S(t)$$

$$\Rightarrow \frac{dP_S(t)}{dt} = -V(t)\, P_S(t) \quad \Rightarrow \quad P_S(t) = P_S(t_0)\, e^{-\int_{t_0}^{t} ds\, V(s)}$$

Now "follow" the Brownian particle along its trajectory $x(\tau)$
$\Rightarrow$ the probability that at time t it is still alive is given by:
$$P_S[x(\tau)] = \exp\left\{-\int_0^t d\tau\, V(x(\tau),\tau)\right\}$$

so that, taking into account the probability of the given path $x(\tau)$:

$P_{1|1}$

$$W(x_t, t\,|\,x_0, 0) = \int_{x(0)=x_0}^{x(t)=x_t} d_W x(\tau)\, \exp\left\{-\int_0^t d\tau\, V(x(\tau),\tau)\right\}$$

transition probability
$(x_0, 0) \longrightarrow (x_t, t)$

Taking into account the definition of d_W it is easy to recognize the conditional probability as the Euclidean propagator in an external potential.

Obs: if V is bounded from below (say, >0) $\Rightarrow$ W exists in mathematical sense.

Feynman - Kac theorem:

$$W(x_t,t\,|\,x_0,0) = \int_{x(0)=x_0}^{x(t)=x_t} d_W x(\tau) \quad e^{-\int_0^t ds\ V(x(s))} \qquad (***)$$

satisfies the so-called Bloch equation (diffusion equation with potential)

$$\frac{\partial W}{\partial t} = D\,\frac{\partial^2 W}{\partial x_t^2} - V(x_t)\,W \qquad (4*)$$

note that this does not have the form of a Kramers-Moyal expansion! (the rhs is not the divergence of a current ⇒ prob. is not conserved

and $\quad W(x_t, t\to 0\,|\,x_0,0) = \delta(x_t - x_0)$

[W is said to be the fundamental solution or Green's function of the eq.]

Idea: We know the eq. satisfied by $W^{(V=0)}$ (diffusion!) ⇒ express $W^{(V\neq 0)}$ in terms of $W^{(V=0)}$ ...

Obs:

(i) $\quad \int_{-\infty}^{\infty} dx_s \ W(x_t,t\,|\,x_s,s)\, W(x_s,s\,|\,x_0,0) = W(x_t,t\,|\,x_0,0)$

(Ex) prove it; this is obvious for $V=0$

(ii) for $t\to 0$ (***) reduces to the conditional Wiener measure (i.e. the integrand → 1) so that the condition (4*) $\delta(\dots)$ follows immediately.

(iii) (trick) $\quad e^{-\int_0^t d\tau\ V(x(\tau))} = 1 - \int_0^t d\tau\ \underbrace{V(x(\tau))\ e^{-\int_0^{\tau} ds\ V(x(s))}}_{-\frac{d}{d\tau} e^{-\int_0^{\tau} ds\ V(x(s))}}$!

Inserting this equation in the (***):

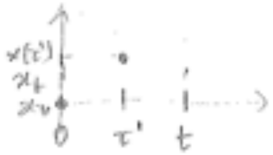
$$W(x_t,t\,|\,x_0,0) = \underset{\substack{\uparrow\\ V\equiv 0}}{W_0(x_t,t\,|\,x_0,0)} - \int_{x(0)=x_0}^{x(t)=x_t} d_W x(\tau) \int_0^t d\tau\ V(x(\tau))\ e^{-\int_0^{\tau} ds\ V(x(s))}$$

well-defined integral, so that we can exchange the order of integration

Thus:

$$\int_{x(0)=x_0}^{x(t)=x_t} d_W x(\tau) \left(\int_0^t d\tau' \right) V(x(\tau')) \, e^{-\int_0^{\tau'} ds \, V(x(s))} =$$

$$= \int_0^t d\tau' \int_{x(0)=x_0}^{x(t)=x_t} d_W x(\tau) \; V(x(\tau')) \, e^{-\int_0^{\tau'} ds \, V(x(s))}$$



$$= \int_0^t d\tau' \left[\int_{-\infty}^{\infty} dx_{\tau'} \int_{x(0)=x_0}^{x(\tau')=x_{\tau'}} d_W x(\tau) \int_{x(\tau')=x_{\tau'}}^{x(t)=x_t} d_W x(\tau) \; V(x_{\tau'}) \, e^{-\int_0^{\tau'} ds \, V(x(s))} \right]$$

