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Spring College on the Physics of Complex Systems

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Stochastic processes and applications
Lectures 6 & 7

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Discrete stochastic interacting particle systems.

(and mapping onto a FT: Doi and Delitto > 1976)

in this class there is the large family of reaction - diffusion systems which modelize, e.g., a variety of different phenomena from chemical reactions to directed percolation [Comment on it? examples?]

- Assume that the system is on a lattice and the config. is specified by the occupation number of each lattice site, assumed to be an integer = # of particles.

  If more than one species is present ⇝ occupation variables for each of them.
- Assume that particles react when they meet and diffuse

  ↓

  atoms/molecules in chemistry

  individuals in population dyn

  other degrees of freedom of a system (# of domain walls... in a magnet...)
- for simplicity we assume time-independent reaction and diffusion rates.

The dynamics is assigned via the Master Equation:

$$\frac{\partial P(\alpha;t)}{\partial t} = \sum_{\beta} \left[P(\beta,t)\, W(\beta\to\alpha) - P(\alpha,t)\, W(\alpha\to\beta) \right]$$

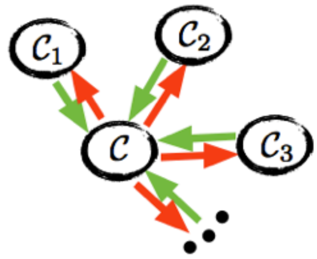
Fock-space representation of the ME:

I illustrate it for the simple case of irreversible binary annihilation process $A+A \xrightarrow{\lambda} A$ (simple annihilation to distinguish it from binary annihilation $A+A\to\emptyset$)

⇝ this is a on-site reaction (we might consider a single site... the generalization to more sites is straightforward...)

and the occupation of the site is labeled by an integer n

Master Equation:



$$\frac{\partial P(\mathcal{C},t)}{\partial t} = \sum_{\mathcal{C}'} \Big\{ \underbrace{\boxed{W(\mathcal{C}' \mapsto \mathcal{C}) P(\mathcal{C}',t)}}_{\text{gain}} \underbrace{\boxed{- W(\mathcal{C} \mapsto \mathcal{C}') P(\mathcal{C},t)}}_{\text{loss}} \Big\}$$

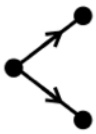
$$= - \sum_{\mathcal{C}'} \mathcal{L}_{\mathcal{C},\mathcal{C}'} P(\mathcal{C}',t)$$

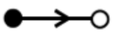
$$\mathcal{L}_{\mathcal{C},\mathcal{C}'} = -W(\mathcal{C}' \mapsto \mathcal{C}) + \sum_{\mathcal{C}''} W(\mathcal{C} \mapsto \mathcal{C}'') \delta_{\mathcal{C},\mathcal{C}'}$$

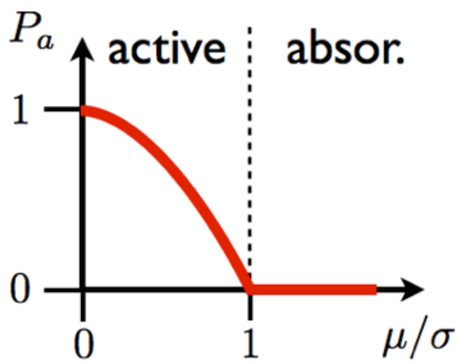
$\mathcal{C} = n \in \mathbb{N}$

$\mathcal{C} = \{n_1, n_2, n_3, \ldots\} = \{n_i\}_i \subset \mathbb{N}^{\#\text{sites}}$

-Reaction-diffusion
-Chemical reactions
-Directed percolation
-Epidemiology
-.......

(Branching)  $A \overset{\sigma}{\mapsto} A + A$

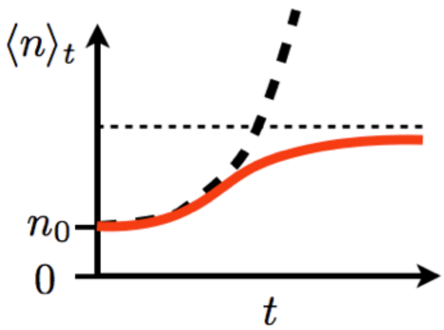
(Decay)  $A \overset{\mu}{\mapsto} \emptyset$



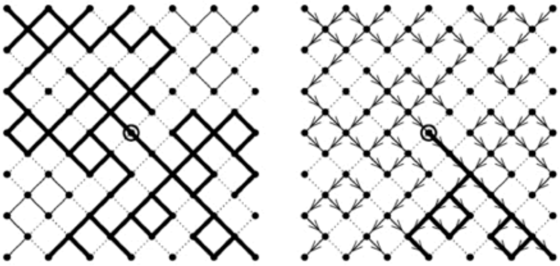
$$P_a \sim (\sigma - \mu)^{\beta = 1}$$

$$\tau_c \sim |\mu - \sigma|^{-1}$$

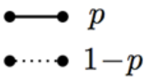
(Coagulation)
(Lethal competition)  $A + A \overset{\lambda}{\mapsto} A$



