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Coherent States and Wavelets: A Unified Approach Pt 1

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Coherent States and Wavelets: A Unified Approach – I

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DEDICATION

I would like to humbly dedicate this set of lectures to the memory of Prof. Abdus Salam (1926 – 1996), the original inspirer and one of the founding fathers of the ICTP. His vision, indefatigable energy and selfless dedication to his dream of creating a world class research centre, primarily to help physicists in "development challenged" countries to step into the international arena of theoretical physics as equal players, has led to a flowering of generations of talented physicists and mathematicians from there. So many of us owe our professional careers to that idea that it is impossible to express adequately our indebtedness to the greatness of the man.

Scientific thought and its creation is the common and shared heritage of mankind.

- *Abdus Salam*

Abstract

In this series of lectures we shall study the group theoretical and functional analytic links between the continuous wavelet and the coherent sate transforms. We begin with a look at the canonical coherent states, which in a way gave rise to the entire field of coherent states and also pointed up the group theoretical basis of wavelet analysis.

Contents

References

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Notation

- $\mathfrak{H}$ = Hilbert space, assumed separable, infinite or finite dimensional.

- Scalar product of $\phi, \psi \in \mathfrak{H}$

$$\begin{array}{ccc} \langle \phi \mid \psi \rangle & = & (\psi, \phi) \\ \nearrow & & \nwarrow \\ \text{physicists} & & \text{mathematicians} \end{array}$$

- For $\phi, \psi \in \mathfrak{H}$, the rank one operator $T = |\phi\rangle\langle\psi|$ is defined to be:

$$T\chi = \langle \psi \mid \chi \rangle \phi, \qquad \chi \in \mathfrak{H}$$

- Operator integrals will be assumed to converge weakly:

$$f : X \longrightarrow \mathcal{L}(\mathfrak{H}), \qquad \text{and} \qquad (X, \mu) = \text{measure space}$$

Notation

then

$$I = \int_X f(x)\, d\mu(x)$$

is assumed to converge in the sense that

$$\int_X \langle \phi | f(x) \psi \rangle \, d\mu(x) < \infty, \quad \phi, \psi \in \mathfrak{H}$$

- $\mathcal{B}(X)$ = set of all Borel sets of X.

I will systematically use the physicists' notation!

Unless otherwise stated, we shall use the natural system of units, in which $c = \hbar = 1$.

Canonical coherent states

We start out by looking at the quintessential example of coherent states – the canonical coherent states.

It is fair to say that the entire subject of coherent states developed by analogy from this example.

This set of states, or rays in the Hilbert space of a quantum mechanical system, was originally discovered by Schrödinger in 1926, as a convenient set of quantum states for studying the transition from quantum to classical mechanics.

They are endowed with a remarkable array of interesting properties. Apart from initiating the discussion, this will also help us in motivating the various mathematical directions in which one can try to generalize the notion of a CS.

The also encapsulate the group theoretical basis for the development of the continuous wavelet transform.

Minimal uncertainty states

The quantum kinematics of a free *n*-particle system is based upon the existence of an irreducible representation of the canonical commutation relations (CCR),

$$[Q_i,\ P_j] = iI\delta_{ij}, \quad i, j = 1, 2, \ldots, n,$$

on a Hilbert space $\mathfrak{H}$. (Here I denotes the identity operator on $\mathfrak{H}$).

If n is finite, then according to the well-known uniqueness theorem of von Neumann, up to unitary equivalence, there exists only one irreducible representation of the CCR by self-adjoint operators, on a (separable, complex) Hilbert space.

Furthermore, the CCR imply that for any state vector ψ in $\mathfrak{H}$ (note, $\|\psi\| = 1$), the Heisenberg *uncertainty relations* hold:

$$\langle \Delta Q_i \rangle_\psi \ \langle \Delta P_i \rangle_\psi \geq \frac{1}{2}, \quad i = 1, 2, \ldots, n,$$

Minimal uncertainty states

where, for an arbitrary operator A on $\mathfrak{H}$,

$$\langle \Delta A \rangle_{\psi} = [\langle \psi | A^2 \psi \rangle - |\langle \psi A \psi \rangle|^2]^{\frac{1}{2}}$$

is its standard deviation in the state ψ .

