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Stochastic processes and applications
Lectures 8 & 9

Andrea Gambassi
SISSA & INFN

Obs: (a) So far we have inserted the observable $\mathcal{O}$ at the "end" of the evolution:

$$\langle \mathcal{O} \rangle (T) = \langle \mathcal{P} | \mathcal{O}(\{a_i^\dagger a_i\}) | \Phi(T) \rangle = \langle \mathcal{P} | \mathcal{O}(\{a_i^\dagger a_i\}) e^{-HT} | \Phi(0) \rangle$$

but one might be actually interested in -two- (or more!) time correlation functions in which the observable $\mathcal{O}$ is introduced at different times ... e.g. $n_i \equiv a_i^\dagger a_i$

$$\langle n_i(T) n_j(T' < T) \rangle = \langle \mathcal{P} | n_i \, e^{-H(T-T')} n_j \, e^{-HT'} | \Phi(0) \rangle$$

or, in general:

$$\langle \mathcal{O}_1(\{a_i^\dagger a_i\})(T) \, \mathcal{O}_2(\{a_i^\dagger a_i\})(T') \rangle = \langle \mathcal{P} | \mathcal{O}_1(\{a_i^\dagger a_i\}) e^{-H(T-T')} \mathcal{O}_2(\{a_i^\dagger a_i\}) e^{-HT'} | \Phi(0) \rangle$$

Exercise: write down the corresponding expression as a path integral ... (treating properly the boundary terms)

$$\mathcal{O}_2(\{a_i^\dagger a_i\}) \to \mathcal{O}_2(\{\phi_{T'}^* \phi_{T'}\})$$

(b) We have considered only one site: the construction can be done at each site independently and leads to:

$$\prod_t{}' \int \frac{d\phi_t \, d\phi_t^*}{\pi} \rightsquigarrow \prod_{t,x} \int \frac{d\phi_{t,x} \, d\phi_{t,x}^*}{\pi}$$

and the spatial variable then enters in the action A_T and A_0 with the sum over all the sites of the lattice...

(c) If one is interested in the collective properties at large length scales, one can perform a continuum limit of the actions A_T, A_0, by letting the lattice spacing $a \to 0$, ending up in the typical field-theoretical description in terms of quantities $\phi(x,t)$ $\phi^*(x,t)$ (or, alternatively, $\tilde{\phi}(x,t)$)

take, e.g., the case of the reaction-diffusion system with binary annihilation reaction $A+A \xrightarrow{\lambda} \emptyset$

$$H = H_D + H_R$$

(exercise!)

where
$$H_D = \sum_{\langle ij \rangle} D(a_j^+ - a_i^+)(a_j - a_i)$$
$$H_R = -\lambda \sum_i (a_i^2 - a_i^{+2} a_i^2)$$

$$H_D(\phi_i, \phi_i^*) = \sum_{\langle ij \rangle} D(\phi_{j,t}^* - \phi_{i,t}^*)(\phi_{jt} - \phi_{it})$$

the hopping term turns naturally into diffusion

!!
$$\rightsquigarrow D a^2 \int d^dx \, (\nabla \phi^*(x,t))(\nabla \phi(x,t))$$

$$H_R(\phi_i, \phi_i^*) = -\lambda \sum_i (\phi_i^2 - \phi_i^{*2} \phi_i^2)$$
$$\rightsquigarrow -\lambda \int d^dx \, [\phi^2(x,t) - \phi^{*2}(x,t)\phi^2(x,t)]$$

$\rightsquigarrow$
$$A_T[\phi, \tilde\phi] \underset{(\text{bulk})}{=} \int_0^T dt \int d^dx \Big\{ \tilde\phi \, \partial_t \phi + \underbrace{Da^2}_{\bar D} (\nabla\tilde\phi)(\nabla\phi) - \lambda [\phi^2 - (1+\tilde\phi)^2 \phi^2] \Big\}$$

where $\tilde\phi = \tilde\phi(x,t)$; $\phi = \phi(x,t)$

$$= \int_0^T dt \int d^dx \, \{ \tilde\phi(\partial_t - \bar D \nabla^2)\phi - \lambda\phi^2[1 - (1+\tilde\phi)^2] \}$$

Summing up: From the ME we have constructed a microscopic field-theory for stochastic reaction-diffusion processes, without invoking any assumptions on the form of correlations or of the noise (internal!) $\rightsquigarrow$ to describe the stoch dynamics we need two fields: $\phi, \tilde\phi$

$\Rightarrow$ this FT can be used for further, systematic coarse-graining, e.g., to construct a perturbative expansion, which then can be combined with RG methods and analyses in order to extract the scaling behaviour

Summarizing:

See A. Kamenev: "Field Theory & Non-equilib. Systems" Cambridge, 2011. ch. 4.

(B) $A \xrightarrow{\sigma} 2A$

(D) $A \xrightarrow{\mu} \phi$

(C) $2A \xrightarrow{\lambda} A$

$$\Rightarrow \quad H_R(a^+, a) = (1-a^+)[\sigma a^+ a - \mu a - \lambda a^+ a^2]$$

$$\Rightarrow \quad A_T[\phi, \phi^*] = \int dt\, d^d r \left\{ \phi^*(\partial_t - D\nabla^2)\phi + (1-\phi^*)[\sigma \phi\phi^* - \mu\phi - \lambda \phi^* \phi^2] \right\}$$

$$\phi^* \partial_t \phi + H(\phi^*, \phi)$$

$$\Big\downarrow \ \phi^* = 1 + \tilde{\phi}$$

$$\tilde{\phi}\, \partial \phi + H(1+\tilde{\phi}, \phi) \quad \text{up to boundary terms..}$$

Consider now, the 0-dim. case

$$\phi(x,t) \to q(t)$$
$$\tilde{\phi}(x,t) \to p(t)$$

$$H(1+\tilde{\phi}, \phi) \to -\mathcal{H}(p,q) \equiv H(1+p, q)$$

$\Rightarrow$ note that: $\quad A_T \to S_t = \int^{t} dt' \left[p\dot{q} - \mathcal{H}(p,q) \right]$ | Obs: $\mathcal{H}(0,q) = 0$. (cons. of probabil.)

$\Rightarrow$ action of an Hamiltonian system!

$$P_{MM}(q,t \mid q_0, t_0) = \int_{\substack{q(t_0)=q_0 \\ q(t)=q}} \mathcal{D}q\, \mathcal{D}p \; e^{-S_t[p,q]}$$

For studying rare events or as a first approximation, the previous functional integral can be evaluated in the stationary path approximation; the stationary path is determined by requiring: (WKB in QM)

$$\frac{\delta S}{\delta p(t)} = 0 \quad \Rightarrow \quad \dot{q} - \partial_p \mathcal{H}(p,q) = 0$$

$$\frac{\delta S}{\delta q(t)} = 0 \quad \Rightarrow \quad \dot{p} + \partial_q \mathcal{H}(p,q) = 0 \quad \text{(up to bound. terms)}$$

$\Rightarrow$ Hamiltonian eqs. of motion!

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Note that: $\mathcal{H}$ is independent of t on the optimal path!
(the "energy" is conserved by the dynamics)

In fact: $\frac{d}{dt}\mathcal{H}(p(t),q(t)) = \partial_p \mathcal{H} \underbrace{\dot{p}}_{-\partial_q \mathcal{H}} + \partial_q \mathcal{H} \underbrace{\dot{q}}_{\partial_p \mathcal{H}} = 0$

In the stationary state: $\dot{q} \to 0$
$p\dot{q} \to 0$
$\Rightarrow$ if $\mathcal{H}(p(t),q(t)) \not\to 0$ then the probability of the path is exponentially small in time, as:

$S_t \propto t$

$\Rightarrow$ for $t \to +\infty$ the optimal trajectory $(q_0,t_0) \to (q,t)$ is always on the curve $\mathcal{H}=0$

How can we visualize the optimal trajectories? In the "phase space" (p,q)

$\rightsquigarrow$ given that the long-time behavior is described by the curves at $\mathcal{H}=0$, consider first this case;

(i) Because of cons. of probability: $\mathcal{H}(0,q)=0$

(ii) if an absorbing state is present $\Rightarrow$ $W(n\to ?) \propto n$

$\Rightarrow \mathcal{H}(p,0)=0$

Accordingly there are at least 2 lines @ $\mathcal{H}=0$: $p=0$ & $q=0$.

For $p=0$ the corresp. stationary path eq. is:

$\dot{q} = \partial_p \mathcal{H}(p,q)\Big|_{p=0}$ and corresponds (ex!) to the <u>rate</u> equation

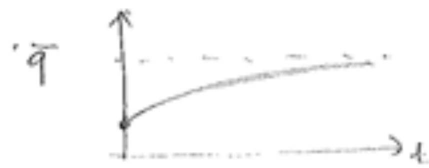
$\rightsquigarrow$ mean-field solution which neglects fluctuations & discreteness of q.