Percolation



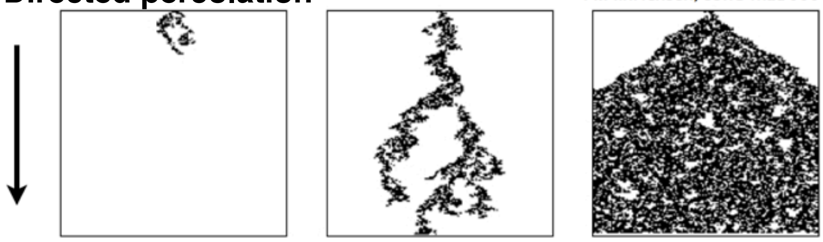
isotropic bond percolation *directed bond percolation*



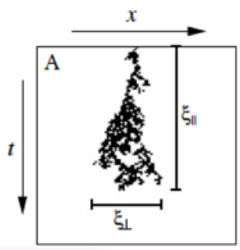
H.Hinrichsen, cond-mat/0001070

Directed percolation

H.Hinrichsen, cond-mat/000107



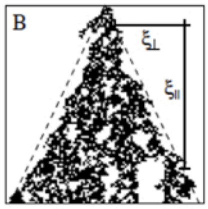
low p	$p = p_c$	high p
$p < p_c$		$p > p_c$



$$\xi_{\perp} \sim (p - p_c)^{-\nu_{\perp}}$$

$$\xi_{\|} \sim (p - p_c)^{-\nu_{\|}}$$

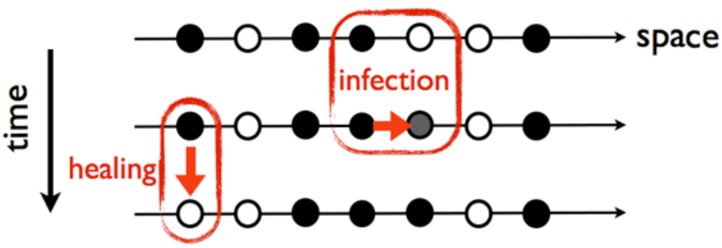
$$\xi \sim (T - T_c)^{-\nu}$$



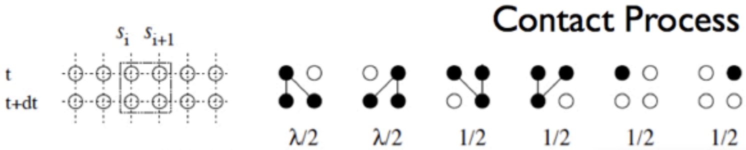
Infection spreading

○ ☺ healty

● ☹ sick

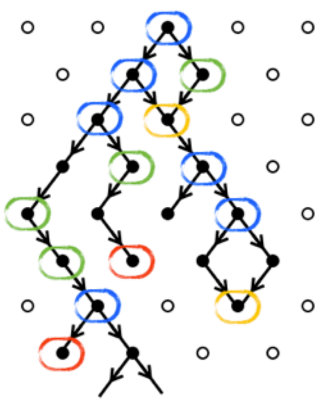


Contact Process



H.Hinrichsen, cond-mat/0001070

Directed Percolation & Contact Process:



Forest fire.....

(B) $A \overset{\sigma}{\mapsto} A + A$

(D) $A \overset{\mu}{\mapsto} \emptyset$

(C) $A + A \overset{\lambda}{\mapsto} A$

(diff) D

The ME for this process is: $w(n\to n-1) = \lambda n(n-1)$

$$\frac{\partial P(n,t)}{\partial t} = \underbrace{w(n+1\to n)}_{\lambda(n+1)n} P(n+1,t) - \underbrace{w(n\to n-1)}_{\lambda n(n-1)} P(n,t)$$

and one might consider an initial Poisson distribution

$$P(n) = \frac{n_0^n}{n!} e^{-n_0}$$

Obs: the reactions all change the site occupation number n by integer values, a Fock space repres. (borrowed from QM) is particularly useful.

Introduce the HO (or bosonic ladder) operator algebra at each site of the lattice:

$$\begin{cases} [a_i, a_j] = 0 \\ [a_i^+, a_j^+] = 0 \\ [a_i, a_j^+] = \delta_{ij} \end{cases}$$

with a "vacuum state" $|0\rangle$ such that: $\boxed{a_i|0\rangle = 0}$

From a^+ we construct the particle number $\boxed{N_i \equiv a_i^+ a_i}$

eigenstates $N_i|n_i\rangle = n_i|n_i\rangle$

such that $a_i^+|n_i\rangle = |n_i+1\rangle$

starting from the vacuum $|0\rangle$ one construct the 1, 2, 3, ... particle states by applying a^+

(exercise $\Rightarrow$) $a_i|n_i\rangle = n_i|n_i-1\rangle$

which corresponds to the annihilation of one of the n particles in the state.

do it! exercise: Calculate $\langle n'|n\rangle$

For this reason:

$$\left.\begin{array}{l} a \to \text{destruction} \\ a^+ \to \text{creation} \end{array}\right\} \text{operator}$$