As already pointed out by Schrödinger, there exists an entire family of states, $\eta^{\mathbf{s}}$ in the Hilbert space, labelled by a vector parameter $\mathbf{s} = (s_1, s_2, \ldots, s_n)$, $s_i > 0$, each one of which saturates the uncertainty relations

$$\langle \Delta Q_i \rangle_{\eta^{\mathbf{s}}} \langle \Delta P_i \rangle_{\eta^{\mathbf{s}}} = \frac{1}{2}, \quad i = 1, 2, \ldots, n.$$

We call these vectors minimal uncertainty states (MUSTs).

In the configuration space, or Schrödinger representation of the CCR, in which

$$\mathfrak{H} = L^2(\mathbb{R}^n, d\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2, \ldots, x_n),$$
$$(Q_i \psi)(\mathbf{x}) = x_i \psi(\mathbf{x}), \qquad (P_i \psi)(\mathbf{x}) = -i \frac{\partial}{\partial x_i} \psi(\mathbf{x}),$$

Minimal uncertainty states

the MUSTs, $\eta^{\mathbf{s}}$, are just the Gaussian wave packets

$$\eta^{\mathbf{s}}(\mathbf{x}) = \prod_{i=1}^{n} (\pi s_i^2)^{-\frac{1}{4}} \ \exp[-\frac{x_i^2}{2s_i^2}].$$

Not surprisingly, quantum systems in these states display behaviour very close to classical systems. More generally, there exists a larger family of states, namely gaussons or gaussian pure states which exhibits the minimal uncertainty property.

These latter states $\eta_{\mathbf{q},\mathbf{p}}^{U,V}$ are parametrized by two vectors, $\mathbf{q} = (q_1, q_2, \ldots, q_n),\ \mathbf{p} = (p_1, p_2, \ldots, p_n)\ \in \mathbb{R}^n$ and two real $n \times n$ matrices U and V, of which U is positive definite. In the Schrödinger representation,

$$\eta_{\mathbf{q},\mathbf{p}}^{U,V}(\mathbf{x}) = \pi^{-\frac{n}{4}} [\det U]^{\frac{1}{4}} \ \exp\left[i(\mathbf{x} - \frac{\mathbf{q}}{2}) \cdot \mathbf{p}\right] \ \exp\left[-\frac{1}{2}(\mathbf{x} - \mathbf{q}) \cdot (U + iV)(\mathbf{x} - \mathbf{q})\right].$$

Minimal uncertainty states

In the optical literature, states of the type $\eta^{U,V}_{\mathbf{q},\mathbf{p}}$, for which U is a diagonal matrix but *not* the identity matrix, are called squeezed states.

Note that when $\mathbf{q} = \mathbf{p} = 0$ and U is diagonal, with eigenvalues $1/s_i^2$, $i = 1, 2, \ldots, n$, the gaussons are exactly the MUSTs above.

Moreover, if T denotes the orthogonal matrix which diagonalizes U, i.e., $TUT^{-1} = D$, where D is the matrix of eigenvalues of U, then defining the vectors $\mathbf{x}' = T\mathbf{x}$, $\mathbf{q}' = T\mathbf{q}$, $\mathbf{p}' = T\mathbf{p}$, and the matrix $V' = TVT^{-1}$, we may rewrite $\eta^{U,V}_{\mathbf{q},\mathbf{p}}(\mathbf{x})$ as

$$\eta^{D,V'}_{\mathbf{q}',\mathbf{p}'}(\mathbf{x}') = \pi^{-\frac{n}{4}}[\det D]^{\frac{1}{4}} \exp[i(\mathbf{x}' - \frac{\mathbf{q}'}{2}) \cdot \mathbf{p}'] \exp[-\frac{1}{2}(\mathbf{x}' - \mathbf{q}') \cdot (D + iV')(\mathbf{x}' - \mathbf{q}')].$$

It is clear from this relation that, if Q_i', P_i', $i = 1, 2, \ldots, n$, are the components of the rotated vector operators, $\mathbf{Q}' = T^{-1}\mathbf{Q}$, $\mathbf{P}' = T^{-1}\mathbf{P}$, where, $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_n)$, $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ are the vector operators of position and momentum, respectively.