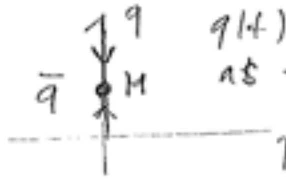
Consider, e.g., the model recalled above: the rate equation is $\langle n \rangle = q$

$$\dot q = (\sigma - \mu) q - \lambda q^2$$

$\Rightarrow$ for $\sigma > \mu$ there is a stable fixed point at

$$\bar q = \frac{\sigma - \mu}{\lambda}$$

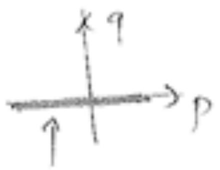




$q(t) \to \bar q$ as t grows $(p = 0)$

while for $\sigma < \mu$, $q(t \to 0)$ exponentially fast.

Note, however, that even for $\sigma > \mu$ the only possible asymptotic state is the absorbing one, due to the fact that even if $\langle n \rangle_t \simeq \bar q$ initially, a fluctuation will always drive the system into the <u>absorbing state</u> $n = 0$ $\Rightarrow$ no actual phase transition in 0d for $\lambda \neq 0$ (for $\lambda = 0$ this is possible because $n(t) \nearrow +\infty$ at $t \nearrow +\infty$) [$\Rightarrow$ the point M is metastable...]



$\dot p = -\partial_q \mathcal{H}|_{q=0}$

in higher-dimensions the phase transition active/absorbing exists also for $\lambda \neq 0$ because diffusion can reintroduce fluctuations in the sites where the particles were previously extinct.

$\Rightarrow$ the optimal path is eventually the one with $q = 0$

For the specific model above, one finds:

$$\mathcal{H}(p, q) = p \left[\sigma(1+p) - \mu - \lambda(1+p) q \right] q$$

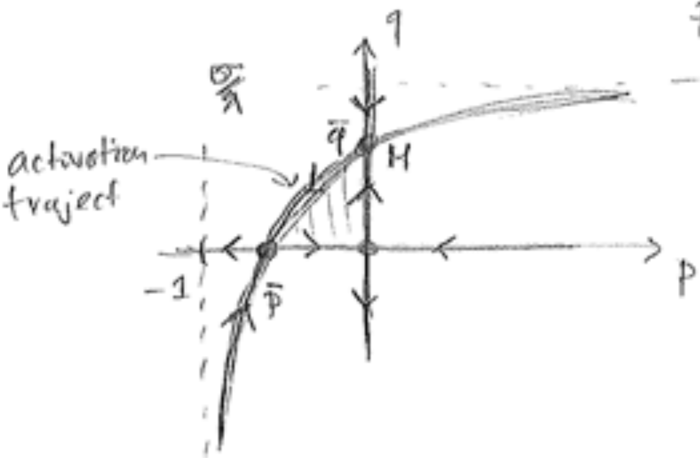
and therefore, in addition to $p = 0$ & $q = 0$ we have also www.ictp.it

the zero-energy line ("activation trajectory")

$$[\ldots] = 0 \Rightarrow \quad \sigma(1+p) - \mu - \lambda(1+p) q_a(p) = 0$$

$$\Rightarrow \quad q_a(p) = \frac{\sigma}{\lambda} - \frac{\mu}{\lambda}\frac{1}{1+p}$$

with $q_a(p=0) = \bar{q}$ while $q_a(p) = 0$ for $p = \bar{p} = \frac{\mu}{\sigma} - 1 < 0$ for $\sigma > \mu$



$\Rightarrow$ the vector field is as in the figure.

<u>Q</u>: if we start close to the metastable state, how long will it take before extinction?

Rate of extinction: $\propto e^{-S_{ex}}$

where $S_{ex} = \int dt \, [\, \underbrace{p}_{<0}\, \underbrace{\dot{q}}_{<0} - \underbrace{\mathcal{H}(p,q)}_{=0} \,]\Big|_{p = p_a(q)}$ where: $p_a(q) = \frac{\mu}{\sigma - \lambda q} - 1$

$$= \int_{\bar{q}}^{0} p_a \, dq \quad \Rightarrow \text{ area of the triangle!}$$

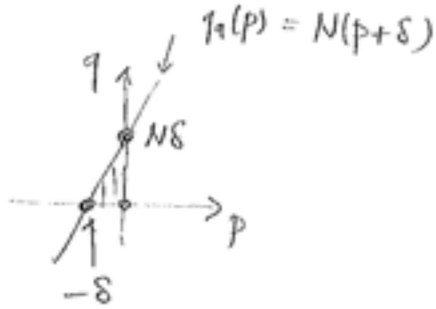
Note that if $|\bar{p}| \ll 1 \Rightarrow$ along the activation trajectory $p \simeq 0$ and one can neglect the curvature of the curve:

$$\Rightarrow \quad \mathcal{H}(p,q) = p[\, \sigma - \mu + \sigma p - \lambda \underbrace{(1+p) q}_{\simeq q \text{ (neglect curvature)}} \,]\, q$$

Accordingly:

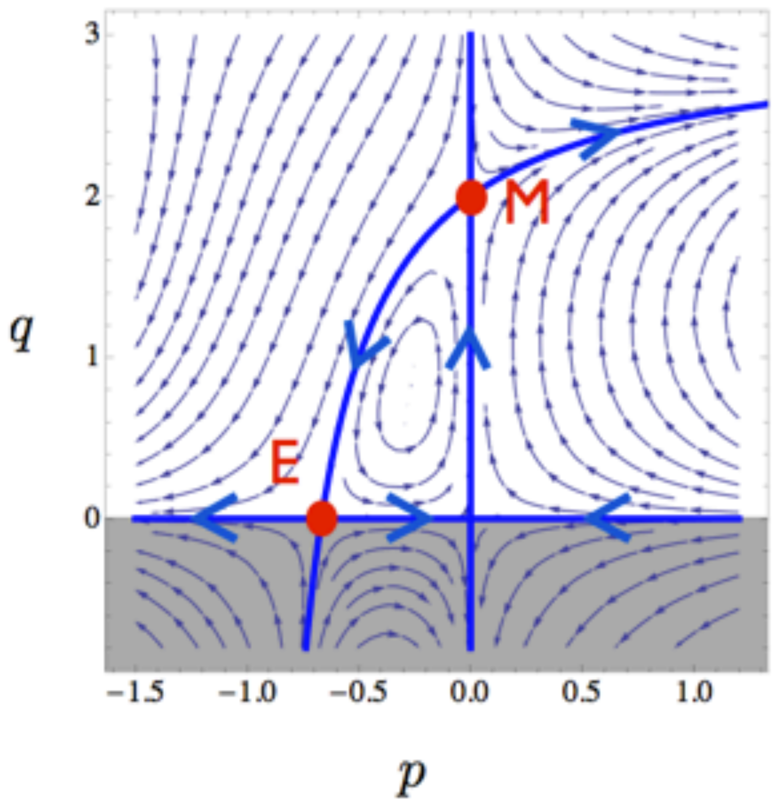
$$\mathcal{H}(p,q) = p\,[\,\sigma - \mu + \sigma p - \lambda q\,]\,q$$

$$= \sigma p \Big[\underbrace{\frac{\sigma-\mu}{\sigma}}_{\equiv \delta} + p - \frac{q}{\underbrace{\sigma/\lambda}_{\equiv N}} \Big]\, q$$

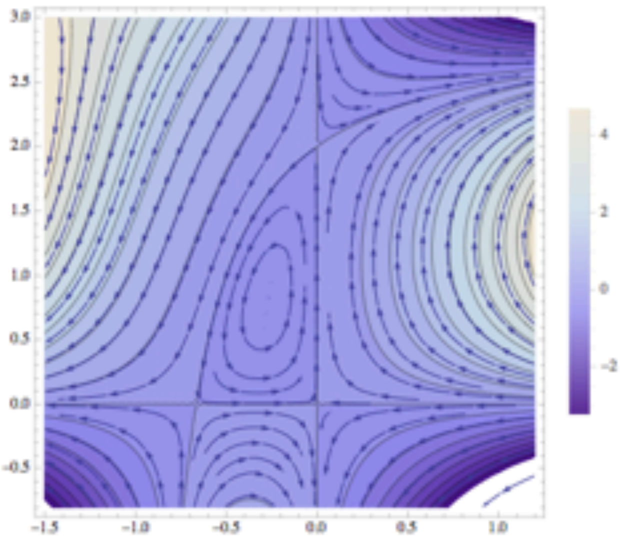


$$\Rightarrow \quad S_{ex} = \frac{1}{2}\,\delta N^2$$

$\rightsquigarrow$ define $\quad \tau_{ex} \doteq \sqrt{\frac{2\pi}{N}}\,\frac{1}{\delta^2}\, e^{N\delta^2/2}$ $\qquad$ (see Kamenev sec. 4.10)



$\mu = 1$
$\sigma = 3$
$\lambda = 1$



$$\mathcal{H}(p,q) = p\,[\sigma(1+p) - \mu - \lambda(1+p)q]\,q$$

Example of what one can get with this approach:

Go back to the population dynamics described at p. 25.

(a) branching $A \xrightarrow{\sigma} 2A$

(b) decay $A \xrightarrow{\mu} \emptyset$

(c) lethal competition (coagulation) $2A \xrightarrow{\lambda} A$

we said that this model (we solved the ME exactly) has a non-equilibrium phase transition between the absorbing phase for $\mu > 0$ and the active phase for $\mu < 0$
↳ see APPLICATIONS!