Fock space:

$$\begin{cases} [a_i, a_j] = 0 \\ [a_i^\dagger, a_j^\dagger] = 0 \\ [a_i, a_j^\dagger] = \delta_{ij} \end{cases}$$

$$a_i|0_i\rangle = 0$$

$$a_i^\dagger|n_i\rangle = |n_i + 1\rangle$$

$$\Downarrow$$

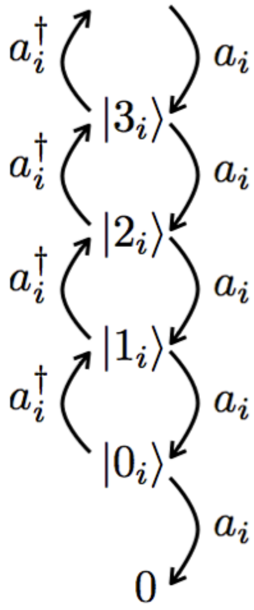
$$a_i|n_i\rangle = n_i|n_i - 1\rangle$$

$$N_i \equiv a_i^\dagger a_i$$

(✍) $N_i|n_i\rangle = n_i\,|n_i\rangle$

$\langle n|n'\rangle = ?$

Bosonic ladder



$\{|n_1, n_2, n_3, \ldots\rangle\}$

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \qquad \{\psi_1, \psi_2, \ldots\} \Rightarrow \psi = \sum_n c_n\psi_n$$

$$\Downarrow$$

$$i\hbar\frac{\partial c_n}{\partial t} = \sum_m H_{n,m}c_m$$

$$\frac{\partial P(\{n\},t)}{\partial t} = -\sum_{\{n'\}} \mathcal{L}_{\{n\},\{n'\}}P(\{n'\},t)$$

$$\psi_n \Longleftrightarrow |\{n\}\rangle$$

$$c_n(t) \Longleftrightarrow P(\{n\},t)$$

$$\psi \Longleftrightarrow \boxed{|\Phi(t)\rangle \equiv \sum_{\{n\}} P(\{n\},t)\;|\{n\}\rangle}$$

To rewrite the ME we introduce a formal state vector

$$|\Phi(t)\rangle = \sum_{\{n_i\}} P(\{n_i\};t)\, |\{n_i\}\rangle$$

$|\{n_i\}\rangle$ they form a base of the configuration space...

which is a linear combination of all possible states of the system weighted by the time-dependent configurat. probability

⇝ Simple manipulations then transform the linear time evolution prescribed by the ME into an "imaginary-time" Schrödinger eq.:

$$\frac{\partial}{\partial t} |\Phi(t)\rangle = -H |\Phi(t)\rangle$$ with suitable H defined by its matrix element $\langle\{n_i'\}|H|\{n_i\}\rangle$

⇝ formal solution $|\Phi(t)\rangle = e^{-Ht}|\Phi(0)\rangle$

$H \rightarrow$ Liouville time evolution operator.

For on-site reaction processes, different sites do no "talk to each other" $\Rightarrow$ $H_R = \sum_i H_i(a_i^+, a_i)$

Note: It is convenient always to express H in the normal order i.e. with all destruction operators to the right and all creation operators to the left

(this can be done by suitably commuting a^+a ... in non-ordered terms)

Example: "Binary annihilation process," (simple annihilation at a single site...) take $\sum_{n=0}^{\infty}$ ME $|n\rangle$:

$$\underbrace{\sum_{n=0}^{\infty} \frac{\partial P(n,t)}{\partial t}|n\rangle}_{\frac{\partial}{\partial t}|\Phi(t)\rangle} = \sum_{n=0}^{\infty}\left[\lambda(n+1)n\, P(n+1,t) - \lambda\, n(n-1)P(n,t)\right]|n\rangle$$

for this term we need to express $|n\rangle$ as $|n+1\rangle$

$a|n+1\rangle = (n+1)|n\rangle$

and the extra factor n can be produced via $N = a^+a$

$a|n\rangle = n|n-1\rangle$

$a^2|n\rangle = n(n-1)|n-2\rangle$

$a^{+2}a^2|n\rangle = n(n-1)\underbrace{a^{+2}|n-2\rangle}_{|n\rangle}$

Accordingly:

$$\sum_{n=0}^{\infty} \lambda (n+1)\, n\, P(n+1,t)\, |n\rangle =$$

$$= \sum_{n=-1}^{\infty} \lambda \underbrace{(n+1)\, n\, P(n+1,t)\, |n\rangle}_{(a^+ a)\, a\, |n+1\rangle} =$$

$$= \sum_{n=0}^{\infty} \lambda\, a^+ a^2\, |n\rangle\, P(n,t) = \lambda\, a^+ a^2\, |\Phi(t)\rangle$$

$$\sum_{n=0}^{\infty} (-\lambda)\, \underbrace{n(n-1)\, P(n,t)\, |n\rangle}_{a^{+2} a^2 |n\rangle} = -\lambda\, a^{+2} a^2\, |\Phi(t)\rangle$$

$$\Rightarrow \quad -H_R = \lambda\, a^+ a^2 - \lambda\, a^{+2} a^2 = -\lambda\, (a^{+2} - a^+)\, a^2$$

$$\Rightarrow \quad H_R = \lambda\, (a^{+2} - a^+)\, a^2$$

In order to determine these Hamiltonians it might be useful to remember that:

$$a^{+\ell} a^k\, |n+k-\ell\rangle = (n+k-\ell)(n+k-\ell-1) \cdots (n-\ell+1)\, |n\rangle$$