$$\langle \Delta Q_i' \rangle_{\eta^{U,V}_{\mathbf{q},\mathbf{p}}} \langle \Delta P_i' \rangle_{\eta^{U,V}_{\mathbf{q},\mathbf{p}}} = \frac{1}{2}, \quad i = 1, 2, \ldots, n.$$

To examine some properties of the MUSTs, take $n = 1$, and define the creation and annihilation operators,

$$a^\dagger = \frac{1}{\sqrt{2}}(s^{-1}Q - isP), \qquad a = \frac{1}{\sqrt{2}}(s^{-1}Q + isP),$$
$$[a,\ a^\dagger] = 1.$$

Using these operators and the MUST η^s, for a fixed $s \in \mathbb{R}$, we can generate a very interesting class of other MUSTs.

The MUST as a coherent state

To do so, define the complex variable

$$z = x + iy = \frac{1}{\sqrt{2}}(s^{-1}q + isp), \qquad (q, p) \in \mathbb{R}^2$$

and write

$$\eta^s = |0\rangle.$$

Note that $a|0\rangle = 0$.

Also let $\{|n\rangle\}_{n=0}^{\infty}$ be the normalized eigenstates of the number operator $N = a^\dagger a$:

$$N|n\rangle = n|n\rangle, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle, \quad \langle m|n\rangle = \delta_{mn}.$$

Then, for all $z \in \mathbb{C}$, the set of states in $\mathfrak{H}$,

$$|z\rangle = \exp\,[-\frac{|z|^2}{2} + za^\dagger]\,|0\rangle = \exp\,[-\frac{|z|^2}{2}]\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle,$$

The MUST as a coherent state

have the eigenvalue property

$$a|z\rangle = z\ |z\rangle.$$

It is straightforward to verify that each one of these states $|z\rangle$ is again a MUST.
Suppose now that we have a quantized electromagnetic field (in a box), and let $a_k^\dagger, a_k,\ \ k = 0, \pm 1, \pm 2, \ldots$, be the creation and annihilation operators for the various Fourier modes k. Then in the states

$$|\{z_k\}\rangle = \bigotimes_k |z_k\rangle,$$

the electromagnetic field behaves "classically". More precisely, the correlation functions for the field factorize in these states.

The MUST as a coherent state

Thus, let $x = (\mathbf{x}, t)$ be a space-time point and $\mathbf{E}^+(x)$ the positive frequency part of the quantized electric field (note: $\mathbf{E}^-(x) = \mathbf{E}^+(x)^*$ is the negative frequency part of the field).

Then,

$$\mathbf{E}^+(x)\,|\{z_k\}\rangle = \underline{\mathcal{E}}(x)\,|\{z_k\}\rangle,$$

where $\underline{\mathcal{E}}$ is a 3-vector valued function of x, giving the observed field strength at the point x.

Let ρ be the density matrix,

$$\rho = |\{z_k\}\rangle\langle\{z_k\}|,$$

and $G^{(n)}_{\mu_1,\mu_2,\ldots,\mu_{2n}}$ the correlation functions,

$$G^{(n)}_{\mu_1,\mu_2,\ldots,\mu_{2n}}(x_1, x_2, \ldots, x_{2n}) = \mathrm{Tr}[\rho E^-_{\mu_1}(x_1)\ldots E^-_{\mu_n}(x_n)E^+_{\mu_{n+1}}(x_{n+1})\ldots E^+_{\mu_{2n}}(x_{2n})],$$

The MUST as a coherent state

where $E^{\pm}_{\mu_k}$ denotes the μ_k-th component of $\mathbf{E}^{\pm}$. It is then easily verified that

$$G^{(n)}_{\mu_1,\mu_2,\ldots,\mu_{2n}}(x_1, x_2, \ldots, x_{2n}) = \prod_{k=1}^{n} \overline{\mathcal{E}}_{\mu_k}(x_k) \prod_{\ell=n+1}^{2n} \mathcal{E}_{\mu_\ell}(x_\ell).$$

It is because of this factorizability property that the states $|\{z_k\}\rangle$ or the MUSTs $|z\rangle$ were called coherent states.

