Read carefully the following problems and provide correct and complete answers to 5 questions chosen among the following 16, possibly within the same problem.

A. Cauchy random walker

In one spatial dimension consider a random walker which starts from $x = 0$ and proceeds by consecutive (random) jumps drawn independently from the density

$$p(\Delta x) = \frac{1}{\pi a} \frac{1}{(\Delta x/a)^2 + 1} \quad \text{with} \quad a > 0. \tag{1}$$

Indicate by Δx_i the value of the i-th jump and by $X_N = \sum_{i=1}^{N} \Delta x_i$ the position after N steps.

(1) Determine the probability density of the variable X_N. Does the central limit theorem apply to X_N? Why?

Knowing that the $P_{1|1}$ for a Cauchy process satisfies Chapmann-Kolmogorov equation, one concludes that the sum of two variables with Cauchy distribution has still a Cauchy distribution (stability of the law). This can be directly seen by calculating the characteristic function of the distribution:

$$\langle e^{it\Delta x} \rangle = \int_{-\infty}^{\infty} d\Delta x \, p(\Delta x) e^{it\Delta x} = \frac{a}{\pi} \int_{-\infty}^{\infty} d\Delta x \frac{e^{it\Delta x}}{(\Delta x - ia)(\Delta x - ia)} = e^{-a|t|}, \tag{2}$$

where the residue theorem has been used for the calculation. Accordingly, the characteristic function of the sum X_N is given by $\langle e^{itX_N} \rangle = \prod_{i=1}^{N} \langle e^{it\Delta x_i} \rangle = e^{-Na|t|}$, where we used the independence of the jumps. By comparison with Eq. (2) (of by inverting the characteristic function) one concludes that X_N has a Cauchy distribution with $a \mapsto Na$. The central limit theorem clearly does not apply to the present case (and therefore X_N has not a Gaussian distribution) because the variance of $p(\Delta x)$ is not finite.

(2) Define $\Delta_N \equiv \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_N\}$ the maximum forward step taken by the walker. Write down the expression for the probability $P(\Delta_N < x)$. Determine its behavior for large N, under the assumption that, typically, $\Delta_N \gg a$ as N increases and plot the corresponding (Fréchet) distribution. *[Hint:* $(1-t)^N \simeq e^{-Nt}$ *for* $t \ll 1$*.]*

In order for Δ_N to be smaller than a certain x, all Δx_i have to be smaller than x and therefore $P(\Delta_N < x) = \prod_{i=1}^{N} P(\Delta x_i < x) = \left[\int_{-\infty}^{x} d\Delta x \, p(\Delta x)\right]^N$. As N increases, this probability does not vanish only for $x \gg a$ and, correspondingly, *the integral approaches 1: it is therefore convenient to emphasize this fact by writing* $\int_{-\infty}^{x} \mathrm{d}\Delta x\, p(\Delta x) = 1 - \int_{x}^{\infty} \mathrm{d}\Delta x\, p(\Delta x) \simeq 1 - a/(\pi x)$ *for* $x \gg a$*. As a result (see also the hint)* $P(\Delta_N < x) \simeq \mathrm{e}^{-Na/(\pi x)}$*, which is known as Fréchet distribution. What matters in order to determin the law of the maximum* Δ_N *is only the asymptotic behavior of* $p(\Delta x)$ *for large* Δx*.*

B. Diffusion with evaporation

On a one-dimensional lattice (with lattice spacing a) consider a particle A which can either: (i) jump to the left or to the right of its current position at site i, with the same rate Δ; (ii) evaporate $A \xrightarrow{\varepsilon_i} 0$, with a transition rate ε_i, leaving the lattice forever. We are interested in the master equation for the evolution of the probability $P(i,t)$ of finding the particle at site i at time t.

(3) If $\Delta = 0$, what is the effect of evaporation on $P(i,t)$? *[Hint: Connect* $\mathrm{d}P(i,t)/\mathrm{d}t$ *to* ε_i*.]* Write down the expression for the transition rates W associated to diffusion and evaporation.

The evaporation removes the particle from the lattice and therefore it reduces the probability $P(i,t)$ *of finding the immobile particle originally at site* i *still there after a time* t*. In particular* $\mathrm{d}P(i,t)/\mathrm{d}t = -\varepsilon_i P(i,t)$*. Heuristically, one can associate a sort of "transition rate" (which is however, negative)* $-\varepsilon_i$ *to evaporation at site* i *while diffusion proceeds with the usual transition rates* Δ*.*

(4) Write the master equation for the probability $P(i,t)$. Does this master equation conserve the total probability $\sum_i P(i,t)$? Why?