Exercise: Determine the quasi-Hamiltonian (Liouville operator) of this model:

reaction part for a single site: $H_R(a, a^\dagger) = (1-a^\dagger)[\sigma a^\dagger a - \mu a - \lambda a^\dagger a^2]$

$$\rightsquigarrow H_R(\phi, \phi^*) = (1-\phi^*)[\sigma\phi\phi^* - \mu\phi - \lambda\phi^*\phi^2]$$

$\rightsquigarrow$ the field-theoretical action reads: (bulk part)

$$A_T[\phi, \phi^*] = \int_0^T dt\, d^dx \left\{ \phi^*(\partial_t - D\nabla^2)\phi + (1-\phi^*)[\sigma\phi\phi^* - \mu\phi - \lambda\phi^*\phi^2] \right\}$$

Now we can proceed via the usual methods for a FT...
we have only to take into account that the theory contains two independent fields.

First of all determine the extremal solutions (MF!) (neglect boundaries.)

$$\frac{\delta A_T}{\delta\phi} = 0 \Rightarrow (-\partial_t - D\nabla^2)\phi^* + (1-\phi^*)[\sigma\phi^* - \mu - 2\lambda\phi^*\phi] = 0$$

$$\frac{\delta A_T}{\delta\phi^*} = 0 \Rightarrow (\partial_t - D\nabla^2)\phi + [-\sigma\phi\phi^* + \mu\phi + \lambda\phi^*\phi^2] + (1-\phi^*)[\sigma\phi - \lambda\phi^2] = 0$$

If you look for a homogeneous solution in space we can forget the ∇^2 term; then the first equation is trivially solved by $\phi^* = \overline{\phi}^* = 1$

The second equation becomes, for $\phi = \bar{\Phi}$

$$\partial_t \bar{\Phi} = (\sigma - \mu)\bar{\Phi} - \lambda \bar{\Phi}^2$$

(Note that neglecting the fluctuations around $\bar{\Phi}$ ⇒
$\langle n \rangle = \langle \phi \rangle = \bar{\Phi}$)

⇝ the rate equation, as expected, emerges as the mean-field approximation of the complete theory
(one might consider also diffusion ...)

→ (a) for $\lambda = 0$

$\bar{\Phi}(t) \simeq \exp\{-(\mu - \sigma)t\}$ i.e. → 0 in the absorbing phase
→ ∞ in the active phase

(b) for $\lambda \neq 0$, instead one can explicitly solve the equation (exercise!) finding that:
(note that the quadratic term prevents the density from "exploding")

$\mu > \sigma$ (absorbing): $\bar{\Phi}(t \to \infty) \sim e^{-(\mu - \sigma)t}$

$\mu < \sigma$ (active) : $\bar{\Phi}(t \to \infty) = \bar{\Phi}_s = \dfrac{\sigma - \mu}{\lambda}$ (saturation)

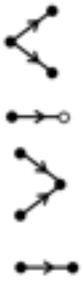
$\mu = \sigma$ ("critical" point) : $\bar{\Phi}(t \to \infty) \sim \dfrac{1}{\lambda t}$

What about fluctuations?

As in the case of II order phase transitions, fluctuations are expected to:

(a) modify the phase diagram (eg the actual location of the critical point ...)
⇝ but this is a non-universal aspect in the sense that it depends on the μ-scopic details of the system (e.g. the way the field theory is regularized!)

$$\mathcal{A}_T[\phi,\phi^*] \equiv \int_0^T \mathrm{d}t \int \mathrm{d}^d x \left\{ \phi^*(\partial_t - \bar{D}\nabla^2)\phi + H_R(\phi^*,\phi) \right\}$$



(B) $A \xrightarrow{\sigma} A + A$

(D) $A \xrightarrow{\mu} \emptyset$

(C) $A + A \xrightarrow{\lambda} A$

(diff) D

(✍) $H_R(\phi^*,\phi) = (1-\phi^*)[\sigma\phi\phi^* - \mu\phi - \lambda\phi^*\phi^2]$

$$\begin{cases} \left.\dfrac{\delta \mathcal{A}[\phi,\phi^*]}{\delta\phi(x,t)}\right|_{\mathrm{MF}} = 0 & \Longrightarrow \bar{\phi}^* \equiv 1 \\ & \mathrm{MF}: \begin{cases} \phi = \bar{\phi}(t) \\ \phi^* = \bar{\phi}^*(t) \end{cases} \\ \left.\dfrac{\delta \mathcal{A}[\phi,\phi^*]}{\delta\phi^*(x,t)}\right|_{\mathrm{MF}} = 0 & \Longrightarrow \partial_t\bar{\phi} = (\sigma-\mu)\bar{\phi} - \lambda\bar{\phi}^2 \end{cases}$$

(b) Modify the power-law behaviour predicted by mean-field! (but now, with universal aspects, i.e., indep. of many μ-scopic details of the theory! ⇝ RG etc..)

But: fluctuations are relevant only in low spatial dimension $d < d_c$

$\uparrow$ upper critical dimensionality of the model.

[In equilibrium: Ginzburg criterion..]

For convenience we rescale the fields as follow:

$$\phi = \sqrt{\frac{\sigma}{\lambda}}\,\varphi$$

$$\phi^* = \underset{\substack{\uparrow \\ \text{MF sol}}}{1} + \sqrt{\frac{\lambda}{\sigma}}\,\tilde{\varphi}$$

good combinatoric factor for quadratic terms.

$$\Rightarrow \underset{\substack{\uparrow \\ \text{apart from boundary terms}}}{A_T =} \int dt\, d^d x \left\{ \tilde{\varphi}\,(\partial_t - D\nabla^2 + r)\varphi - \frac{u}{2}(\tilde{\varphi} - \varphi)\,\tilde{\varphi}\varphi + \lambda\,\tilde{\varphi}^2\varphi^2 \right\}$$

where: $r \equiv \mu - \sigma$ MF distance from the critical point

$u \equiv 2\sqrt{\lambda\sigma}$: effective coupling of the theory...

Consider the "action"

$$A = \int dt\, d^dx \left\{ \tilde{\varphi}(\partial_t - D\nabla^2 + \tau)\varphi - \frac{\mu}{2}(\tilde{\varphi} - \varphi)\tilde{\varphi}\varphi + \lambda \tilde{\varphi}^2 \varphi^2 \right\}$$

add a scale τ

Study the effect of short-wavelength fluctuations on those occurring at larger scales!

We would like to implement Wilson's RG scheme, which consists of 3 steps:

(i) The theory is originally defined with a large-momentum cutoff $\Lambda \sim \frac{1}{a}$ (a = lattice spacing)

In the momentum space we can separate each field ϕ into large- / small- momentum components $\phi_>(k)$, $\rho\Lambda < |k| < \Lambda$; $\phi_<(k)$, $|k| < \rho\Lambda$ for $0 < \rho < 1$

$\Rightarrow \phi(k) = \phi_>(k) + \phi_<(k)$

(ii) Integrate out the large-momentum component $\phi_>$ and determine the effective action for the small-momentum components $\phi_<$ $\Rightarrow$ now the resulting action has a cutoff $\rho\Lambda$, i.e. a larger lattice spacing $\frac{a}{\rho} > a$

$\rightsquigarrow$ in order to iterate the procedure of integrating out modes, it is therefore convenient to proceed to:

(iii) rescaling of coordinates, and fields:

introduce:

$$x' = \rho x \qquad (k' = \rho^{-1} k)$$

$$t' = \rho^2 t \qquad (\omega' = \rho^{-2}\omega)$$

$$\phi(x,t) = \rho^{\frac{d+\eta}{2}} \phi'(x', t')$$

Standard case: $\frac{d-2+\eta}{2}$

then eventually rename the variables $(\phi', x', t') \to (\phi, x, t)$ and go back to point (ii).

$\rightsquigarrow$ Note that after each single step, the values of the effective parameters change $\rightsquigarrow$ RG "flow".

Consider the Gaussian theory: $u = \lambda = 0$

$\Rightarrow$ $\phi_<$ and $\phi_>$ modes are decoupled and therefore the integration over $\phi_>$ gives just a constant and does not modify the action for the slow modes...

One ends up with:

$$\int_{(\rho\Lambda)^{-1}} dt\, d^dx \quad \tilde{\varphi}(\tau\partial_t - D\nabla^2 + r)\varphi$$

(iii)

$$= \int_{\Lambda^{-1}} dt'\, \rho^{-z}\, d^dx'\, \rho^{-d}\, \rho^{\frac{d+\eta}{2}\times 2} \times$$

$$\times\, \tilde{\varphi}'(\tau\rho^{z}\partial_{t'} - D\rho^2\nabla'^2 + r)\varphi'$$

rename

$$= \int_{\Lambda^{-1}} dt\, d^dx \quad \rho^{\eta}\, \tilde{\varphi}(\tau\partial_t - D\rho^{2-z}\nabla^2 + r\rho^{-z})\varphi \Rightarrow \begin{cases} \tau' = \rho^{\eta}\tau \\ D' = \rho^{2-z+\eta} D \\ r' = \rho^{\eta-z} r \end{cases}$$

Note that: in this theory $\tilde{\varphi}, \varphi$ play the same role in the interaction, so one heuristically expect them to have the same scaling behav. (this can be made more precise...)

$\Rightarrow$ in order to be a fixed point: $\eta = 0$, $z = 2$

and $r = 0$; otherwise: $r' = r\rho^{-z} > r$

$\Rightarrow$ as we iterate the steps $r \to +\infty$.