However, in the current mathematical literature (though not always in the optical literature), the term coherent state is used to designate an entire array of other mathematically related states, which do not necessarily display either the factorizability property or the minimal uncertainty property.

We shall reserve the term canonical coherent states for these MUSTs .

The MUST as a coherent state

In order to bring out some additional properties of the canonical CS $|z\rangle$, let us write

$$|\overline{z}\rangle = \eta^s_{\sigma(q,p)},$$

where $\overline{z}$ is the complex conjugate of z and z and q, p are related as above. The significance of the σ in this notation will become clear in a while.

A short computation shows that

$$\begin{aligned}\langle \eta^s_{\sigma(q,p)}|Q|\eta^s_{\sigma(q,p)}\rangle &= q, \\ \langle \eta^s_{\sigma(q,p)}|P|\eta^s_{\sigma(q,p)}\rangle &= p.\end{aligned}$$

In other words, the MUST $\eta^s_{\sigma(q,p)}$ is a translated Gaussian wave packet, centered at the point q in position and p in momentum space.

Explicitly, as a vector in $L^2(\mathbb{R}, dx)$,

$$\eta^s_{\sigma(q,p)}(x) = (\pi s^2)^{-\frac{1}{4}}\ \exp\left[-i\left(\frac{q}{2} - x\right)p\right]\ \exp\left[-\frac{(x-q)^2}{2s^2}\right].$$

Some group theoretical properties

A group theoretical property of $|z\rangle$ emerges if we use the Baker-Campbell-Hausdorff identity,

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}\; e^{A} e^{B},$$

for two operators A, B, the commutator $[A, B]$ of which commutes with both A and B, and the fact that $a^n|0\rangle = 0, \quad n \geq 1$, to write $|z\rangle$ as

$$|z\rangle = \exp[-\frac{1}{2}|z|^2]\; e^{za^\dagger} e^{\overline{z}a}|0\rangle = e^{za^\dagger - \overline{z}a}|0\rangle.$$

In terms of (q, p) we have

$$\eta^s_{\sigma(q,p)} = e^{i(pQ-qP)}\eta^s \equiv U(q, p)\eta^s,$$

where, $\forall (q, p) \in \mathbb{R}^2$, the operators $U(q, p) = e^{i(pQ-qP)}$ are, of course, unitary.

Some group theoretical properties

Moreover, we have the integral relation,

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |\eta^s_{\sigma(q,p)}\rangle\langle\eta^s_{\sigma(q,p)}| \; dqdp = I,$$

The convergence of the above integral is in the weak sense, i.e., for any two vectors ϕ, ψ in the Hilbert space $\mathfrak{H}$,

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \langle\phi|\eta^s_{\sigma(q,p)}\rangle\langle\eta^s_{\sigma(q,p)}|\psi\rangle \; dqdp = \langle\phi|\psi\rangle.$$

This relation is called the resolution of the identity generated by the canonical CS.
The operators $U(q, p)$ arise from a unitary, irreducible representation (UIR) of the Weyl-Heisenberg group, G_{WH}, which is a central extension of the group of translations of the two-dimensional Euclidean plane.

Some group theoretical properties

The UIR in question is the unitary representation of G_{WH} which integrates the CCR . An arbitrary element g of G_{WH} is of the form

$$g = (\theta, q, p), \quad \theta \in \mathbb{R}, \quad (q, p) \in \mathbb{R}^2,$$

with multiplication law,

$$g_1 g_2 = (\theta_1 + \theta_2 + \xi((q_1, p_1); (q_2, p_2)),\ q_1 + q_2,\ p_1 + p_2),$$

where ξ is the multiplier function

$$\xi((q_1, p_1); (q_2, p_2)) = \frac{1}{2}(p_1 q_2 - p_2 q_1).$$

Any infinite-dimensional UIR, U^λ, of G_{WH} is characterized by a real number $\lambda \neq 0$ and may be realized on the same Hilbert space $\mathfrak{H}$, as the one carrying an irreducible representation of the CCR:

Some group theoretical properties

$$U^\lambda(\theta, q, p) = e^{i\lambda\theta} U^\lambda(q, p) := e^{i\lambda(\theta - \frac{pq}{2})} e^{i\lambda pQ} e^{-i\lambda qP}.$$

If $\mathfrak{H} = L^2(\mathbb{R}, dx)$, these operators are defined by the action

$$(U^\lambda(\theta, q, p)\phi)(x) = e^{i\lambda\theta} e^{i\lambda p(x - \frac{q}{2})}\phi(x - q), \qquad \phi \in L^2(\mathbb{R}, dx).$$

Thus, the three operators, I, Q, P, appear now as the infinitesimal generators of this representation and are realized as:

$$(Q\phi)(x) = x\phi(x), \quad (P\phi)(x) = -\frac{i}{\lambda}\frac{\partial \phi}{\partial x}(x), \qquad [Q, P] = \frac{i}{\lambda} I.$$

For our purposes, we take for λ the specific value, $\lambda = \frac{1}{\hbar} = 1$, and simply write U for the corresponding representation.

We now take the phase subgroup of G_{WH}:

$$\Theta = \{g = (\theta, 0, 0) \mid \theta \in \mathbb{R}\}.$$

Some group theoretical properties

Then the left coset space G_{WH}/Θ can be identified with $\mathbb{R}^2$ and a general element in it parametrized by (q, p).

In terms of this parametrization, G_{WH}/Θ carries the invariant measure

$$d\nu(q, p) = \frac{dqdp}{2\pi}.$$

The function

$$\sigma : G_{WH}/\Theta \to G_{WH}, \quad \sigma(q, p) = (0, q, p),$$

then defines a section in the group G_{WH}, now viewed as a fibre bundle, over the base space G_{WH}/Θ, having fibres isomorphic to Θ.

Thus, the family of canonical CS is the set,

$$\mathfrak{S}_\sigma = \{\eta^s_{\sigma(q,p)} = U(\sigma(q, p))\eta^s \mid (q, p) \in G_{WH}/\Theta\},$$

Some group theoretical properties

and the resolution of the identity becomes

$$\int_{G_{WH}/\Theta} |\eta^s_{\sigma(q,p)}\rangle\langle\eta^s_{\sigma(q,p)}| \, d\nu(q,p) = I.$$

In other words, the CS $\eta^s_{\sigma(q,p)}$ are labelled by the points (q,p) in the homogeneous space G_{WH}/Θ of the Weyl-Heisenberg group,

and they are obtained by the action of the unitary operators $U(\sigma(q,p))$, of a UIR of G_{WH}, on a fixed vector $\eta^s \in \mathfrak{H}$.

The resolution of the identity equation is then a statement of the square-integrability of the UIR, U, with respect to the homogeneous space G_{WH}/Θ.

This way of looking at coherent states turns out to be extremely fruitful.

Some group theoretical properties

Indeed, one could ask if it might not be possible to use this idea to generalize the notion of a CS and to build families of such states, using UIR's of groups other than the Weyl-Heisenberg group, making sure in the process that the basic ingredients that went into this construction are also present in the general setting.

This is indeed possible, and such an approach yields a powerful generalization of the notion of a coherent state.

Two remarks are in order before proceeding.

First and not surprisingly, the same canonical CS may be obtained from the oscillator group $H(4)$, which is the group with the Lie algebra generated by $\{a, a^\dagger, N = a^\dagger a, I\}$.

Secondly, it is interesting that the canonical CS are widely used in signal processing, where they generate the so-called windowed Fourier transform or Gabor transform.

This is a hint that CS will have an important role in classical physics as well as in quantum physics, and as a matter of fact they may be viewed as a natural bridge between the two.

Furthermore, some of the functional analytic properties of the CS, that we will now study, also turn out to be useful in the context of non-commutative geometries, in particular, non-commutative quantum mechanics.

Some functional analytic properties

The resolution of the identity leads to some interesting functional analytic properties of the CS, $\eta^s_{\sigma(q,p)}$. These properties can be studied in their abstract forms and be used to obtain a generalization of the notion of a CS, but now independently of any group theoretical implications.