$$= \int_0^t d\tau' \left[\int_{-\infty}^{\infty} dx_{\tau'} \; V(x_{\tau'}) \underbrace{\int_{x(0)=x_0}^{x(\tau')=x_{\tau'}} d_W x(\tau) \, e^{-\int_0^{\tau'} ds \, V(x(s))}} \; \underbrace{\int_{x(\tau')=x_{\tau'}}^{x(t)=x_t} d_W x(\tau)} \right]$$

$$= \int_0^t d\tau' \left[\int_{-\infty}^{\infty} dx_{\tau'} \; V(x_{\tau'}) \quad W(x_{\tau'}, \tau' | x_0, 0) \quad W_0(x_t, t | x_{\tau'}, \tau') \right]$$

Thus:

$$W(x_t, t | x_0, 0) = \underbrace{W_0(x_t, t | x_0, 0)} - \int_0^t d\tau' \int_{-\infty}^{\infty} dx_{\tau'} \; \underbrace{W_0(x_t, t | x_{\tau'}, \tau')} \, V(x_{\tau'}) \, W(x_{\tau'}, \tau' | x_0, 0)$$

they satisfy by construction the diffusion equation: $W_0(x_t, t | x_0, 0) = \frac{1}{\sqrt{4\pi D t}} \exp\left\{ -\frac{1}{4D} \frac{(x_t - x_0)^2}{t} \right\}$

(Dyson-like equation!)

$$\partial_t W_0 = D \, \partial^2_{x_t} W_0$$

So:

$$\partial_t W(x_t, t | x_0, 0) = D \, \partial^2_{x_t} W_0(x_t, t | x_0, 0) - \int_{-\infty}^{\infty} dx_{\tau'} \overbrace{W_0(x_t, t | x_{\tau'}, t)}^{\delta(x_t - x_{\tau'})} V(x_{\tau'}) \, W(x_{\tau'}, t | x_0, 0)$$

$$- \int_0^t d\tau' \int_{-\infty}^{\infty} dx_{\tau'} \; D \, \partial^2_{x_t} W_0(x_t, t | x_{\tau'}, \tau') \; V(x_{\tau'}) \, W(x_{\tau'}, \tau' | x_0, 0)$$

$$= D \, \partial^2_{x_t} \left[W_0 - \int_0^t d\tau' \int_{-\infty}^{\infty} dx_{\tau'} \; W_0 \, V \, W \right] +$$

$$- V(x_t) \, W(x_t, t | x_0, 0)$$

$$\Rightarrow \quad \partial_t W = D \, \partial^2_{x_t} W - V W$$

Summing up:

$$W(x_t,t|x_0,0) = \int_{x(0)=x_0}^{x(t)=x_t} d_W\, x(\tau)\; e^{-\int_0^t ds\, V(x(s))}$$

$$\text{"="} \int_{x(0)=x_0}^{x(t)=x_t} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{4\pi D\, d\tau}}\; e^{-S_W}$$

where: $S_W = \int_0^t ds \left[\frac{1}{4D} \dot{x}^2(s) + V(x(s)) \right]$

⇝ analogous of the Euclidean action of a system with Hamiltonian: $H = Dp^2 + V$

and:

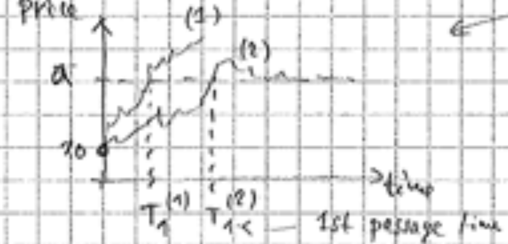
$$\partial_t W = \left[D\, \partial^2_{x_t} - V \right] W$$

(imaginary time Schrödinger eq. for a system with Hamiltonian H)

Application: FIRST PASSAGE TIMES / first crossing time ...

In general: (we shall discuss specifically the case of the Wiener process...) ↓ and later on of non-Markovian Gaussian processes.

ex: (a) price of a share...

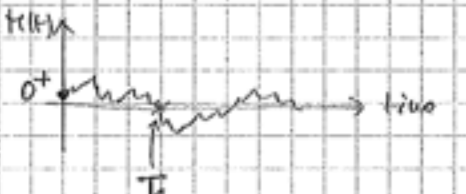


Note: we have in mind the case of continuous trajectories... One can ask of course the same for discontinuous ones... (eg. Cauchy..)