Consider, now, the behavior of the non-quadratic terms at the Gaussian fixed point: (i.e., assume u to be small and perturbative)

$$u\int_{(\rho\Lambda)^{-1}} dt\, d^dx \quad \tilde{\varphi}^2\varphi \underset{\substack{\eta=0\\ z=2}}{=} u\int_{\Lambda^{-1}} dt'\, \rho^{-2}\, d^dx'\, \rho^{-d}\, \rho^{\frac{d}{2}\times 3}\, \tilde{\varphi}'^2\varphi'$$

$$\overset{!}{=} \underbrace{u\rho^{\frac{d}{2}-2}}_{\text{rename}} \int_{\Lambda^{-1}} dt\, d^dx \quad \tilde{\varphi}^2\varphi$$

In general: (α)
$u' = u\rho^{\frac{d}{2}-z+\frac{3}{2}\eta}$

$u' \equiv u\rho^{\frac{d}{2}-2} \to u'' = u'\rho^{\frac{d}{2}-2} = u(\rho^{\frac{d}{2}-2})^2 \ldots$ and so on...

$\Rightarrow u \to +\infty$ if $\frac{d}{2} - 2 < 0$ $(d < 4)$

$\Rightarrow u \to 0$ if $\frac{d}{2} - 2 > 0$ $(d > 4)$ Gaussian FP is ok

in this case the Gaussian fixed point is no longer stable and one has to consider the effects of u on it; $d = 4$ is the upper critical dimens. of the model!

Wilson's RG:

(i) separate components

$$\phi(k) = \begin{cases} \phi_>(k), & \rho\Lambda < |k| < \Lambda \\ \phi_<(k), & |k| < \rho\Lambda \end{cases}$$

$$(\tau,\, D,\, r,\, \lambda,\, u; \Lambda)$$

(ii) integrate out $\phi_>(k)$ ⇒ effective theory for $\phi_<(k)$

$$(\tau',\, D',\, r',\, \lambda',\, u'; \rho\Lambda)$$

(iii) rescaling of coordinates & fields:

$$\begin{cases} x' = \rho\, x \\ t' = \rho^z t \\ \phi'(x', t') = \rho^{-(d+\eta)/2}\, \phi_<(x, t) \end{cases} \qquad (\tau'', D'', r'', \lambda'', u''; \Lambda)$$

$(x',\, t',\, \phi') \to (x,\, t,\, \phi)$ ⇒ back to (i)

same for $\tilde{\phi}$

Gaussian fixed point:

$$\int \mathrm{d}t \int_{\Lambda^{-1}} \mathrm{d}^d x\, \tilde{\varphi}(\tau\partial_t - D\nabla^2 + r)\varphi$$

$$\overset{\text{(ii)}}{\mapsto} \int \mathrm{d}t \int_{(\rho\Lambda)^{-1}} \mathrm{d}^d x\, \tilde{\varphi}_<(\underset{\tau'}{\underset{\Downarrow}{\tau}}\, \partial_t - \underset{D'}{\underset{\Downarrow}{D}}\nabla^2 + \underset{r'}{\underset{\Downarrow}{r}})\varphi_<$$

$$\overset{\text{(iii)}}{=} \int \mathrm{d}t'\, \rho^{-z} \int_{\Lambda^{-1}} \mathrm{d}^d x'\, \rho^{-d}\, \rho^{\frac{d+\eta}{2}\times 2} \tilde{\varphi}'(\tau\rho^z\partial_{t'} - D\rho^2\nabla'^2 + r)\varphi'$$

$$= \int \mathrm{d}t \int_{\Lambda^{-1}} \mathrm{d}^d x\, \tilde{\varphi}(\underset{\tau''}{\underset{\Downarrow}{\tau\rho^\eta}}\partial_t - \underset{D''}{\underset{\Downarrow}{D\rho^{2+\eta-z}}}\nabla^2 + \underset{r''}{\underset{\Downarrow}{r\rho^{\eta-z}}})\varphi$$

$$u'' = \rho^{d/2-2} u \qquad\qquad \lambda'' = \rho^{d-2}\lambda$$

$d/2-z+3\eta/2$

Consider now:

$$\lambda \int_{(\rho\Lambda)^{-1}} dt\, d^dx\, \tilde{\varphi}^2 \varphi^2 = \lambda \int_{\Lambda^{-1}} dt'\, \rho^{-2}\, d^dx'\, \rho^{-d}\, \rho^{\frac{d}{2}\times 4}\, \tilde{\varphi}'^2 \varphi'^2$$

rename

$$= \underbrace{\lambda \rho^{d-2}}_{} \int_{\Lambda^{-1}} dt\, d^dx\, \tilde{\varphi}^2 \varphi^2$$

$$\lambda' = \lambda \rho^{d-2} < \lambda \quad \text{for } d>2$$

$$\Rightarrow \lambda'' = \lambda' \rho^{d-2} = \lambda (\rho^{d-2})^2 \quad \text{and so on..}$$

$$\Rightarrow \lambda \downarrow 0 \quad \text{for } d>2$$

$\Rightarrow$ if we work for $d>2$, this coupling can be neglected! (Care! this analysis is actually valid around the Gaussian fixed point, i.e., for $d>4$, strictly speaking... .. however this can be verified a posteriori)

$\Rightarrow$ the leading scaling behaviour can be obtained by setting $\lambda=0$

$\rightarrow$ The theory with $\lambda=0$ is well-known is high-energy physics and is called Reggeon field theory.. it has a peculiar symmetry called Rapidity Reversal: $\varphi(x,t) \Leftrightarrow -\tilde{\varphi}(x,-t)$

$\rightarrow$ This FT characterizes a universality class to which also directed percolation belongs..
There is even the Janssen - Grassberger conjecture:
"any continuous non-eq. phase transition from an active to an absorbing state in a system governed by a Markovian stochastic dynamics that is decoupled from 'slow' variables and in the absence of additional symmetries and randomness should belong to this universality class"

$$\mathcal{A} = \int \mathrm{d}t\mathrm{d}^d x \left\{ \tilde{\varphi}(\partial_t - \bar{D}\nabla^2 + r)\varphi - \frac{u}{2}(\tilde{\varphi} - \varphi)\varphi\tilde{\varphi} \right\}$$

$$\varphi(x,t) \longleftrightarrow -\tilde{\varphi}(x,-t)$$

any (Janssen-Grassberger, '81-'82)

(i) continuous non-eq PT active ➔ 1 abs
(ii) Markovian dyn
(iii) no slow variables
(iv) no symmetries
(v) no randmness

$\Rightarrow$ DP/CP univ. class

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II

Summing up the FT we have to discuss is:

$$A_T = \int dt\, d^dx \left\{ \tilde\varphi (\partial_t - D\nabla^2 + r)\varphi - \frac{u}{2}(\tilde\varphi^2 - \varphi)\tilde\varphi\varphi \right\}$$

r — good combinatorial factor...

we have two fields: $(\varphi, \tilde\varphi)$ instead of the usual single field.

Consider the Gaussian theory $u=0$ (around which we are going to consider perturbations)

$\rightsquigarrow$ from it we read the propagator of the theory!

apart from boundary terms (irrelevant now, not in general!)

$$\tilde\varphi(\partial_t - D\nabla^2 + r)\varphi = \frac{1}{2}\left\{ \tilde\varphi(\partial_t - D\nabla^2 + r)\varphi + \varphi(-\partial_t - D\nabla^2 + r)\tilde\varphi \right\}$$

$$= \frac{1}{2}(\tilde\varphi\ \varphi)\begin{pmatrix} 0 & \partial_t - D\nabla^2 + r \\ -\partial_t - D\nabla^2 + r & 0 \end{pmatrix}\begin{pmatrix}\tilde\varphi \\ \varphi\end{pmatrix}$$

$\rightsquigarrow$ in Fourier transform (ω, q)

$$(-q,-\omega)\quad G^{-1} \equiv \begin{pmatrix} 0 & +i\omega + Dq^2 + r \\ -i\omega + Dq^2 + r & 0 \end{pmatrix}\quad (q,\omega)$$

$$\varphi(t,x) = \int \left(\frac{dq}{2\pi}\right)^d \left(\frac{d\omega}{2\pi}\right) e^{i(xq - \omega t)}\hat\varphi(q,\omega)$$