Let $\tilde{\mathfrak{H}} = L^2(G_{WH}/\Theta,\ d\nu)$ be the Hilbert space of all complex valued functions on G_{WH}/Θ which are square integrable with respect to $d\nu$. Then the resolution of the identity implies that functions $\Phi : G_{WH}/\Theta \to \mathbb{C}$ of the type

$$\Phi(q,p) = \langle \eta^s_{\sigma(q,p)} | \phi \rangle,$$

for $\phi \in \mathfrak{H}$, define elements in $\tilde{\mathfrak{H}}$, and moreover, writing $W : \mathfrak{H} \to \tilde{\mathfrak{H}}$ for the linear map which associates an element ϕ in $\mathfrak{H}$ to an element Φ in $\tilde{\mathfrak{H}}$ (i.e., $W\phi = \Phi$), we see that W is linear isometry:

$$\|W\phi\|^2 = \|\Phi\|^2 = \int_{G_{WH}/\Theta} |\Phi(q,p)|^2 \, d\nu(q,p) = \|\phi\|^2.$$

Some functional analytic properties

The range of this isometry, which we denote by $\mathfrak{H}_K$,

$$\mathfrak{H}_K = W\mathfrak{H} \subset \tilde{\mathfrak{H}},$$

is a closed subspace of $\tilde{\mathfrak{H}}$ and furthermore, it is a reproducing kernel Hilbert space.
To understand the meaning of this, consider the fuction $K(q, p;\ q', p')$ defined on $G_{WH}/\Theta \times G_{WH}/\Theta$:

$$\begin{aligned}
K(q, p;\ q', p') &= \langle \eta^s_{\sigma(q,p)} | \eta^s_{\sigma(q',p')} \rangle \\
&= \exp\left[-\frac{i}{2}(pq' - p'q)\right] \exp\left[-\frac{s^2}{4}(p - p')^2\right] \exp\left[-\frac{1}{4s^2}(q - q')^2\right] \\
&= \exp\left[z\overline{z}' - \frac{1}{2}|z|^2 - \frac{1}{2}|z'|^2\right] \\
&= \langle z' | z \rangle = K(\overline{z}', z),
\end{aligned}$$

Some functional analytic properties

The function K is a reproducing kernel, in view of the property:

$$\Phi(q,p) = \int_{G_{WH}/\Theta} K(q,p;\ q',p')\,\Phi(q',p')\,d\nu(q',p')\ , \quad \forall\Phi\in\mathfrak{H}_K\ .$$

The function K enjoys the properties:

1. Hermiticity,

$$K(q,p;\ q',p') = \overline{K(q',p';\ q,p)}.$$

2. Positivity,

$$K(q,p;\ q,p) > 0.$$

3. Idempotence,

$$\int_{G_{WH}/\Theta} K(q,p;\ q'',p'')K(q'',p'';\ q',p')\,d\nu(q'',p'') = K(q,p;\ q',p').$$

Some functional analytic properties

The above relations hold for all $(q,p),(q',p') \in G_{WH}/\Theta$. Condition 3, of idempotence is also called the square integrability property of K.

All three relations are the transcription of the fact that the orthogonal projection operator $\mathbb{P}_K$ of $\tilde{\mathfrak{H}}$ onto $\mathfrak{H}_K$ is an integral operator, with kernel $K(q,p;\ q',p')$.

The kernel K actually determines the Hilbert space $\mathfrak{H}_K$. Indeed,

$$(W\eta^s_{\sigma(q',p')})(q,p) = K(q,p;\ q',p');$$

in other words, for fixed (q',p'), the function $(q,p) \mapsto K(q,p;\ q',p')$ is simply the image in $\mathfrak{H}_K$ of the CS $\eta^s_{\sigma(q',p')}$ under the isometry W.

Additionally, if Φ is an element of the Hilbert space $\mathfrak{H}_K$, it is necessarily of the form $\Phi(q,p) = \langle \eta^s_{\sigma(q,p)} | \phi \rangle$. The resolution of the identity then implies,

$$\phi = \int_{G_{WH}/\Theta} \Phi(q,p)\eta^s_{\sigma(q,p)}\, d\nu(q,p).$$

Some functional analytic properties

This shows that the set of vectors $\eta^s_{\sigma(q,p)}$, $(q,p) \in G_{WH}/\Theta$, is overcomplete in $\mathfrak{H}$ and hence, since W is an isometry, the set of vectors

$$\xi_{\sigma(q,p)} = W\eta^s_{\sigma(q,p)}, \qquad \xi_{\sigma(q,p)}(q',p') = K(q',p';q,p), \quad \forall (q,p) \in G_{WH}/\Theta$$

is overcomplete in $\mathfrak{H}_K$.