The master equation is

$$\frac{\mathrm{d}P(i,t)}{\mathrm{d}t} = -\varepsilon_i P(i,t) + \Delta P(i+1,t) + \Delta P(i-1,t) - 2\Delta P(i,t). \tag{3}$$

The total probability $\sum_i P(i,t)$ *is not conserved because, assuming an infinite lattice, the sum over all lattice sites of the last three terms vanishes, while this is not necessarily the case for the first term. Physically we know that evaporation does not conserve probability.*

(5) Express the master equation derived above in terms of the coordinate $x_i = i\,a \equiv x$ of the particle at site i. Discuss its continuum limit $a \to 0$. How should Δ and ε_i scale with a in order to have a well-defined continuum limit?

Taking into account that $P(i,t) \mapsto P(x_i \equiv x, t)$*, we have* $P(i \pm 1, t) \mapsto P(x \pm a, t) = P(x,t) \pm a\partial_x P(x,t) + (a^2/2)\partial_x^2 P(x,t) + \mathcal{O}(a^3)$*; accordingly Eq. (3) becomes*

$$\frac{\mathrm{d}P(x,t)}{\mathrm{d}t} = -\varepsilon_x P(x,t) + \mathcal{O}(\varepsilon a) + a^2\Delta\partial_x^2 P(x,t) + \mathcal{O}(a^3\Delta) \tag{4}$$

and a proper continuum limit is obtained for $\varepsilon \sim 1$ and $\Delta \sim a^{-2}$. This latter scaling is expected because it characterizes the scaling of the diffusion process which is recovered for $\varepsilon = 0$.

(6) In the continuum limit, solve the master equation for $P(x,t)$ assuming a spatially constant evaporation rate $\varepsilon(x) \equiv \varepsilon$ and the initial condition $P(x,0) = \delta(x)$.

In order to solve the differential equation, it is sufficient to note that $P(x,t)$ is expected to decrease exponentially in time because of the evaporation and therefore it is convenient to look for a solution of the form $P(x,t) = \mathrm{e}^{-\varepsilon t} p(x,t)$. Substituting this expression in the original equation one finds that $p(x,t)$ satisfies the diffusion equation and therefore $p(x,t) = \exp\{-x^2/(4Dt)\}/\sqrt{4\pi Dt}$ where $D \equiv a^2\Delta$. Alternatively, Eq. (4) can be solved by considering its Fourier transform in space, which turns it into a ordinary differential equation in time.

(7) Using the result of the previous point, calculate $\langle x(t)\rangle$ and $\langle x^2(t)\rangle$ and plot them as a function of t. Determine the time t_M at which the spreading $\ell(t) \equiv \sqrt{\langle x^2(t)\rangle}$ is maximum and the corresponding value $\ell_M \equiv \ell(t_M)$. Interpret these results.

The solution $P(x,t)$ found at the previous point is nothing but the probability density of a standard diffusion process with an overall extra factor $\mathrm{e}^{-\varepsilon t}$; accordingly, one concludes that $\langle x(t)\rangle = 0$ whereas $\langle x^2(t)\rangle_{\varepsilon\neq 0} = \mathrm{e}^{-\varepsilon t}\langle x^2(t)\rangle_{\varepsilon=0} = \mathrm{e}^{-\varepsilon t} 2Dt$. This function vanishes at time $t = 0$, grows till it reaches a maximum at $t_M = 1/\varepsilon$, and then decreases exponentially to zero. The maximal spreading ℓ_M for a single particle is obtained for $\ell_M = \ell(t = t_M) = \sqrt{2D/(\mathrm{e}\varepsilon)}$. Note that $\langle x^2(t)\rangle$ decreases for $t > t_M$ because this analysis assumes that an evaporated particle does not contribute to $\langle x^2(t)\rangle$ and the probability of the particle not being evaporated at time t decreases exponentially. Alternatively, one could calculate the spreading by accounting for the contribution of the evaporated particle. [1]In fact, at a certain time t, the particle is not evaporated with probability $P_a(t) = \mathrm{e}^{-\varepsilon t}$ and in that case it is characterized by a spreading $\ell^2_{ne}(t) \equiv \langle x^2(t)\rangle_{\varepsilon=0} = 2Dt$; on the other hand, the particle might have been already evaporated, and this occurs with probability $1 - \mathrm{e}^{-\varepsilon t}$. The contribution of an evaporated particle to the spreading, given by $\langle x^2(t_e)\rangle_{\varepsilon=0} = 2Dt_e$, depends on the time t_e at which it evaporated, the density of which is $p_e(t) = -\mathrm{d}P_a(t)/\mathrm{d}t = \varepsilon\mathrm{e}^{-\varepsilon t}$. As a result the contribution of evaporated particles to the average spreading is $\ell^2_e(t) \equiv \int_0^t \mathrm{d}t_e\, p_e(t_e)\langle x^2(t_e)\rangle_{\varepsilon=0} = 2D(1-\mathrm{e}^{-\varepsilon t} - \varepsilon t\,\mathrm{e}^{-\varepsilon t})/\varepsilon$. The total spreading is then given by $\ell^2_T(t) \equiv \langle x^2\rangle = \ell^2_{ne}(t)P_a(t) + [1-P_a(t)]\ell^2_e(t) = 2D[(1-\mathrm{e}^{-\varepsilon t})^2 + \varepsilon t\,\mathrm{e}^{-2\varepsilon t}]/\varepsilon$, which tends to the constant $\ell^2_T(t\to\infty) = 2D/\varepsilon$ for $\varepsilon \neq 0$, while it reproduces the result of pure diffusion for $\varepsilon = 0$. Note that the second contribution to ℓ^2_T in brackets corresponds to the one previously calculated.