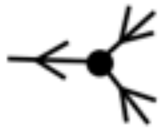
$$\left\langle \begin{pmatrix}\tilde\varphi \\ \varphi\end{pmatrix}_{(+q,+\omega)} (\tilde\varphi\ \varphi)_{(q',\omega')} \right\rangle = \underbrace{\ \ }_{(2\pi)^d\delta(q+q')\,2\pi\delta(\omega+\omega')} G = \begin{pmatrix} 0 & \frac{1}{+i\omega + Dq^2 + r} \\ \frac{1}{-i\omega + Dq^2 + r} & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{cases} \langle \varphi(-q,-\omega)\,\varphi(q',\omega')\rangle = 0 \\ \langle \tilde\varphi(-q,-\omega)\,\tilde\varphi(q',\omega')\rangle = 0 \\ \langle \tilde\varphi(-q,-\omega')\,\varphi(q,\omega)\rangle = \dfrac{1}{+i\omega + Dq^2 + r} \qquad \varphi \overset{(q,\omega)}{\longleftarrow\!\!\!\sim\!\sim\!\sim} \tilde\varphi \\ \langle \varphi(-q,-\omega)\,\tilde\varphi(+q',\omega')\rangle = \dfrac{1}{-i\omega + Dq^2 + r} \qquad \varphi \overset{(q,\omega)}{\longrightarrow\!\!\!\sim\!\sim\!\sim} \tilde\varphi \end{cases}$$

$$
\begin{cases}
\langle\tilde\varphi(\vec q,\omega)\tilde\varphi(\vec q\,',\omega')\rangle_0 = 0\\
\langle\varphi(\vec q,\omega)\varphi(\vec q\,',\omega')\rangle_0 = 0\\
\langle\tilde\varphi(\vec q,\omega)\varphi(\vec q\,',\omega')\rangle_0 = \dfrac{\delta^d(\vec q+\vec q\,')\delta(\omega+\omega')}{i\tau\omega + Dq^2 + r}
\end{cases}
$$

$$
\langle\tilde\varphi(\vec q\,',t')\varphi(\vec q,t)\rangle_0 = \Theta(t-t')\frac{\mathrm{e}}{\tau}^{-(Dq^2+r)\frac{(t-t')}{\tau}}\delta^d(\vec q+\vec q\,')
$$

$\frac{u}{2}$ 

$-\frac{u}{2}$ 

$\tilde\varphi\tilde\varphi\varphi$ $\qquad$ $\tilde\varphi\varphi\varphi$

In the time domain:

$$\langle \tilde\varphi(q,t)\,\varphi(q',t')\rangle = \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{d\omega'}{2\pi} e^{-i\omega' t'} \langle \tilde\varphi(q,\omega)\,\varphi(q',\omega')\rangle$$

$$= (2\pi)^d \delta^d(q+q') \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{1}{i\omega + Dq^2 + r}$$

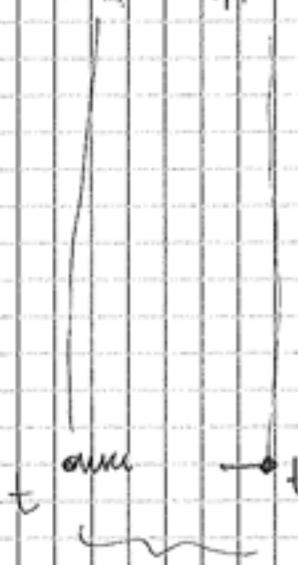
$$= (2\pi)^d \delta^d(q+q') \frac{1}{i} \Theta(t'-t) \frac{2\pi i}{2\pi} e^{-(Dq^2+r)(t'-t)};$$

$$= (2\pi)^d \delta^d(q+q') \Theta(t'-t)\, e^{-(Dq^2+r)(t'-t)}$$

Denom. $i\omega + Dq^2 + r = i[\omega - i(Dq^2+r)]$

$t-t'<0$ $\times$ $\theta(q^2+r)$

$t-t'>0$



$= 0$ for $t'<t$

⇒ time increases from the ~~ -leg to the —• leg.

⇝ we indicate it with an arrow:

$t < t'$

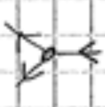
⇝ the vertices connected by → are ordered in time if one considers the mixed (q,t) representation...

⇝ as usual momenta are conserved at vertices, which are <u>local</u> in time...

$$-\frac{u}{2}(\tilde\varphi - \varphi)\tilde\varphi\varphi$$

(opposite sign e^{-A_φ})

$\tilde\varphi\tilde\varphi\varphi$: – 2 outgoing arrows
– 1 incoming arrow

$+\frac{u}{2}$ 

$\tilde\varphi\varphi\varphi$: – 1 outgoing arrow
– 2 incoming arrows

$-\frac{u}{2}$ 

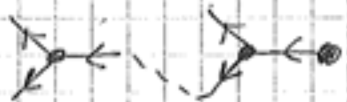
→ the symmetry (rapidity reversal) ensures that the two vertices are going to be renormalized in the same way... (i.e., the renormalized versions of the two couplings is the same...)

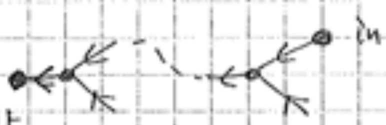
Propagator: $\Gamma^{(1,1)}$ ← # of $\tilde{\varphi}$-legs, # of φ-legs

(Consider only 1-particle irreducible graph, which cannot be disconnected by cutting a line)

$\left(\leftarrow \right)^{-1}$ - 1 loop diagram
↓
we need at least 2 vertices to close a diagram:

WILSON RG:
H. Hinrichsen
Adv. in Physics
49(7), 815,
see in particular
p. 871

$\left(+\frac{u}{2}\right)^2$  ?

$\left(-\frac{u}{2}\right)^2$ 

?

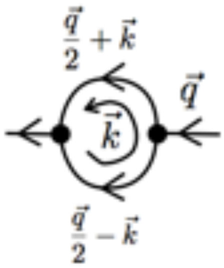
$\left(-\frac{u}{2}\right)\left(+\frac{u}{2}\right)$ 

$\left(-\frac{u}{2}\right)\left(+\frac{u}{2}\right)$  $\times 2$

$\left(-\frac{u}{2}\right)\left(+\frac{u}{2}\right) \times 2 \times$ 
↑
Combinatoric

$\varphi\varphi$

$\tilde{\varphi}\tilde{\varphi}$

$\Gamma^{(1,1)} = i\tau\omega + Dq^2 + r -$  $(-u/2)(u/2) \times 2$

$$\frac{1}{\tau}\int_{S_\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{i\tau\omega + Dq^2/2 + 2Dk^2 + 2r} \qquad \int_{S_\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d} = K_d \int_{\rho\Lambda}^{\Lambda} \mathrm{d}k\, k^{d-1}$$

$$\stackrel{\rho=1-\ell}{\simeq} K_d \frac{\Lambda^{d-1}}{\tau}\ell\Lambda \frac{1}{i\tau\omega + Dq^2/2 + 2D\Lambda^2 + 2r}$$

Propagator:

$$S_\Lambda = \{ \vec{k} \mid \rho\Lambda < |\vec{k}| < \Lambda \}$$

$$= \theta(t-t') \int_{S_\Lambda} \left(\frac{dk}{2\pi}\right)^d e^{-[D(\frac{\vec q}{2}+\vec k)^2 + r](t-t')} \, e^{-[D(\frac{\vec q}{2}-\vec k)^2 + r]\frac{(t-t')}{\tau}} \times \frac{1}{\tau}$$

$$\overset{!}{=} \theta(t-t') \int_{S_\Lambda} \left(\frac{dk}{2\pi}\right)^d e^{-\left(D\frac{q^2}{2} + 2Dk^2 + 2r\right)\frac{(t-t')}{\tau}} \times \frac{1}{\tau^2}$$

$\Rightarrow$ in frequency space:

$$\frac{1}{\tau} \int_{S_\Lambda} \left(\frac{dk}{2\pi}\right)^d \frac{1}{i\omega\tau + D\frac{q^2}{2} + 2Dk^2 + 2r}$$

$$\downarrow$$

$$\frac{1}{\tau} \int_{\rho\Lambda}^{\Lambda} \frac{\Omega_d}{(2\pi)^d} dk\, k^{d-1} \frac{1}{i\omega\tau + D\frac{q^2}{2} + 2Dk^2 + 2r} \overset{\text{for } \rho \to 1}{\simeq} \qquad \text{consider } \rho = 1 - l, \; l \to 0$$

$$\simeq \frac{1}{\tau} \cdot \frac{\Omega_d}{(2\pi)^d} \Lambda^{d-1} \times \Lambda\overbrace{(1-\rho)}^{l} \frac{1}{i\omega\tau + D\frac{q^2}{2} + 2D\Lambda^2 + 2r}$$

$$\frac{\Omega_d}{(2\pi)^d} \equiv K_d = \frac{2\pi^{d/2}}{(2\pi)^d \Gamma(d/2)} = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}$$

$$\Rightarrow \Gamma^{(1,1)} = i\omega + Dq^2 + r - 2\left(-\frac{u^2}{4}\right) \int_{S_\Lambda} \left(\frac{dk}{2\pi}\right)^d \frac{1}{i\omega\tau + D\frac{q^2}{2} + 2Dk^2 + 2r} \times \frac{1}{\tau}$$

$$\overset{!}{=} i\omega + Dq^2 + r + \frac{u^2}{2} K_d \frac{\Lambda^d}{\tau} l \frac{1}{i\omega\tau + D\frac{q^2}{2} + 2D\Lambda^2 + 2r} + O(u^4)$$

Note that (1) $\Gamma^{(1,1)}(q=0, \omega=0) = \underbrace{r + \frac{u^2}{2} K_d \frac{\Lambda^d}{\tau} l \frac{1}{2D\Lambda^2 + 2r}}_{r'} \rightarrow \equiv r S_2$