$T_1 = T_1(a)$ is a random variable...

Question: How is it distributed?

(b) fluctuations of the magnetization of an Ising model

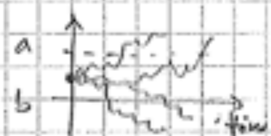


? Distribution
? Average value
⋮

(c) extinction of a population (branching & decay..)

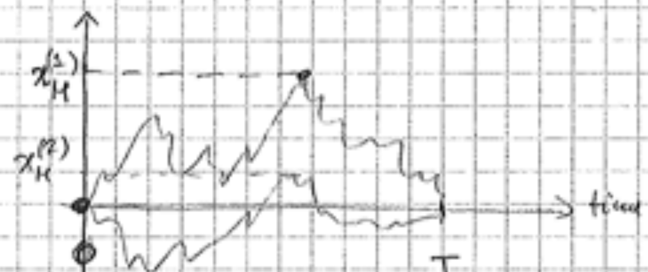
Related problems:

(i) escape of a diffusing particle from a segment: ⟶



(applications in several fields of physics)

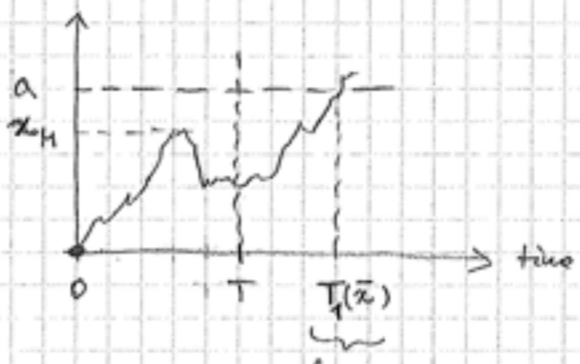
(ii) Distribution of extremal points:



The process starts from 0, and we observe it for a time T → depending on the realization, the value of the maximum x_M achieved changes..

⟹ x_M is a random variable!

? How is x_M distributed? How is this problem connected to the first passage?



x_M maximum $\Leftrightarrow$ in $[0,T]$ the process does not reach any $a > x_M$ $\Rightarrow$ it reaches them only at times $> T$

$\Downarrow$

$$\text{Prob}(x_M < a) = \underbrace{\text{Prob}(T_1(a) > T)}_{\text{probability of the first passage time.}}$$

Elementary approach: (reflection principle)

Consider a path γ which originates from 0 at time $t=0$ and reaches $x_T > a$ at time T

$\Rightarrow$ it first crosses $x = a$ at time $T_1 < T$

Construct now $\bar{\gamma}$, which is reflected after T_1 $\rightarrow$ it ends in the reflected point $\bar{x}_T = a - (x_T - a)$

Because of: (i) Markov property
(ii) invariance under reflection } think of it on the discrete...

$\Rightarrow$ γ and $\bar{\gamma}$ have the same probability

$\Rightarrow$ for every path γ which connects 0 to $x_T > a$ there is an associated path $\bar{\gamma}$", with the same probability, which connects 0 to $\bar{x}_T < a$ and has $T_1 < T$, where T_1 is the 1st crossing time of γ.

$\Rightarrow$ Note that also the converse is true.

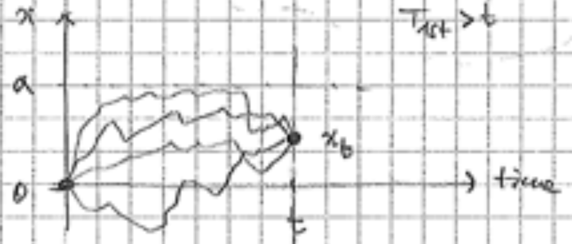
$$\Rightarrow \quad \text{Prob}(x_T < a,\ T_1(a) < T) = \text{Prob}(x_T > a,\ T_1(a) < T) \qquad (*)$$

Now:
$$\text{Prob}(T_1(a) < T) = \text{Prob}(T_1(a) < T,\ x_T > a) + \underbrace{\text{Prob}(T_1(a) < T,\ x_T \le a)}_{= (*)}$$

$$= 2\,\text{Prob}(T_1(a) < T,\ x_T > a) \underset{x_T > a \Rightarrow T_1(a) < T\,!}{=} 2\,\text{Prob}(x_T > a)$$