[1] Thanks to Dr. Sayed-Allaei for raising this point.

C. Binary annihilation process

Consider a population consisting of individuals A, each of which might undergo the following processes, with the specified rates:

(i)	$A \xrightarrow{\sigma} A + A$	branching,
(ii)	$A + A \xrightarrow{\lambda} \emptyset,$	binary annihilation.

The reaction in (ii) occurs among all possible *pairs* of individuals of the population.

(8) Describe qualitatively the expected behavior of this population, depending on the values of $\sigma \geq 0$ and $\lambda \geq 0$.

The dynamics of this population results from the competition between (i) *and* (ii). *In a first approximation, this dynamics is described by the rate equation* $\dot{n} = \sigma n - 2\lambda n^2$ *for the average population size* n. *Given that the rate of* (i) *is* $\propto n$, *while the one of* (ii) *is* $\propto n^2$, (ii) *will always dominate in large populations, while* (i) *will do so in small populations. As a result, we expect that they balance at some stationary state which is achieved as long as* $\sigma, \lambda \neq 0$ *and which is characterized by the stationary value* $n_s = \sigma/(2\lambda)$ *predicted by the rate equation. Note that the case under study here differs from the one discussed in the lectures with* (ii) *replaced by a death* $A \xrightarrow{\mu} \emptyset$ *and therefore one does not expect a phase transition. The stationary state alluded to before, however, is destabilized by fluctuations (at last in low space dimensionality) and the population will eventually reach the absorbing state. For* $\sigma = 0$ *and* $\lambda \neq 0$, *instead, one expects a fast extinction, whereas for* $\sigma \neq 0$ *and* $\lambda = 0$ *an exponential increase of the population.*

(9) Write down the transition rates $W(n \to n+1)$ and $W(n \to n-2)$ for the number n of individuals in the population, associated to (i) and (ii). Is there an absorbing state for the dynamics?

The transition rates are given by $W(n \to n+1) = \sigma\, n$ *and* $W(n \to n-2) = \lambda n(n-1)$ *(with a possible factor 1/2 adsorbed in the definition of* λ.*)* $n = 0$ *is clearly an absorbing state because* $W(0 \to \ldots) = 0$.

(10) Write down the master equation for the probability $P_n(t)$ of having n individuals at time t. Determine the evolution equation of the average population $\langle n \rangle_t \equiv \sum_{n=0}^{\infty} n P_n(t)$ at time t.

The master equation is

$$\partial_t P_n = \sigma(n-1)P_{n-1} + \lambda(n+2)(n+1)P_{n+2} - [\sigma n + \lambda n(n-1)]\, P_n, \qquad (5)$$

where the first term on the r.h.s. is present only for $n \geq 1$. *By multiplying the* *previous expression by n and summing over n, one finds*

$$\begin{aligned}\partial_t \langle n\rangle_t &= \sum_{n=0}^{\infty}\left\{\sigma n(n-1)P_{n-1} + \lambda(n+2)(n+1)nP_{n+2} - \left[\sigma n^2 + \lambda n^2(n-1)\right]P_n\right\}\\ &= \sigma\langle (n+1)n\rangle + \lambda\langle \underbrace{n(n-1)(n-2)}_{n^2(n-1)-2n(n-1)}\rangle - \left[\sigma\langle n^2\rangle + \lambda\langle n^2(n-1)\rangle\right]\\ &= \sigma\langle n\rangle - 2\lambda\langle n(n-1)\rangle = (\sigma+2\lambda)\langle n\rangle - 2\lambda\langle n^2\rangle.\end{aligned} \tag{6}$$

(11) Introduce the *mean-field approximation* $\langle n^2\rangle_t \simeq \langle n\rangle_t^2$. Solve the evolution equation for $\langle n\rangle_t$ and discuss the qualitative features of the result as a function of $\sigma \geq 0$ and $\lambda \geq 0$.