(2) $$\frac{\partial}{\partial q^2} \Gamma^{(1,1)}\Big|_{\substack{\omega=0\\ q=0}} \equiv D' = D - \frac{u^2}{2} K_d \frac{\Lambda^d}{\tau} l \underbrace{\frac{D}{2} \frac{1}{(2D\Lambda^2 + 2r)^2}}_{= S_1}$$

$$\Gamma^{(1,1)} = i\tau\omega + Dq^2 + r + \frac{u^2 K_d \Lambda^d}{2\tau}\ell \frac{1}{i\tau\omega + Dq^2/2 + 2D\Lambda^2 + 2r}$$

$$r' \equiv \Gamma^{(1,1)}(q=0, \omega=0) \quad = r + \frac{u^2 K_d \Lambda^d}{2\tau}\ell \frac{1}{2D\Lambda^2 + 2r} \equiv r S_2$$

$$D' \equiv \left. \frac{\partial}{\partial q^2}\Gamma^{(1,1)} \right|_{q=0,\omega=0} \quad = D - \overbrace{\frac{u^2 K_d \Lambda^d}{2\tau}\ell \frac{D}{2(2D\Lambda^2 + 2r)^2}}^{\equiv S_1}$$

$$\tau' \equiv \left. \frac{\partial}{\partial (i\omega)}\Gamma^{(1,1)} \right|_{q=0,\omega=0} \quad = \tau - \overbrace{\frac{u^2 K_d \Lambda^d}{2\tau}\ell \frac{\tau}{(2D\Lambda^2 + 2r)^2}}^{= 2S_1}$$

(3) $\frac{\partial}{\partial(i\omega)} \Gamma^{(1,1)}(q;\omega)\Big|_{\substack{q=0\\ \omega=0}} \equiv \tau' = \tau - \underbrace{\frac{u^2}{2} k_d \frac{\Lambda^d}{\cancel{\tau}}\, \ell\, \cancel{\tau} \frac{1}{(2D\Lambda^2+2r)^2}}_{2\tau \ell S_1}$

(introduce a τ: $\omega \to \omega\tau$)

Vertex:

(we eventually consider $\vec{q}_1 = \vec{q}_2 = 0$)

$$\int \frac{d\omega'}{2\pi} \int_{S_\Lambda} \left(\frac{dk}{2\pi}\right)^d \frac{1}{i\omega'\tau + D(q_1+k)^2 + r} \; \frac{1}{-i\omega'\tau + Dk^2 + r} \; \frac{1}{-i\omega'\tau + D(q_2-k)^2 + r}$$

$\omega' = \frac{i}{\tau}\left[D(q_1+k)^2 + r\right]$ $\quad$ $\omega' = -\frac{i}{\tau}(Dk^2+r)$ $\quad$ $\omega' = -i\frac{D}{\tau}\left[(q_2-k)^2+r\right]$

$$= \int_{S_\Lambda} \left(\frac{dk}{2\pi}\right)^d \frac{i}{i\tau} \; \frac{1}{D(q_1+k)^2 + Dk^2 + 2r} \; \frac{1}{D(q_1+k)^2 + D(q_2-k)^2 + 2r}$$

consider $\vec{q}_1 = \vec{q}_2 = 0$

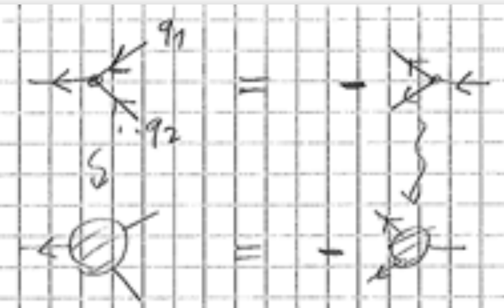
$$= \int_{S_\Lambda} \left(\frac{dk}{2\pi}\right)^d \frac{1}{\tau} \frac{1}{(2Dk^2+2r)^2} \simeq k_d \Lambda^d \ell \, \frac{1}{\tau} \frac{1}{(2D\Lambda^2+2r)^2}$$

$$\Gamma^{(1,2)} = -u + \left(\frac{u}{2}\right)\left(-\frac{u}{2}\right)^2 \times 8 \left(\quad \times \overset{q_1 \leftrightarrow q_2}{2} \right)$$

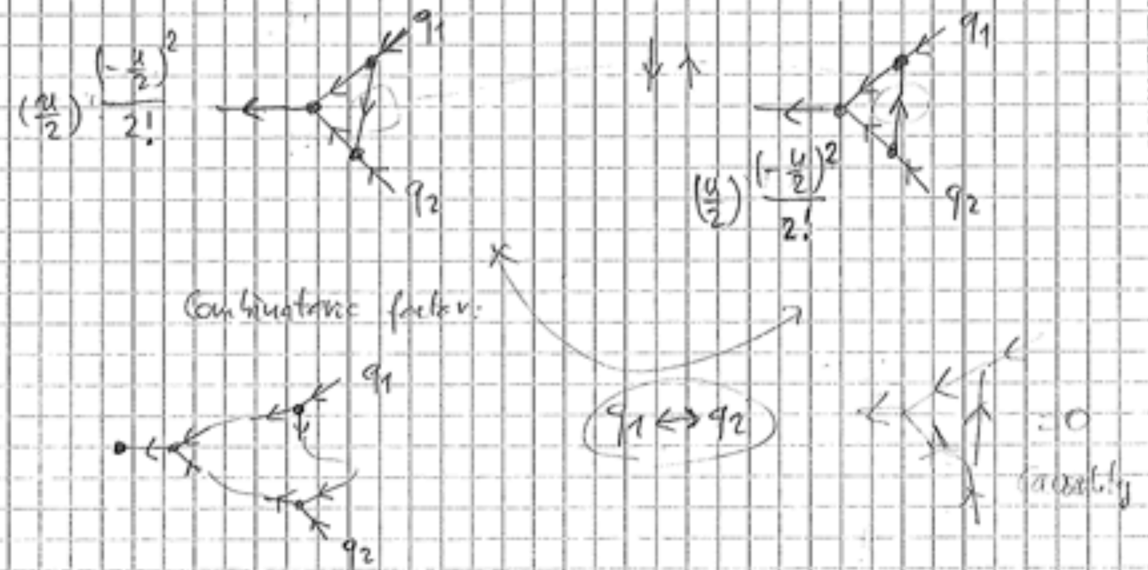
$$= -u + 2u^3 \, k_d \Lambda^d \ell \, \frac{1}{\tau} \frac{1}{(2D\Lambda^2+2r)^2} \to 8uS_1\ell$$

define the "new" coupling as $-\Gamma^{(1,2)}$ $\Rightarrow$

Vertex ;

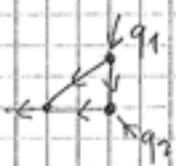


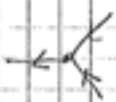
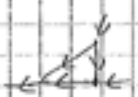
symmetry.

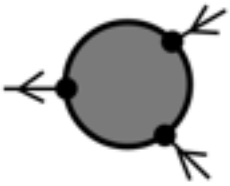
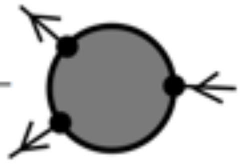


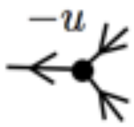
$2 \times 2 \times 2 \times 2$

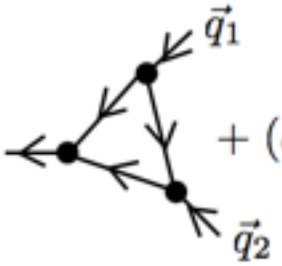
Calculate :



$\Rightarrow \quad \Gamma^{(1,2)} =$  $+ \; 2^4 \, \frac{1}{2} \left(\frac{u}{2}\right)\left(-\frac{u}{2}\right)^2$  $+ \; (q_1 \leftrightarrow q_2)$

$\Gamma^{(1,2)} =$  $= -$  $= -\Gamma^{(2,1)}$

$=$ 

$+ \ (u/2)(-u/2)^2 \times 8$ 

$+ \ (\vec{q}_1 \longleftrightarrow \vec{q}_2)$

$$\int \frac{\mathrm{d}\omega'}{2\pi} \int_{S_\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{i\tau\omega' + Dk^2 + r} \frac{1}{[-i\tau\omega' + Dk^2 + r]^2}$$

$$= \int_{S_\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\tau} \frac{1}{[2Dk^2 + 2r]^2} \simeq \frac{K_d \Lambda^d}{\tau} \ell \frac{1}{(2D\Lambda^2 + 2r)^2}$$

$$\Gamma^{(1,2)} = -u + \overbrace{2u^3 \frac{K_d \Lambda^d}{\tau} \ell \frac{1}{(2D\Lambda^2 + 2r)^2}}^{= 8uS_1}$$

$$u' \equiv -\left.\Gamma^{(1,2)}\right|_{q_i = 0, \omega_i = 0} = u - 8uS_1 \ell$$

$$\rho = 1 - \ell, \quad \ell \to 0$$

$$\begin{cases} \tau'' = \rho^{\eta} \, \tau' & \simeq (1 - \eta\ell) \, \tau' \\ D'' = \rho^{2 - z + \eta} \, D' & \simeq [1 - (2 - z + \eta)\ell] \, D' \\ r'' = \rho^{\eta - z} \, r' & \simeq [1 - (\eta - z)\ell] r' \\ u'' = \rho^{d/2 - z + 3\eta/2} \, u' & \simeq [1 - (d/2 - z + 3\eta/2)\ell] \, u' \end{cases}$$

$$u' = -\Gamma^{(1,2)}\big|_{q_1=q_2=0}$$

$$= u - \left(2u^3 K_d \Lambda^d \ell \frac{1}{\tau} \frac{1}{(2D\Lambda^2+2r)^2}\right) \longrightarrow 8uS_1\ell$$