<u>Exercise</u>: (a) Determine the Hamiltonian H associated with the irreversible process $kA \to \ell A$

Do it!

with $W(n \to n+\ell-k) = \lambda\, n(n-1) \cdots (n-k+1)$

(b) in particular consider the case of <u>binary annihilation</u> $A + A \xrightarrow{\lambda} \emptyset$ (useful later on)

- We add now the other relevant ingredient: <u>Diffusion</u>

we first consider the case of <u>simple hopping</u>;

Consider two sites, labeled by i & j and the hopping $i \to j$ with rate D; the two sites are described by the joint probability distribution $P(n_i, n_j)$

(each particle can hop independently of the others with rate D...)

and the ME is: $\overbrace{D \times \#\text{ of particles}}$

$$\frac{\partial}{\partial t} P(n_i, n_j, t) = D(n_i+1) P(n_i+1, n_j-1, t) - D n_i P(n_i, n_j, t)$$

$$\left.\begin{matrix} n_i+1 \\ n_j-1 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} n_i \\ n_j \end{matrix}\right. \qquad \left.\begin{matrix} n_i \\ n_j \end{matrix}\right\} \rightarrow \left\{\begin{matrix} n_i-1 \\ n_j+1 \end{matrix}\right.$$

In this case: $|\Phi\rangle = \sum_{n_i n_j = 0}^{\infty} P(n_i n_j, t) \, |n_i, n_j\rangle$

taking into account that $|n\rangle = a^+ |n-1\rangle$

$(n+1) |n\rangle = a \, |n+1\rangle$

$$\Rightarrow \quad \frac{\partial}{\partial t} |\Phi(t)\rangle = \sum_{n_i n_j} \Big\{ D \underbrace{(n_i+1) |n_i n_j\rangle}_{= a_j^+ a_i |n_i+1, n_j-1\rangle} P(n_i+1, n_j-1, t) - D \underbrace{n_i |n_i n_j\rangle}_{= \underbrace{a_i^+ a_i}_{\text{number operator}} |n_i n_j\rangle} P(n_i n_j t)$$

$$= \left(D a_j^+ a_i - D a_i^+ a_i \right) |\Phi(t)\rangle$$

$$\Rightarrow \quad H_{i \to j} = -D \left(a_j^+ - a_i^+ \right) a_i$$

If diffusion occurs in both directions with the same rate (unbiased diffusion)

$$H_D = H_{i \to j} + H_{j \to i} = -D \left(a_j^+ - a_i^+ \right) a_i - D \left(a_i^+ - a_j^+ \right) a_j$$

$$= +D \left(a_j^+ - a_i^+ \right) \left(a_j - a_i \right)$$

if the diffusion occurs between nearest neighbours: $|i-j| = 1$ in $d > 1$

<u>Summing up:</u>

$A + A \xrightarrow{\lambda} A$ — Binary annihilation with diffusion

$$H = D \sum_{\langle ij \rangle} (a_j^+ - a_i^+)(a_j - a_i) + \lambda \sum_i (a_i^{+2} - a_i^+) a^2$$

(or) $A + A \xrightarrow{\lambda} \emptyset$ — simple annihilation:

$$H = D \sum_{\underbrace{\langle ij \rangle}_{nn}} (a_j^+ - a_i^+)(a_j - a_i) - \lambda \sum_i \left(a_i^2 - a_i^{+2} a_i^2 \right)$$

<u>Obs:</u> the Hamiltonian H is <u>not</u> quadratic!

$\rightsquigarrow$ non-trivial theory!!

Exercise: Determine the Hamiltonian H for the Lotka – Volterra model
(you need to introduce two families of operators, one for each species..)

Obs: H is not Hermitean..

but: if the rates satisfy detailed balance
$\Rightarrow$ H may be made symmetric and real by a similarity transformation (as in the case of F.P.

Relation between forward & backward evolution?!

$\longrightarrow$

Averages of observables: these are the things we are eventually interested in...

How do we calculate them?

Trick: via the projection state $\langle P|$, defined as:

$$\langle P| \equiv \langle 0| \prod_i e^{a_i}$$

Properties: (exercise)

(i) $\langle P|0\rangle = 1$

(ii) $\langle P|a_i^+ = \langle P|$

why? $[e^{a_i}, a_j^+] = e^{a_i}\delta_{ij}$

from: $[a^k, a^+] = k\, a^{k-1}$

$a^+ \to -\frac{\partial}{\partial a}$

exercise: for later convenience prove that:

$$e^{\lambda a} a^+ = (a^+ + \lambda) e^{\lambda a}$$

$$\Rightarrow e^{\lambda a} f(a^+) = f(a^+ + \lambda) e^{\lambda a}$$

$$\text{and} \quad f(a)\, e^{\lambda a^+} = e^{\lambda a^+} f(a + \lambda)$$

(iii) $\langle P|n_i\rangle = 1 \quad \forall n_i$

Averages:

QM: $$\langle \mathcal{O} \rangle_{\psi(t)} \equiv \langle \psi(t) | \mathcal{O} | \psi(t) \rangle = \sum_{m,n} c_m^*(t) \mathcal{O}_{m,n} c_n(t)$$

SP: $$\langle \mathcal{O}(\{n\}) \rangle_t \equiv \sum_{\{n\}} \mathcal{O}(\{n\}) P(\{n\}, t)$$

$$\neq \langle \Phi(t) | \mathcal{O}(\{a^+ a\}) | \Phi(t) \rangle$$

$$\langle \mathcal{P} | \sum_{\{n\}} \mathcal{O}(\{n\}) P(\{n\}, t) | \{n\} \rangle$$

$$\boxed{\langle \mathcal{P} | \{n\} \rangle = 1 \quad \forall \{n\}}$$

(✍) $$\langle \mathcal{P} | \equiv \langle 0 | \prod_i \mathrm{e}^{a_i}$$

$$\langle \mathcal{P} | \mathcal{O} | \Phi(t) \rangle$$

(✍) $$\langle \mathcal{P} | a_i^\dagger = \langle \mathcal{P} |$$

(a) $$[\mathrm{e}^{a_i}, a_j^\dagger] = \mathrm{e}^{a_i} \delta_{ij} \qquad \Leftarrow \qquad [a^k, a^\dagger] = k a^{k-1}$$

$$a^\dagger \mapsto -\partial_a$$

(b) $$\mathrm{e}^{\lambda a}\, a^\dagger = (a^\dagger + \lambda)\, \mathrm{e}^{\lambda a}$$

$$\mathrm{e}^{\lambda a}\, f(a^\dagger) = f(a^\dagger + \lambda)\, \mathrm{e}^{\lambda a}$$

$$f(a)\, \mathrm{e}^{\lambda a^\dagger} = \mathrm{e}^{\lambda a^\dagger}\, f(a + \lambda)$$

Accordingly, given an observable $\mathcal{O}(\{n_i\})$ which depends on the occupation numbers (e.g. the numbers themselves...), one has:

$$\langle \mathcal{O} \rangle (t) = \sum_{\{n_i\}} \mathcal{O}(\{n_i\}) P(\{n_i\},t) =$$

$$= \langle \mathcal{P} | \, \mathcal{O}(\{a_i^+ a_i\}) \underbrace{|\Phi(t)\rangle}_{\sum_{\{n_i\}} P(\{n_i\},t)\,|\{n_i\}\rangle}$$

and $\langle \mathcal{P} | \{n_i\} \rangle = 1$

$\Rightarrow$ probability conservation implies that:

$1 = \langle 1 \rangle (t) = \langle \mathcal{P} | \Phi(t) \rangle$

$\Rightarrow \; = \langle \mathcal{P} | e^{-Ht} | \Phi(0) \rangle = 1 \qquad \forall \, \Phi(0)$

$\Rightarrow \quad \langle \mathcal{P} | H = 0$

Now we assume that H is ordered normally so that:

$H = H(a_i^+, a_i)$

$\Rightarrow$ using the result of the previous exercise: $\langle \mathcal{P} | a_i^+ = \langle \mathcal{P} |$

$\Rightarrow \quad 0 = \langle \mathcal{P} | H(a_i^+, a_i) = \langle \mathcal{P} | H(1, a_i)$

$\Rightarrow \quad H(a_i^+ \to 1, a_i) = 0$

projecting on the right with suitable states

which is actually the case both for the H of pure annihilation + diffusion and for the Lotka-Volterra model...

and this allows a simplification of the expressions, e.g.,

$$n_i(t) = \langle \mathcal{P} | a_i^+ a_i | \Phi(t) \rangle = \langle \mathcal{P} | a_i | \Phi(t) \rangle \simeq \langle a_i (t) \rangle$$

Field-theoretical representation and continuum limit:

Instead of the operator formalism we will follow a well-established route in many-particle theory, in order to construct a path-integral representation of the stochastic dynamics (in the Schrödinger-like form previously discussed). This is done via

Coherent states: a coherent state $|\phi\rangle$ is defined as the eigenvector of "a" with eigenvalue ϕ.

Note that: a non-Hermitian $\Rightarrow$ $\phi \in \mathbb{C}$

$$a|\phi\rangle = \phi|\phi\rangle$$

Exercise: (a) determine $|\phi\rangle$

and normalize it in such a way that

$$\langle\phi|\phi\rangle = 1$$

$\rightsquigarrow$ Solution: $|\phi_i\rangle = \exp\{-\frac{1}{2}|\phi_i|^2 + \phi_i a_i^+\}|0\rangle$

$(a_i|\phi_i\rangle = \phi_i|\phi_i\rangle)$

(b) Show that:

$$\int \frac{d\phi_i^* d\phi_i}{\pi} |\phi_i\rangle\langle\phi_i| = \mathbb{1} \qquad \forall i$$

(c) Show that:

$$\langle\phi|\phi'\rangle = \exp\{-\tfrac{1}{2}|\phi|^2 - \tfrac{1}{2}|\phi'|^2 + \phi^*\phi'\}$$

do it!