Within the mean-field approximation, the previous equation becomes ($n \equiv \langle n\rangle$)

$$\dot{n} = (\sigma+2\lambda)n - 2\lambda n^2, \tag{7}$$

with the initial condition $n(0) = n_0$. For $\lambda = 0$ the solution is clearly a growing exponential $n(t) = n_0 \mathrm{e}^{\sigma t}$ (assume $\sigma = 0$). For $\lambda \neq 0$ it is convenient to rescale the time variable t in order to get rid of the constant of the eventual exponential approach to the stationary state n_s by writing $t = \tau/(\sigma+2\lambda)$ and $g \equiv 2\lambda/(\sigma+2\lambda)$. In these terms the evolution equation becomes $\partial_\tau n = n - gn^2$ and therefore

$$\int \mathrm{d}n \left(\frac{1}{n} + \frac{g}{1-gn}\right) = \tau + const. \quad \Longrightarrow \quad \ln\left(\frac{n}{1-gn}\right) = \tau + const. \tag{8}$$

which yields

$$n(\tau) = \frac{1}{g + c\,\mathrm{e}^{-\tau}} \tag{9}$$

and, by imposing the initial condition $n(\tau = 0) = n_0$ one determines $c = n_0^{-1} - g$ and therefore

$$n(\tau) = \frac{1}{g(1-\mathrm{e}^{-\tau}) + n_0^{-1}\mathrm{e}^{-\tau}}. \tag{10}$$

Independently of the value of n_0, one finds that $n(t \to \infty) \equiv n_\infty = g^{-1} = 1 + \sigma/(2\lambda)$, and the approach to n_∞ occurs exponentially with a time scale set by $1/(\sigma+2\lambda)$.

Add now diffusion with coefficient D, assuming the model to be in d spatial dimensions.

(12) On the basis of the master equation for the process, determine the reaction hamiltonian H_R within the Doi-Peliti formalism and the associated field-theoretical action $\mathcal{A}_T$ on the continuum (neglecting boundary terms).

The reaction Hamiltonian can be determined by multiplying both members of

Eq. (5) by the Fock basis $|n\rangle$ and summing over n, taking into account the definition of the state vector $|\Phi(t)\rangle = \sum_{n=0}^{\infty} P_n(t)|n\rangle$:

$$\begin{aligned}\partial_t|\Phi(t)\rangle =& \sigma\sum_{n=1}^{\infty}(n-1)P_{n-1}|n\rangle + \lambda\sum_{n=0}^{\infty}(n+2)(n+1)P_{n+2}|n\rangle \\ &- \sum_{n=0}^{\infty}\left[\sigma n + \lambda n(n-1)\right]P_n|n\rangle.\end{aligned} \tag{11}$$

In the first term one can replace $(n-1)|n\rangle = a^\dagger(a^\dagger a)|n-1\rangle$, in the second $(n+2)(n+1)|n\rangle = a^2|n+2\rangle$, in the third $n|n\rangle = (a^\dagger a)|n\rangle$, and in the fourth $n(n-1)|n\rangle = (a^\dagger)^2a^2|n\rangle$. Collecting the various contribution one finds

$$\partial_t|\Phi(t)\rangle = \underbrace{\left[\sigma(a^\dagger)^2a + \lambda a^2 - \sigma a^\dagger a - \lambda(a^\dagger)^2a^2\right]}_{-H_R(a^\dagger,a)}|\Phi(t)\rangle. \tag{12}$$

As expected, one immediately verifies that $H_R(a^\dagger = 1, a) = 0$. The corresponding action $\mathcal{A}_T$ on the continuum, neglecting boundary terms, is (make use of the notes)

$$\begin{aligned}\mathcal{A}_T =& \int \mathrm{d}^dx \int \mathrm{d}t \left\{\tilde\phi(\partial_t - D\nabla^2)\phi - H_R(1+\tilde\phi,\phi)\right\} \\ & \int \mathrm{d}^dx \int \mathrm{d}t \left\{\tilde\phi(\partial_t - D\nabla^2 - \sigma)\phi + 2\lambda\tilde\phi\phi^2 - \sigma\tilde\phi^2\phi + \lambda\tilde\phi^2\phi^2\right\}.\end{aligned} \tag{13}$$

(13) Determine the rate equations for the evolution of the system as the the mean-field equations derived from $\mathcal{A}_T$. Compare with the result of point (10) and comment. Expand the theory around the mean-field solution and determine the form of the propagators both in the frequency and in the time domain.