Note that the vectors $\xi_{\sigma(q,p)}$ are the same CS as the $\eta^s_{\sigma(q,p)}$, but now written as vectors in the Hilbert space of functions $\mathfrak{H}_K$.

The term overcompleteness is to be understood in the following way: Since $\mathfrak{H}$ is a separable Hilbert space, it is always possible to choose a countable basis $\{\eta_i\}_{i=1}^{\infty}$ in it, and to express any vector $\phi \in \mathfrak{H}$ as a linear combination of these.

By contrast, the family of CS, $\mathfrak{S}_\sigma$ in is labelled by a pair of continuous parameters (q,p), and the resolution of the identity is also a statement of the fact that any vector ϕ can be expressed in terms of the vectors in this family.

As a remark, for a given $\phi \in \mathfrak{H}$, the function $\Phi(q,p) = \langle \eta^s_{\sigma(q,p)} \mid \phi \rangle$ which is an element of $L^2(G_{WH}/\Theta,\ d\nu)$ is also known as the coherent state transform of the vector ϕ.

Some functional analytic properties

Clearly, it should be possible to choose a countable set of vectors $\{\eta^s_{\sigma(q_i,p_i)}\}_{i=1}^{\infty}$ from $\mathfrak{S}_\sigma$ and still obtain a basis for $\mathfrak{H}$. This is in fact possible and many different discretizations exist.

The most familiar situation is that where the set of points $\{q_i, p_i\}$ is a lattice, such that the area of the unit cell is smaller than a critical value (to be sure, the resulting set of CS is then overcomplete).

The determination of adequate subsets $\{q_i, p_i\}$ leads to very interesting mathematical problems, for instance in number theory and in the theory of analytic functions. These considerations are part of the general problem of CS discretization.

The equation $\Phi(q,p) = \langle \eta^s_{\sigma(q,p)} | \phi \rangle$ also implies a boundedness property for the functions Φ in the reproducing kernel Hilbert space $\mathfrak{H}_K$. Indeed,

$$|\Phi(q,p)| \leq \|\eta\| \, \|\phi\|, \quad \forall (q,p) \in G_{WH}/\Theta,$$

Some functional analytic properties

Thus, the vectors in $\mathfrak{H}_K$ are all bounded functions. More importantly, the linear map

$$E_K(q, p) : \mathfrak{H}_K \to \mathbb{C}, \qquad E_K(q, p)\Phi = \Phi(q, p),$$

which simply evaluates each function $\Phi \in \mathfrak{H}_K$ at the point (q, p), and hence called an evaluation map, is continuous.

This can in fact can be taken to be the defining property of a reproducing kernel Hilbert space and used to arrive at a family of coherent states.

The CS $\eta^s_{\sigma(q,p)}$, along with the resolution of the identity relation can be used to obtain a useful family of localization operators on the phase space $\Gamma = G_{WH}/\Theta$.

Indeed, the relations $\langle \eta^s_{\sigma(q,p)} | Q | \eta^s_{\sigma(q,p)} \rangle = q$ and $\langle \eta^s_{\sigma(q,p)} | P | \eta^s_{\sigma(q,p)} \rangle = p$, which we obtained earlier, tend to indicate that the CS $\eta^s_{\sigma(q,p)}$ do in some sense describe the localization properties of the quantum system in the phase space Γ.