$\Rightarrow$ overcomplete set of vectors

Now, take the evolution operator e^{-Ht}:

$$e^{-Ht} = \underset{\text{in proper sense}}{\lim_{\Delta t \to 0}} (1 - H\Delta t)^{t/\Delta t} =$$

$$= \lim_{\Delta t \to 0} \underbrace{(1 - H\Delta t)(1 - H\Delta t) \cdots (1 - H\Delta t)}_{\frac{t}{\Delta t} \text{ times}}$$

From operators to path integral:

Coherent states: $a|\phi\rangle = \phi|\phi\rangle \qquad \phi \in \mathbb{C}$

(✍) (a) determine $|\phi\rangle \;\mid\; \langle\phi|\phi\rangle = 1$

$$|\phi\rangle = \exp\left\{-\frac{1}{2}|\phi|^2 + \phi a^\dagger\right\}|0\rangle$$

(b) $$\int \frac{\mathrm{d}\phi_i \mathrm{d}\phi_i^*}{\pi}\,|\phi_i\rangle\langle\phi_i| = \mathbb{I}_i$$

(c) $$\langle\phi|\phi'\rangle = \exp\left\{-\frac{1}{2}|\phi|^2 - \frac{1}{2}|\phi'|^2 + \phi^*\phi'\right\}$$

$$\mathrm{e}^{-HT} = \underbrace{(1-H\Delta t)\ldots(1-H\Delta t)\ldots(1-H\Delta t)(1-H\Delta t)}$$

$T/\Delta t$

$T \qquad t+\Delta t \qquad t \qquad 2\Delta t \qquad \Delta t \qquad 0$

$$\int \frac{\mathrm{d}\phi_T \mathrm{d}\phi_T^*}{\pi}|\phi_T\rangle\langle\phi_T| \qquad \int \frac{\mathrm{d}\phi_0 \mathrm{d}\phi_0^*}{\pi}|\phi_0\rangle\langle\phi_0|$$

$$\int \frac{\mathrm{d}\phi_{\Delta t} \mathrm{d}\phi_{\Delta t}^*}{\pi}|\phi_{\Delta t}\rangle\langle\phi_{\Delta t}|$$

$$\int \frac{\mathrm{d}\phi_t \mathrm{d}\phi_t^*}{\pi}|\phi_t\rangle\langle\phi_t|$$

$$\int \frac{\mathrm{d}\phi_{t+\Delta t} \mathrm{d}\phi_{t+\Delta t}^*}{\pi}|\phi_{t+\Delta t}\rangle\langle\phi_{t+\Delta t}|$$

$$\langle\phi_{t+\Delta t}|(1-H\Delta t)|\phi_t\rangle$$

Now insert: (Consider a single site. for more sites it is enough to add $\prod_i \phi_t^{(i)}$)

$$(1-\Delta t\, H)$$

$$\int \frac{d^2\phi_{t+\Delta t}}{\pi} |\phi_{t+\Delta t}\rangle\langle\phi_{t+\Delta t}| \qquad \int \frac{d^2\phi_t}{\pi} |\phi_t\rangle\langle\phi_t|$$

$\Rightarrow$ we have to evaluate the matrix element:

$$\langle\phi_{t+\Delta t}|(1-\Delta t\, H)|\phi_t\rangle =$$

$$= e^{-\frac{1}{2}|\phi_{t+\Delta t}|^2-\frac{1}{2}|\phi_t|^2} \langle 0| e^{\phi^*_{t+\Delta t} a} (1-\Delta t\, H) e^{\phi_t a^+}|0\rangle$$

$$= \quad {}'' \quad \Big[\langle 0| e^{\phi^*_{t+\Delta t} a} e^{\phi_t a^+}|0\rangle + \\ - \Delta t \langle 0| e^{\phi^*_{t+\Delta t} a} H(a^+,a) e^{\phi_t a^+}|0\rangle \Big]$$

But: $e^{\lambda a} f(a^+) = f(a^+ + \lambda)\, e^{\lambda a}$

$$\Rightarrow (a) \langle 0| e^{\phi^*_{t+\Delta t} a} e^{\phi_t a^+}|0\rangle = \langle 0| e^{\phi_t(a^+ + \phi^*_{t+\Delta t})} e^{\phi^*_{t+\Delta t} a}|0\rangle$$

$$= e^{\phi_t \phi^*_{t+\Delta t}}$$

$$(b) \langle 0| e^{\phi^*_{t+\Delta t} a} H(a^+,a) e^{\phi_t a^+}|0\rangle =$$

$$= \langle 0| H(a^+ + \phi^*_{t+\Delta t}, a) e^{\phi_t(a^+ + \phi^*_{t+\Delta t})} e^{\phi^*_{t+\Delta t} a}|0\rangle$$