Path-integral approach: → Note that: $\{x(\tau)\} \mid x(\tau) \le a$ for $0 \le \tau \le t$
$\Updownarrow$
$T_{1st} > t$

We have to "count" the # of traj. with $x(\tau) < a$ $\forall \tau \in [0,t]$



or better, to determine the Wiener measure of the ensemble of traj:

$$\varphi \equiv \underbrace{\mathrm{Prob}\left(\{x(\tau)\} \mid x(\tau) \le a,\ 0 \le \tau \le t \right) = \mathrm{Prob}(T_{1st} > t)}_{\text{according to our definitions}}$$

T_{1st}: first crossing time

How do we characterize this ensemble of traj?

$\Rightarrow$ via the characteristic function:

$$\text{traj: } x(\tau) \Rightarrow \mathcal{X}[x(\tau)] = \begin{cases} 1 & \text{if } x(\tau) \le a \quad \forall\, 0 \le \tau \le t \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \varphi \equiv \mathrm{Prob}\left(\{x(\tau)\} \mid x(\tau) \le a,\ 0 \le \tau \le t \right) = \underbrace{\int_{\substack{x(0)=0 \\ x(t)\ \text{arbitrary}}} d_W x(\tau)}_{\text{unconditioned WM}}\ \mathcal{X}[x(\tau)] = \langle \mathcal{X}[x(\tau)] \rangle_W$$

Expectation on the Wiener measure

$$= \int_{-\infty}^{\infty} dx_t \underbrace{\int_{\substack{x(0)=0 \\ x(t)=x_t}} d_W x(\tau)}_{\text{conditioned WM}}\ \mathcal{X}[x(\tau)]$$

Q: $\mathcal{X}[x(\tau)] = ?$

$$\mathcal{X}[x(\tau)] = e^{-\int_0^t d\tau\, V_a(x(\tau))}$$

$$\text{with} \quad V_a = \begin{cases} 0 & \text{for } x(\tau) < a \\ \infty & \text{for } x(\tau) \ge a \end{cases}$$

$\Rightarrow$ hard-wall potential!

$$\Rightarrow \quad \varphi = \int_{-\infty}^{\infty} dx_t \underbrace{\int_{\substack{x(0)=0\\ x(t)=x_t}} d_W x(\tau) \; \exp\left\{ -\int_0^t d\tau \, V_a(x(\tau)) \right\}}_{= \; W(x_t,t|0,0)}$$

satisfies Bloch eq.:

$$\partial_t W = \left[D\partial^2_{x_0} - V_a \right] W$$

How do we solve this?

QM: $t = i\tau$

$+i\partial_\tau K = \left[-D\partial^2_{x_t} + V_a \right] K$

$\Rightarrow$ propagator in the $\frac{1}{2}\infty$ semi-infinite line

(i) Clearly $W(x_t,t|0,0) = 0$ for $x_t > a$

(ii) $W(x_t,t|0,0)$ has to be a continuous function of x_t

$\Rightarrow \quad W(x_t = a, t|0,0) = 0.$

(iii) for $x_t < a \Rightarrow \quad \partial_t W = D\partial^2_{x_t} W$

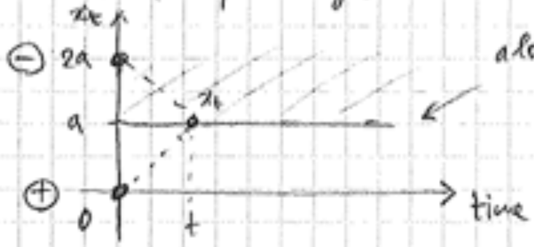
we know the solution $W^{(0)}$ with the proper initial condition: $W^{(0)}(x_0,0|0,0) = \delta(x_0)$

$$W^{(0)}(x_t,t|0,0) = \frac{1}{\sqrt{4\pi D t}} \exp\left\{ -\frac{x_t^2}{4Dt} \right\}$$

$\Rightarrow$ we should satisfy the BC (ii).. the eq. is linear

$\Rightarrow$ linear superposition of solution is a solution..

Method of images:



along this line the distance from the two sources is the same...