Now we implement the rescaling according to:

$$\tau'' = \rho^{\eta}\tau' \simeq (1-\eta\ell)\tau' \qquad \rho = 1-\ell \quad \text{and} \quad \ell \to 0$$

$$D'' = \rho^{2-z+\eta} D' \simeq \left[1-(2-z+\eta)\ell\right] D'$$

$$r'' = \rho^{\eta-2} r' \simeq \left[1-(\eta-2)\ell\right] r'$$

$$u'' = \rho^{\frac{d}{2}-2+\frac{3}{2}\eta} u' \simeq \left[1-\left(\tfrac{d}{2}-2+\tfrac{3}{2}\eta\right)\ell\right] u'$$

Accordingly, by composing the two transformations one finds:

$$\tau'' \simeq (1-\eta\ell)\tau' = (1-\eta\ell)(\tau - 2\tau\ell S_1)$$

$$\simeq \tau\left[1-(\eta+2S_1)\ell\right]$$

$$\partial_\ell \tau \equiv \frac{\tau''-\tau}{\ell} = -\tau(\eta+2S_1)$$

$$D'' \simeq \left[1-(2-z+\eta)\ell\right] D' = \left[1-(2-z+\eta)\ell\right] D(1-\ell S_1)$$

$$\simeq D\left[1-(2-z+\eta+S_1)\ell\right]$$

$$\partial_\ell D \equiv \frac{D''-D}{\ell} = -D\left[2-z+\eta+S_1\right]$$

$$r'' = [1-(\eta-z)]r' = [1-(\eta-z)\ell][r + r\ell S_2] = r[1-(\eta-z-S_2)\ell]$$

$$\Rightarrow \partial_\ell r \equiv \frac{r''-r}{\ell} = -r(\eta-z-S_2)$$

and, eventually:

$$u'' = [1-(\tfrac{d}{2}-z+\tfrac{3}{2}\eta)\ell]u' = [1-(\tfrac{d}{2}-z+\tfrac{3}{2}\eta)\ell]u[1-8S_1\ell] \stackrel{!}{=} [1-(\tfrac{d}{2}-z+\tfrac{3}{2}\eta+8S_1)\ell]u$$

$$\Rightarrow \partial_\ell u = \frac{u''-u}{\ell} = -u\left(\tfrac{d}{2}-z+\tfrac{3}{2}\eta+8S_1\right)$$

The previous equations tell us how the parameters of the action change upon integrating out the short wave length modes.

Out of the parameters (τ, D, r, u) we can always choose the normalization of the fields $\varphi, \tilde{\varphi}$ and of time such that τ and D do not change with ℓ.

$\Rightarrow \partial_\ell \tau = 0 \Rightarrow \boxed{\eta = -2S_1}$

& $\partial_\ell D = 0 \Rightarrow 2-z+\eta+S_1 = 0 \Rightarrow 2-z-S_1=0$

$\Rightarrow \boxed{z = 2-S_1}$

The remaining equations have to be solved:

$$\partial_\ell r = -r(\eta-z-S_2) = -r(-2S_1-2+S_1-S_2) = +r(2+S_1+S_2)$$

$$\partial_\ell u = -u(\tfrac{d}{2}-2+\tfrac{3}{2}\eta+8S_1) \stackrel{!}{=} -u(\tfrac{d}{2}-2+S_1-3S_1+8S_1) \qquad d=4-\epsilon$$

$$= -u(-\tfrac{\epsilon}{2}+6S_1) = u(\tfrac{\epsilon}{2}-6S_1)$$

RG flow:

$$\begin{cases} \partial_\ell \tau = -(\eta + 2S_1)\tau \\ \partial_\ell D = -(2 - z + \eta + S_1)D \\ \partial_\ell r = -(\eta - z - S_2)r \\ \partial_\ell u = -\left(\frac{d}{2} - z + \frac{3\eta}{2} + 8S_1\right)u \end{cases}$$

$$\boxed{\eta = -2S_1}$$

$$\boxed{z = 2 - S_1}$$

$$\begin{cases} \partial_\ell r = (2 + S_1 + S_2)r \\ \partial_\ell u = \left(\frac{\epsilon}{2} - 6S_1\right)u \end{cases}$$

$$\epsilon = 4 - d$$

$$\tilde{r} \equiv \frac{r}{D\Lambda^2}$$

$$g \equiv \frac{u^2 K_d \Lambda^d}{4\tau(2D\Lambda^2)^2} = \frac{K_d}{16}\frac{u^2\Lambda^{d-4}}{D^2\tau}$$

Remember that:

$$S_1 \equiv \frac{u^2}{4} K_d \frac{\Lambda^d}{\tau} \frac{1}{(2D\Lambda^2+2r)^2} \qquad \text{and} \qquad S_2 \equiv \frac{u^2}{2} K_d \frac{\Lambda^d}{\tau} \frac{1}{r} \frac{1}{(2D\Lambda^2+2r)}$$

For investigating the flow it is convenient to introduce dimensionless quantities

$$g \equiv \frac{u^2}{4} \frac{K_d}{\tau} \frac{\Lambda^d}{(2D\Lambda^2)^2} = \frac{u^2}{16 D^2 \tau} K_d \Lambda^{d-4}$$

$$\tilde{r} \equiv \frac{r}{D\Lambda^2}$$

In these terms: $\quad S_1 = g \dfrac{1}{(1+\tilde{r})^2}$

while: $$S_2 = 2 \underbrace{\left(\frac{u^2}{4} \frac{K_d}{\tau} \frac{\Lambda^d}{(2D\Lambda^2)^2}\right)}_{g} \underbrace{\frac{2D\Lambda^2}{r}}_{\frac{1}{\tilde{r}}} \frac{1}{1+\tilde{r}} = 4g \frac{1}{\tilde{r}(1+\tilde{r})}$$

and: $$\partial_\ell \tilde{r} = \frac{\partial_\ell r}{D\Lambda^2} = \frac{1}{D\Lambda^2} r(2+S_1+S_2) = \tilde{r}(2+S_1+S_2)$$

$$\partial_\ell g = 2u \underbrace{\partial_\ell u}_{u\left(-\frac{\epsilon}{2}+3S_1\right)} \times \frac{1}{4} \frac{K_d}{\tau} \frac{\Lambda^d}{(2D\Lambda^2)^2} = 2g\left(\frac{\epsilon}{2} - 6S_1\right) = g(\epsilon - 12 S_1)$$

Accordingly:
$$\partial_\ell \tilde{r} = \tilde{r}(2+S_1+S_2) = \tilde{r}\left(2 + \frac{g}{(1+\tilde{r})^2} + \frac{4g}{\tilde{r}(1+\tilde{r})}\right)$$

$$\partial_\ell g = g\left[\epsilon - 12 \frac{g}{(1+\tilde{r})^2}\right]$$

the flow of which is given in the figures...

$$\begin{cases} \partial_\ell \tilde{r} = (2 + S_1 + S_2)\, \tilde{r} \\ \partial_\ell g = (\epsilon - 12 S_1)\, g \end{cases}$$