$$= \langle 0| e^{\phi_t a^+} H(a^+ + \phi^*_{t+\Delta t}, a+\phi_t) e^{\phi_t \phi^*_{t+\Delta t}} e^{\phi^*_{t+\Delta t} a}|0\rangle$$

$$= \langle 0| H(a^+ + \phi^*_{t+\Delta t}, a+\phi_t)|0\rangle e^{\phi_t \phi^*_{t+\Delta t}}$$

$$= H(\phi^*_{t+\Delta t}, \phi_t)\, e^{\phi_t \phi^*_{t+\Delta t}}$$

$$\langle \phi_{t+\Delta t} | \Big[1 - H(a^\dagger, a)\Delta t \Big] | \phi_t \rangle$$

$$\langle 0 | \mathrm{e}^{\phi^*_{t+\Delta t} a} \Big[1 - H(a^\dagger, a)\Delta t \Big] \mathrm{e}^{\phi_t a^\dagger} | 0 \rangle$$

$$\langle 0 | \Big[1 - H(a^\dagger + \phi^*_{t+\Delta t}, a)\Delta t \Big] \mathrm{e}^{\phi_t (a^\dagger + \phi^*_{t+\Delta t})} \mathrm{e}^{\phi^*_{t+\Delta t} a} | 0 \rangle$$

$$\langle 0 | \mathrm{e}^{\phi_t a^\dagger} \Big[1 - H(a^\dagger + \phi^*_{t+\Delta t}, a + \phi_t)\Delta t \Big] | 0 \rangle \mathrm{e}^{\phi_t \phi^*_{t+\Delta t}}$$

$$\Big[1 - H(\phi^*_{t+\Delta t}, \phi_t)\Delta t \Big] \mathrm{e}^{\phi_t \phi^*_{t+\Delta t}}$$

$$\exp\left\{ -\frac{1}{2}|\phi_t|^2 - \frac{1}{2}|\phi_{t+\Delta t}|^2 + \phi_t \phi^*_{t+\Delta t} - \Delta t H(\phi^*_{t+\Delta \phi_t}, \phi_t) \right\}$$

Accordingly:

$$\langle \phi_{t+\Delta t}|1-\Delta t\, H|\phi_t\rangle =$$

$$= e^{-\frac{1}{2}|\phi_{t+\Delta t}|^2-\frac{1}{2}|\phi_t|^2}\, e^{\phi_t\phi^*_{t+\Delta t}}\left[1-\Delta t\, H(\phi^*_{t+\Delta t},\phi_t)\right]$$

$$= \exp\Big\{\underbrace{-\tfrac{1}{2}|\phi_{t+\Delta t}|^2-\tfrac{1}{2}|\phi_t|^2+\phi_t\phi^*_{t+\Delta t}}\;-\Delta t\, H(\phi^*_{t+\Delta t},\phi_t)\Big\}+O((\Delta t)^2)$$

$$-\frac{1}{2}(\phi_{t+\Delta t}-\phi_t)\phi^*_{t+\Delta t}+\frac{1}{2}\phi_t(\phi^*_{t+\Delta t}-\phi^*_t) =$$

$$=\left(-\frac{1}{2}\partial_t\phi_t\,\phi^*_t+\frac{1}{2}\phi_t\,\partial_t\phi^*_t\right)\Delta t+O((\Delta t)^2)$$

$$= \exp\Big\{-\Delta t\left[\frac{1}{2}\phi^*_t\partial_t\phi_t-\frac{1}{2}\phi_t\partial_t\phi^*_t+H(\phi^*_t,\phi_t)\right]+O((\Delta t)^2)\Big\}$$

For $\Delta t\to 0$

$$e^{-tH}\rightsquigarrow \prod_t\int\frac{d\phi_t\, d\phi^*_t}{\pi}\exp\Big\{-\int dt\left[\phi^*_t\partial_t\phi_t+H(\phi^*_t,\phi_t)\right]\Big\}$$

in the "bulk"

$\rightsquigarrow$ we have to be careful with the boundaries...

Indeed:

$$\langle O\rangle(t) = \langle P|O(a^\dagger,a)\,\overbrace{e^{-Ht}|\Phi(0)\rangle}^{|\Phi(t)\rangle}$$

$|\phi_0\rangle\langle\phi_0|$

$|\phi_T\rangle\langle\phi_T|\ldots$

$$\langle\phi_T|e^{-Ht}|\phi_0\rangle$$

$$\Big\downarrow\ \Delta t\to 0$$

$$\exp\Big\{-\int_0^T dt\left[\frac{1}{2}\phi^*_t\partial_t\phi_t-\frac{1}{2}\phi_t\partial_t\phi^*_t+H(\phi^*_t,\phi_t)\right]\Big\}$$

we have to discuss:

final contribution:

$$\langle P | \mathcal{O}(a^+, a) | \phi_T \rangle =$$

$$= \langle 0 | e^{a} \, \mathcal{O}(a^+, a) \, e^{\phi_T a^+} | 0 \rangle \; e^{-\frac{1}{2}|\phi_T|^2}$$

$$= \langle 0 | \mathcal{O}(a^+ + 1, a) \, e^{\phi_T (a^+ + 1)} \underbrace{e^{a} | 0 \rangle}_{|0\rangle} \; e^{-\frac{1}{2}|\phi_T|^2}$$