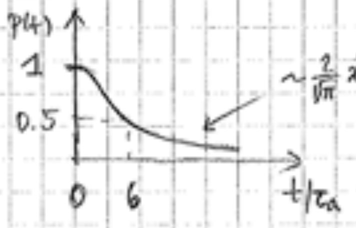
$$W^{(0)}(\underset{a}{x_t},t|0,0) = W^{(0)}(\underset{a}{x_t},t|2a,0)$$

(here enters the assumption of invariance under reflection, as in the previous approach!)

Accordingly:

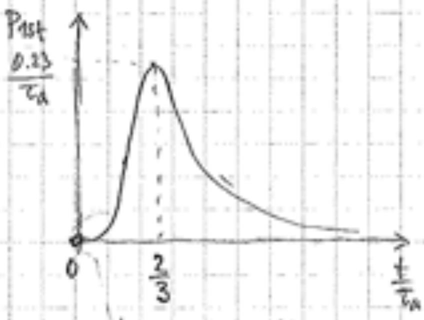
$$W(x_t,t|0,0) = W^{(0)}(x_t,t|0,0) - \underbrace{W^{(0)}(x_t,t|2a,0)}_{W^{(0)}(x_t-2a,t|0,0)} \quad \text{for } x_t \le a$$
$$= 0 \quad \text{for } x_t > a$$

$$\Rightarrow \quad p = \int_{-\infty}^{\infty} dx_t\, W = \int_{-\infty}^{a} dx_t \left[W^{(0)}(x_t,t|0,0) - W^{(0)}(x_t-2a,t|0,0)\right]$$
$$= \int_{-\infty}^{a} dx_t\, W^{(0)} - \int_{-\infty}^{-a} dx_t\, W^{(0)}(x_t,t|0,0)$$
$$= 2\int_0^a dx_t\, W^{(0)}(x_t,t|0,0)$$
$$= \frac{2}{\sqrt{\pi}} \int_0^{a/\sqrt{4Dt}} dx\, e^{-x^2} \qquad \text{def: } \tau_a \equiv \frac{a^2}{4D}$$
$$= \mathrm{Erf}\left(\left(\frac{t}{\tau_a}\right)^{-1/2}\right)$$



See plot below

$$\Rightarrow \quad \mathrm{Prob}(T_{1st} > t) = p$$
$$\Rightarrow \quad p_{1st}(t) = -\frac{d}{dt}\mathrm{Prob}(T_{1st} > t) = -\frac{d}{dt}\frac{2}{\sqrt{\pi}}\int_0^{(t/\tau_a)^{-1/2}} e^{-x^2}$$
$$= \frac{1}{\sqrt{\pi}}\left(\frac{t}{\tau_a}\right)^{-3/2}\frac{1}{\tau_a}\, e^{-\tau_a/t} \sim t^{-3/2}$$



comment on the fact that this would vanish in the discrete version

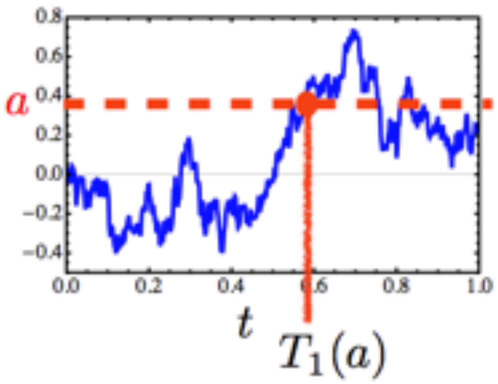
See plot below

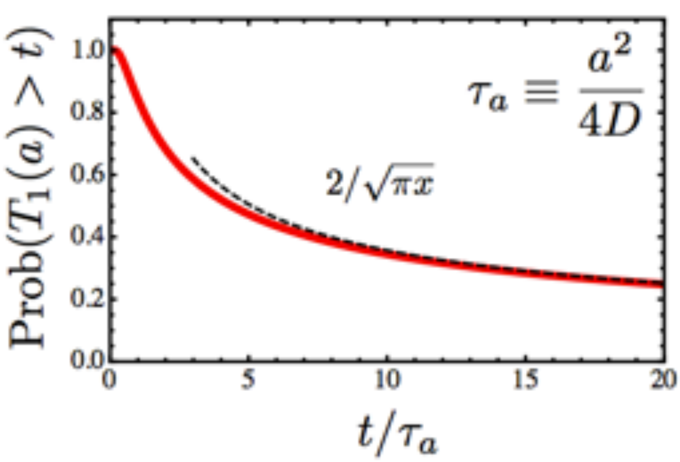
$$\langle T_{1st}\rangle = \int_0^\infty dt\, t\, p_{1st}(t) \sim \int^\infty dt\, t\, t^{-3/2} \to \infty \quad \forall a\,!$$

and therefore for the starting point itself!