$$S_1^* = \epsilon/12$$

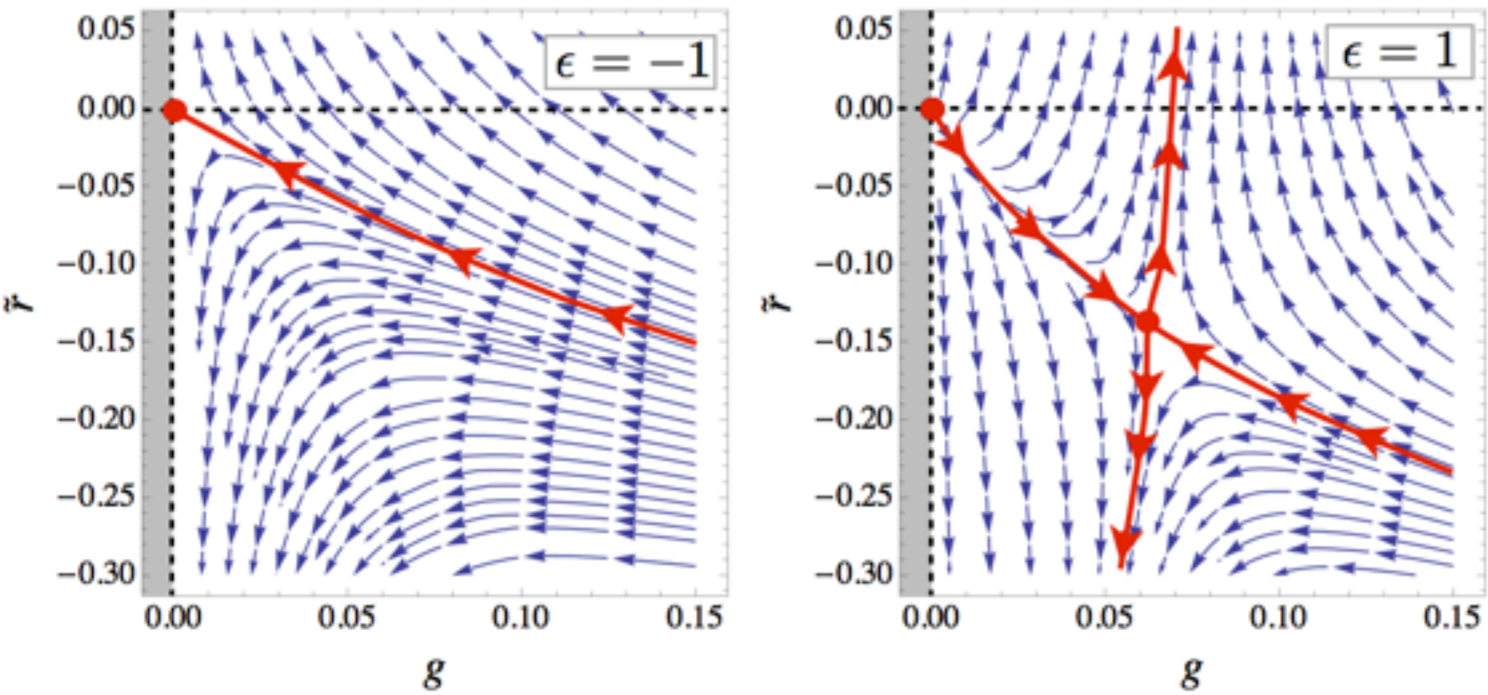
$$S_2^* = -2 - \epsilon/12$$

$$\Longrightarrow (g^*, \tilde{r}^*)$$

$$S_1 = g/(1+\tilde{r})^2$$

$$S_2 = 4\, g/[(1+\tilde{r})\tilde{r}]$$

$$\boxed{\begin{aligned} \eta &= -\epsilon/6 \\ z &= 2 + \epsilon/12 \end{aligned}}$$



The fixed-point values are determined by:

$$\partial_\ell g = 0 \Rightarrow \quad g^* = 0 \quad \text{(Gauss. fixed point)}$$

$$S_1^* = \frac{\epsilon}{12} \Rightarrow \quad \eta = -\frac{\epsilon}{6}, \; z = 2 + \frac{\epsilon}{12}$$

→ Wilson-Fisher fixed point. (Stable only for $d<4$, as one can easily realise..)

$$\partial_\ell \tilde{r} = 0 \Rightarrow \quad S_2^* = -2 - S_1^* = -2 - \frac{\epsilon}{12}$$

As in the case of equilibrium phase transitions & RG, the remaining relevant exponent characterizes the RG flow along the unstable direction (typically the temperature) close to the WF fixed point; solving the equations for $\tilde{r}$ & g at the lowest order in ϵ one finds (ex):

$$\begin{cases} \tilde{r}^* = -\frac{\epsilon}{6} + O(\epsilon^2) \\ g^* = \frac{\epsilon}{12} + O(\epsilon^2) \end{cases}$$

where:

$$\partial_\ell \tilde{r} = \underbrace{\tilde{r}\left(2 + \frac{g}{(1+\tilde{r})^2} + \frac{4g}{\tilde{r}(1+\tilde{r})}\right)}_{f_1(\tilde{r},g)}$$

$$\partial_\ell g = \underbrace{g\left(\epsilon - 12\frac{g}{(1+\tilde{r})^2}\right)}_{f_2(\tilde{r},g)}$$

Define: $\delta r = \tilde{r} - \tilde{r}^*$
$\delta g = g - g^*$

$$\Rightarrow \quad \partial_\ell \delta r = \partial_{\tilde{r}} f_1(\tilde{r},g)\Big|_{\substack{\tilde{r}=\tilde{r}^* \\ g=g^*}} \delta\tilde{r} + \partial_g f_1(\tilde{r},g)\Big|_{\substack{\tilde{r}=\tilde{r}^* \\ g=g^*}} \delta g$$

$$= \left(2 - \frac{\epsilon}{4}\right)\delta\tilde{r} + \left(4 + \frac{\epsilon}{2}\right)\delta g + \text{h.o.}$$

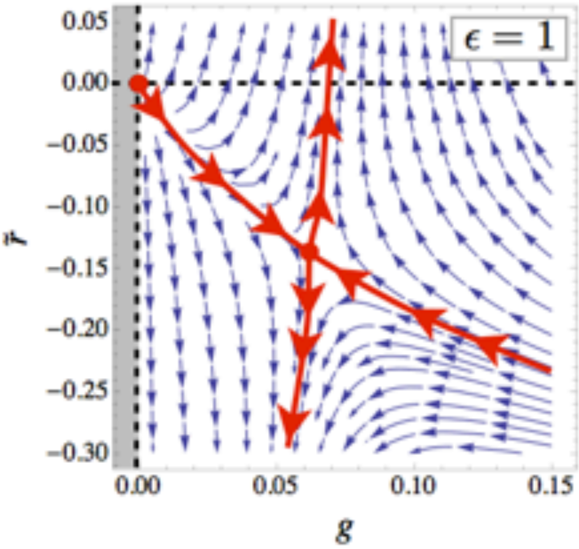
$$\Rightarrow \quad \partial_\ell \delta g = \partial_{\tilde{r}} f_2(\tilde{r},g)\Big|_{\substack{\tilde{r}=\tilde{r}^* \\ g=g^*}} \delta\tilde{r} + \partial_g f_2(\tilde{r},g)\Big|_{\substack{\tilde{r}=\tilde{r}^* \\ g=g^*}} \delta g$$

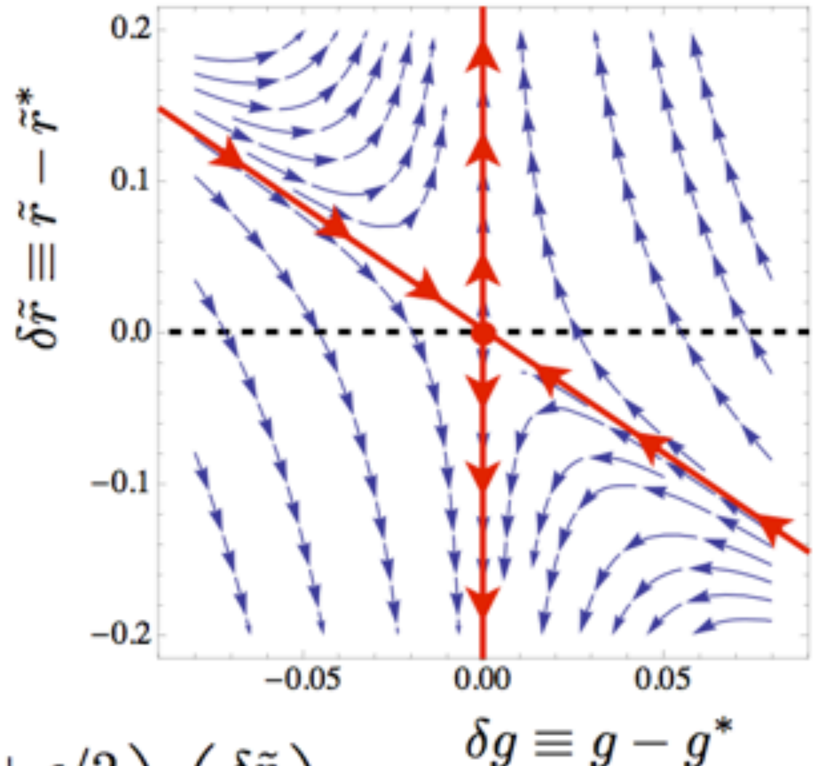
$$= O(\epsilon^2)\,\delta\tilde{r} + (-\epsilon)\,\delta g$$

$$\Rightarrow \quad \partial_\ell \begin{pmatrix} \delta r \\ \delta g \end{pmatrix} = \begin{pmatrix} 2 - \frac{\epsilon}{4} & 4 + \frac{\epsilon}{2} \\ 0 & -\epsilon \end{pmatrix} \begin{pmatrix} \delta r \\ \delta g \end{pmatrix}$$

⇒ we have to determine the eigenvalues of the matrix which are: $-\epsilon$ and $2 - \frac{\epsilon}{4}$.

The negative one corresponds to the stable direction, while $2-\frac{\epsilon}{4}$ to the unstable one and can be identified, by analogy, with $\frac{1}{\nu_\perp} = 2 - \frac{\epsilon}{4} \Rightarrow \nu_\perp = \frac{1}{2} + \frac{\epsilon}{16} + O(\epsilon^2)$





$$\partial_\ell \begin{pmatrix} \delta\tilde{r} \\ \delta g \end{pmatrix} = \begin{pmatrix} 2-\epsilon/4 & 4+\epsilon/2 \\ 0 & -\epsilon \end{pmatrix} \begin{pmatrix} \delta\tilde{r} \\ \delta g \end{pmatrix}$$

$$\delta\tilde{r}(\ell) = \mathrm{e}^{\lambda_1 \ell}\delta\tilde{r}(0)$$
$$= \rho^{-\lambda_1}\delta\tilde{r}(0)$$

$\rho = \mathrm{e}^{-\ell}$

$x' = \rho x$

$$\frac{x}{|\delta\tilde{r}|^{-1/\lambda_1}} \rightarrow \xi_\perp$$

$\nu_\perp = 1/\lambda_1 = 1/2 + \epsilon/16$