$$= \langle 0 | e^{\phi_T a^+} \mathcal{O}(a^+ + 1, a + \phi_T) \, e^{\phi_T} | 0 \rangle \; e^{-\frac{1}{2}|\phi_T|^2}$$

$$= \mathcal{O}(1, \phi_T) \; e^{\phi_T - \frac{1}{2}|\phi_T|^2}$$

initial contribution:

$$\langle \phi_0 | \Phi(0) \rangle = \qquad \text{where } |\Phi(0)\rangle = \sum_n P_0(n) \underset{(a^+)^n|0\rangle}{\underset{\shortparallel}{|n\rangle}}$$

$$= e^{-\frac{1}{2}|\phi_0|^2} \langle 0 | e^{\phi_0^* a} \sum_n P_0(n) \, (a^+)^n | 0 \rangle$$

$$= e^{-\frac{1}{2}|\phi_0|^2} \sum_n P_0(n) \, \langle 0 | (a^+ + \phi_0^*)^n \underbrace{e^{\phi_0^* a} | 0 \rangle}_{|0\rangle}$$

$$= e^{-\frac{1}{2}|\phi_0|^2} \sum_{n=0}^{\infty} P_0(n) (\phi_0^*)^n$$

Combining all the terms:

$$\langle \mathcal{O}(T) \rangle = \langle P | \mathcal{O}(a^+, a) \, e^{-Ht} | \Phi(0) \rangle =$$

$$= \int \mathcal{D}\phi \mathcal{D}\phi^* \; \mathcal{O}(1, \phi_T) \sum_{n=0}^{\infty} P_0(n) (\phi_0^*)^n \; \times$$

$$\times \exp\left\{ -\frac{1}{2}|\phi_0|^2 + \phi_T - \frac{1}{2}|\phi_T|^2 - \int_0^T dt \left[\frac{1}{2} \phi_t^* \partial_t \phi_t - \frac{1}{2} \phi_t \partial_t \phi_t^* + H[\phi_t^*, \phi_t] \right] \right\}$$

$$\partial_t(\phi_t \phi_t^*) - \phi_t^* \partial_t \phi_t$$

yields

$$\frac{1}{2}|\phi_T|^2 - \frac{1}{2}|\phi_0|^2$$

$$\{\ldots\} = \overbrace{-|\phi_0|^2 + \phi_T}\; - \int_0^T dt \left[\phi_t^* \partial_t \phi_t + H(\phi_t^*, \phi_t) \right]$$

Write now:

$$e^{-A_0(\lambda)} \equiv \sum_{n=0}^{\infty} P_0(n)\, \lambda^n \qquad \left[\text{because of normalization } A_0(1) = 0\right]$$

$$\Rightarrow \quad \langle O \rangle(T) = \int \mathcal{D}\phi\, \mathcal{D}\phi^* \; O(1, \phi_T) \; \times$$

$$\times \exp\left\{ - A_T[\phi, \phi^*] - A_0(\phi_0^*) - |\phi_0|^2 \right\}$$

where: $\quad A_T[\phi,\phi^*] = -\phi_T + \int_0^T dt \; \left[\phi_t^* \partial_t \phi_t + H(\phi_t^*, \phi_t)\right]$

If the initial distribution is Poissonian:

$$P_0(n) = \frac{\bar{n}_0^{\,n}}{n!} e^{-\bar{n}_0}$$

$$\Rightarrow \quad e^{-A_0(\lambda)} = \sum_{n=0}^{\infty} \frac{\bar{n}_0^{\,n}}{n!} e^{-\bar{n}_0} \lambda^n = e^{-\bar{n}_0} e^{\lambda \bar{n}_0}$$

$$\Rightarrow \quad A_0(\lambda) = \bar{n}_0 - \lambda \bar{n}_0$$

Note that, as a consequence of probability conservation $H(1, \phi_t) = 0$ $\rightsquigarrow$ it is convenient to introduce the shifted field: $\phi^* = 1 + \tilde{\phi}$

(note that we are integrating independently on $\mathcal{D}\phi\, \mathcal{D}\phi^* \rightsquigarrow \mathcal{D}\phi\, \mathcal{D}\tilde{\phi}$)

$$\Rightarrow \quad A_T[\phi, \tilde{\phi}] = -\phi_T + \int_0^T dt \; \left\{(1+\tilde{\phi}_t)\partial_t \phi_t + H(1+\tilde{\phi}_t, \phi_t)\right\}$$

$$= -\phi_0 + \int_0^T dt \left\{\tilde{\phi}_t \partial_t \phi_t + H(1+\tilde{\phi}_t, \phi_t)\right\}$$

$$A_0(\tilde{\phi}_0) = \bar{n}_0 - (1+\tilde{\phi}_0)\bar{n}_0 = -\tilde{\phi}_0 \bar{n}_0$$

Note that now the "action" takes the form of a bulk action + initial term (ϕ_0, $\tilde{\phi}_0 \bar{n}_0$) which, in the long-time limit should not affect the "bulk" behavior of the system