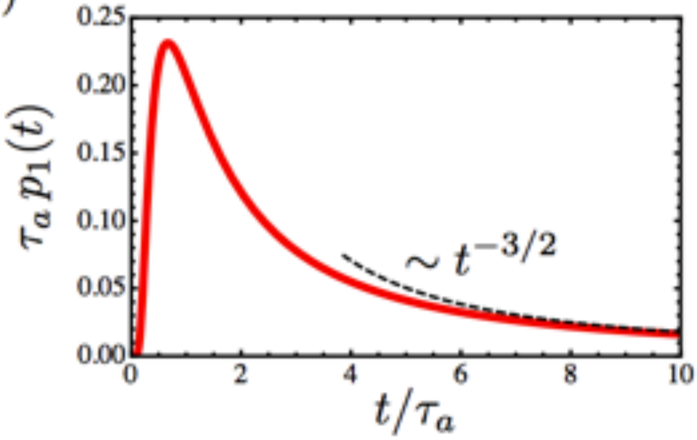
⇒ the time it takes to visit the same point again is <u>on average</u> ∞!

→ On the other hand: the same point is visited ∞-many times! (recurrend in $d=1,2$);





$$p_1(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Prob}(T_1(a) > t)$$



Indeed; this can be easily seen by considering the discrete version:

if the walker starts from 0 $\Rightarrow$ it can be in 0 only after an even number of steps.

Which is the probability p_r to be in 0 after r (even) steps?

$\Rightarrow$ # of steps taken to the right $=$ # of steps to the left $= \frac{r}{2}$

$$\Rightarrow p_r = 2^{-r}\binom{r}{r/2}$$

Define $O_r = \begin{cases} 1 & \text{if the walker is in 0 at the } r\text{-th step} \\ 0 & \text{otherwise} \end{cases}$

Number N of visits to the origin $= \sum_{\substack{r=1 \\ \text{even}}} O_r$ (exclude the first)

$$\Rightarrow \langle N \rangle = \sum_{\substack{r=1 \\ \text{even}}}^{\infty} \langle O_r \rangle = \sum_{\substack{r=1 \\ \text{even}}} p_r = \text{diverges!}$$

(ex: calculate this for a biased RW and show that it is finite...)

— o — o —

A quantity related to p_{1st} is the <u>persistence probability</u>.

$$P(t) = \text{persistence probability} = \text{Prob}\{x_0 \cdot x(\tau) > 0 \quad \forall \tau \in [0,t]\}$$

Clearly:

$$P(t+\Delta t) - P(t) = -\text{prob. of first crossing of zero in the interval } \Delta t = -p_{1st}(t)\,\Delta t$$

$$\Rightarrow \frac{dP(t)}{dt} = -p_{1st}(t)$$

i.e.: $P(t) = \text{Prob}(T_{1st} > t)$ (that could be inferred directly from the definition!)

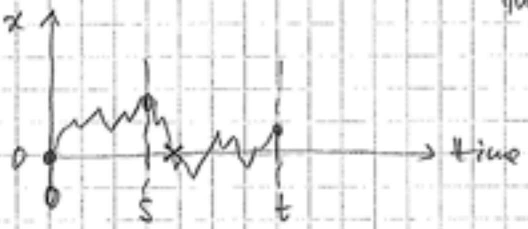
For t large enough ($t \gg \tau_a$), there are cases in which $P(t) \sim t^{-\theta}$ with universal exponent (i.e. indep. of τ_a)

$\Rightarrow \theta$ is called PERSISTENCE EXPONENT (nontrivial!) and actually universal in the sense of crit. phenomena

$\theta = \frac{1}{2}$ for the Wiener process

Ex: Consider a Wiener process which starts from 0.