Assuming a space-independent mean-field field $\tilde\phi$ and ϕ, the saddle-point equations are satisfied by $\tilde\phi \equiv 0$. Accordingly, the remaining equation can be derived from

$$\frac{\delta\mathcal{A}_T}{\delta\tilde\phi} = 0 = (\partial_t - \sigma)\phi + 2\lambda\phi^2. \tag{14}$$

In order to compare it with Eq. (6) one should take into account that $\langle n\rangle = \langle\mathcal{P}|a^\dagger a|\Phi(t)\rangle = \langle\mathcal{P}|a|\Phi(t)\rangle \mapsto \langle\phi\rangle$ (remember the property $\langle\mathcal{P}|a^\dagger = \langle\mathcal{P}|$), whereas $\langle n^2\rangle = \langle\mathcal{P}|a^\dagger a a^\dagger a|\Phi(t)\rangle = \langle\mathcal{P}|a^\dagger(1+a^\dagger a)a|\Phi(t)\rangle \mapsto \langle\phi+\phi^2\rangle$. Accordingly, within the saddle-point approximation we are considering, $\langle n\rangle \mapsto \phi$, while $\langle n^2\rangle \mapsto \phi+\phi^2$. With this identification, Eq. (6) turns into Eq. (14). (The propagators of the theory can be read from the lecture notes, as they are actually independent of the form of the interaction.)

D. Response function

Consider a stochastic system of discrete interacting particles described by an initial state $|\Phi_0\rangle$ in the Fock space, with Hamiltonian H.

(14) Show that the average number n_0 of particles at time t and position x can be expressed as $\langle\phi(x,t)\rangle$ where

$$\langle\cdots\rangle = \int \mathcal{D}\phi\mathcal{D}\phi^* \cdots \mathrm{e}^{-\mathcal{A}[\phi,\phi^*]}; \tag{15}$$

$\mathcal{A}$ contains all the contributions discussed in the lectures, including the one of the initial distribution.

The mean number of particles can be expressed as $\langle\mathcal{P}|a_x^\dagger a_x|\Phi(t)\rangle = \langle\mathcal{P}|a_x|\Phi(t)\rangle$, which is a particular case of the general expression discussed at the lectures for the expectation value of a generic observable $\mathcal{O}(a^\dagger, a)$. The result then follows from the notes and $n_0 = \langle\phi(x,t)\rangle$.

(15) Consider now the case in which a particle is *added* at time 0 and position x' to the state $|\Phi_0\rangle$. Express (in terms of $\langle\mathcal{P}|$, H, a_x, and $a_{x'}^\dagger$) the average number n_1 of particles which can be measured at time t and position x in this perturbed state. Translate this result in terms of the path integral in Eq. (15).

The state $|\Phi_0'\rangle$ after having added a particle can be expressed as $|\Phi_0'\rangle = a_{x'}^\dagger|\Phi_0\rangle$ and therefore $n_1 = \langle\mathcal{P}|a_x^\dagger a_x|\Phi_0'(t)\rangle = \langle\mathcal{P}|a_x \mathrm{e}^{-Ht}|\Phi_0'\rangle = \langle\mathcal{P}|a_x \mathrm{e}^{-Ht} a_{x'}^\dagger|\Phi_0\rangle$. In terms of the path-integral, the latter expression translates as $n_1 = \langle\phi(x,t)\phi^*(x',0)\rangle$ as it can be easily seen by introducing an overcompleteness relation right before the temporal evolution.

(16) On the basis of the previous expressions show that the change $\delta n \equiv n_1 - n_0$ due to adding one particle to the initial state $|\Phi_0\rangle$ is given by

$$\delta n = \langle\phi(x,t)\tilde{\phi}(x',0)\rangle. \tag{16}$$

Accordingly, the correlation of fields on the r.h.s. of this equation expresses the *response function*.

By using the relations found at the previous points, one concludes that $\delta n = n_1 - n_0 = \langle\phi(x,t)\phi^*(x',0)\rangle - \langle\phi(x,t)\rangle = \langle\phi(x,t)[\phi^*(x',0)-1]\rangle = \langle\phi(x,t)\tilde{\phi}(x',0)\rangle$, q.e.d.