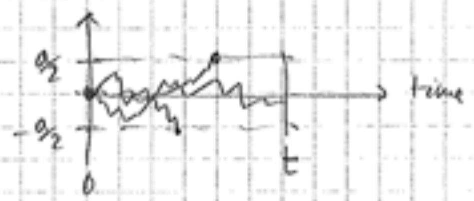
(a) Calculate the probability $P(t,s)$ that the process crosses zero at least once within the time interval $[s,t]$ ⇝ distribution of the zeros of the random walk, this is the so-called arcsin law:



$$P(t,s) = 1 - \frac{2}{\pi} \arcsin \sqrt{\frac{s}{t}}$$

(b) Calculate the distribution of the maximum of the Wiener process in $[0,T]$

(c) Calculate the probability that the trajectory of the Wiener process, up to time t, has never left the strip $[-\frac{a}{2}, \frac{a}{2}]$ and determine the probability density function of the time T of escape



Fluctuations, correlations and field-theoretical methods:

single particle: Although what I have said so far is quite generally valid, I have always considered examples of single-particle (Brownian dynamics) or cases with no spatial structure (population dynamics).

On the other hand, complex behavior often involves (diffusive) spreading, front propagation and spontaneous or induced pattern formation. (Many-body!)

↳ in order to describe all that we need to include spatial degrees of freedom so that the typical average quantities (mesoscopic variables) which describe the macroscopic behaviour of the system become local density fields.

Stochasticity enters in the form of randomly occurring (i) propagation, (ii) interactions and (iii) reactions

Such stochastic processes generate

(a) internal noise (as in the case of population dynamics)

(b) external additive noise ⇕ or multiplicative (as in the Langevin description...) ↳ typically the case when there is a clear separation of time scales.

⇝ the behavior of the system at large scales and long times might be crucially affected by these fluctuations, especially in those cases in which the system becomes per se particularly "soft" to perturbations (as it is the case upon approaching critical points!!) or becomes somehow "collective" or "cooperative".

A quantitative mathematical analysis of complex spatio-temporal structures and cooperative behaviour in stochastic interacting systems with many degrees of freedom typically relies on the study of correlation functions.

Field-theoretic methods have been developed and employed in this context starting from the '70.

→ ingredients:
- usually space-time dependent fields on the continuum (or, in some cases, on a lattice...)
- interaction
- stochastic evolution defined either
  → via a Master Equation
  → via Langevin eqs.

→ methods: those typical of many-body physics and quantum and statistical field theory. (RG etc.)

→ this allows one to construct systematic approximation schemes (eg. perturbative expansions, with respect to some parameter, presumed to be small, which measures the strenght of fluctuations)

lowest-level: mean-field, no fluctuations

→ If one is interested in scale-invariant phenomena, then RG is a powerful method which can be naturally extended to stochastic dynamics (both equilibrium and non-equilibrium) when it is cast in field-theoretical form

Here we focus on the case of dynamics defined via a Master Equation and how it can be mapped onto a field-theoretical problem.

Before doing it, let us focus on another model of population dynamics with interaction (and a surprisingly rich behaviour): the Lotka - Volterra model.

aim: describe emerging periodic oscillations in (Lotka) autocatalytic reactions (Volterra) Adriatic fish population

2 species: $A \to$ predator
$B \to$ prey

dynamics:

$A \xrightarrow{\mu} \phi$ predator death

$B \xrightarrow{\sigma} B+B$ prey proliferation

$A+B \xrightarrow{\lambda} A+A$ predation interaction (predator reproduces only if food is available!)

$\lambda=0 \rightsquigarrow$ decoupling: predator face extinction, prey proliferate.

Describe the populations via their average densities

$$A \to a(t) \quad ; \quad B \to b(t)$$

for $\lambda=0$, they obey a rate equation

$$\begin{cases} \dot{a}(t) = -\mu a(t) \\ \dot{b}(t) = \sigma b(t) \end{cases}$$

leading to exponential extinction / prolifer.

$\lambda \neq 0 \Rightarrow$ competition between the two populations...

↳ to quantify it we should know the probability of finding an A-B pair at time t.

Taking also into account also space, say a lattice on which A and B perform random walks, predation will occurr only if predator and prey occupy adjacent sites.

$\rightsquigarrow$ the evolution eqs for the mean densities have to be amended by the terms: $\pm \lambda \langle a(x,t)\, b(x,t) \rangle$ ensemble averages

Assuming that the populations A and B are uncorrelated (certainly not true)

$\rightsquigarrow$ $\langle a(x,t)\, b(x,t) \rangle = \langle a(x,t) \rangle \langle b(x,t) \rangle$ (mean-field factorization!)

one obtains the so-called deterministic L-V eqs:

$$\begin{cases} \dot{a} = \lambda a b - \mu a \\ \dot{b} = -\lambda a b + \sigma b \end{cases}$$

Obs: $K(t) = \lambda [a+b] - \sigma \ln a - \mu \ln b$ is a constant of the motion ($\dot{K} = 0$)

$\rightsquigarrow$ regular non-linear population oscillations whose frequency and amplitude are determined by initial conditions.

Q: How can we describe this model beyond mean-field?

$\rightsquigarrow$ MC!

$\rightsquigarrow$ FT! (it gives easy access to long-time large-distance properties and scale-invariant behaviours...)

Lotka-Volterra:

A: predator

B: prey

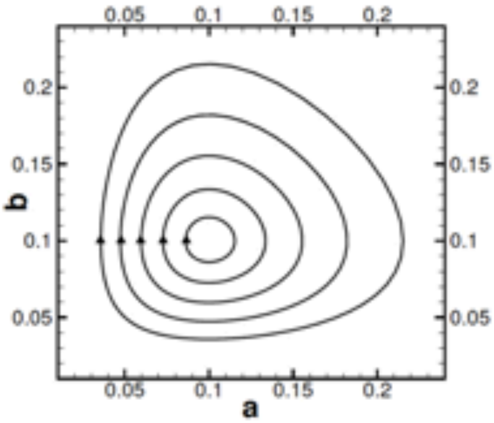
(1880-1949)

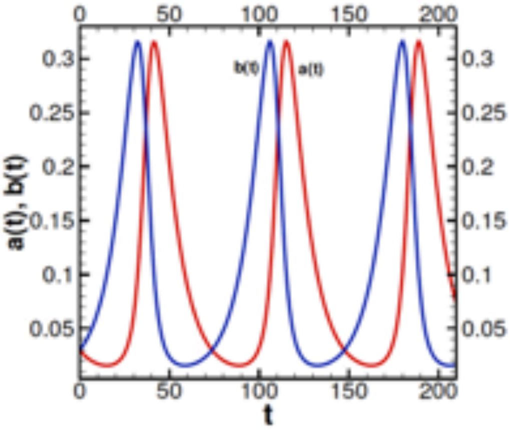
(1860-1940)

$A \xRightarrow{\mu} \varnothing$

$A+B \xRightarrow{\lambda} A+A$

$B+\varnothing \xRightarrow{\sigma} B+B$





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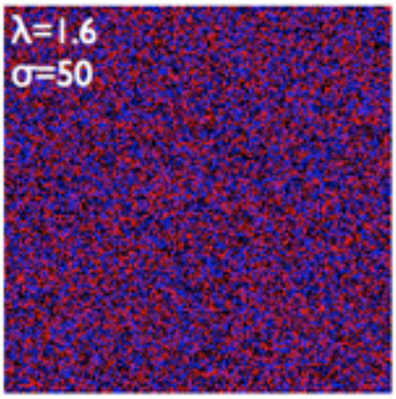

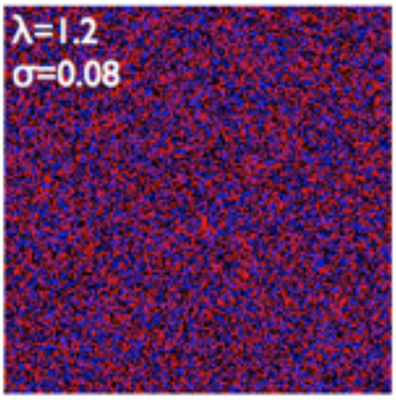

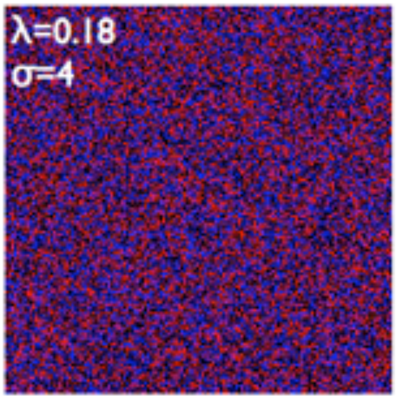

http://www.phys.vt.edu/~tauber/