To pursue this point a little further, denote by Δ an arbitrary Borel set in Γ, considered as a measure space, and let $\mathcal{B}(\Gamma)$ denote the σ-algebra of all Borel sets of Γ.
Define the positive, bounded operator

$$a(\Delta) = \int_{\Delta} |\eta^s_{\sigma(q,p)}\rangle\langle\eta^s_{\sigma(q,p)}| \, d\nu(q,p).$$

This family of operators, as Δ runs through $\mathcal{B}(\Gamma)$, enjoys certain measure theoretical properties:

1. If J is a countable index set and Δ_i, $i \in J$, are mutually disjoint elements of $\mathcal{B}(\Gamma)$, i.e., $\Delta_i \cap \Delta_j = \emptyset$, for $i \neq j$ ($\emptyset$ denoting the empty set), then

$$a(\cup_{i\in J}\Delta_i) = \sum_{i\in J} a(\Delta_i),$$

the sum being understood to converge weakly.

Some functional analytic properties

2. Normalization:

$$a(\Gamma) = I, \qquad \text{also} \quad a(\emptyset) = 0.$$

Such a family of operators $a(\Delta)$ is said to constitute a normalized, positive operator-valued (POV) measure on $\mathfrak{H}$.

Using the isometry W and the CS $\xi_{\sigma(q,p)}$, we obtain the normalized POV-measure $a_K(\Delta)$ on $\mathfrak{H}_K$:

$$a_K(\Delta) = \int_\Delta |\xi_{\sigma(q,p)}\rangle\langle\xi_{\sigma(q,p)}| \, d\nu(q,p) = Wa(\Delta)W^*.$$

Note that

$$a_K(\Gamma) = \int_{G_{WH}/\Theta} |\xi_{\sigma(q,p)}\rangle\langle\xi_{\sigma(q,p)}| \, d\nu(q,p) = \mathbb{P}_K,$$

where $\mathbb{P}_K$ is the projection operator, $\mathfrak{H}_K = \mathbb{P}_K\widetilde{\mathfrak{H}}$.

Some functional analytic properties

If $\Psi \in \mathfrak{H}_K$ is an arbitrary state vector, and $\Psi = W\psi, \;\; \psi \in \mathfrak{H}$, then

$$\langle \Psi | a_K(\Delta)\Psi \rangle = \langle \psi | a(\Delta)\psi \rangle = \int_\Delta |\Psi(q,p)|^2 \, d\nu(q,p).$$

This means that if $\Psi(q,p)$ is considered as being the phase space wave function of the system, then $a_K(\Delta)$ is the operator of localization in the region Δ of phase space.
Of course, to interpret $|\Psi(q,p)|^2$ as a phase space probability density, an appropriate concept of joint measurement of position and momentum has to be developed. Here is an interesting fact reinforcing the interpretation of $a_K(\Delta)$ as a localization operators. On $\mathfrak{H}_K$ define the unbounded operators Q_K , P_K:

$$\begin{aligned} \langle \Psi | Q_K \Phi \rangle &= \int_{\mathbb{R}^2} \overline{\Psi(q,p)} \, q\Phi(q,p) \, d\nu(q,p), \\ \langle \Psi | P_K \Phi \rangle &= \int_{\mathbb{R}^2} \overline{\Psi(q,p)} \, p\Phi(q,p) \, d\nu(q,p), \end{aligned}$$

Some functional analytic properties

on vectors Ψ, Φ chosen from appropriate dense sets in $\mathfrak{H}_K$.

Then it can be shown that

$$[Q_K,\ P_K] = iI_K, \quad I_K = \text{identity operator on } \ \mathfrak{H}_K.$$

Thus, multiplication by q and p, respectively, yield the position and momentum operators on $\mathfrak{H}_K$.

Mathematically, the virtue of the above functional analytic description of the coherent states $\eta^s_{\sigma(q,p)}$ is that it points up another possibility of generalization.

We would like to associate CS to arbitrary reproducing kernel Hilbert spaces. This can be done and the fact can be studied in its own right.

This will also leads to the theory of frames.

A complex analytic viewpoint

To bring out some complex analytic properties of the canonical CS, let $\phi \in \mathfrak{H}$ be an arbitrary vector. Computing its scalar product with the CS $|\overline{z}\rangle$ we get

$$\langle \overline{z}|\phi\rangle = \exp[-\frac{|z|^2}{2}] \sum_{n=0}^{\infty} \frac{\langle n|\phi\rangle}{\sqrt{n!}} z^n = \exp[-\frac{|z|^2}{2}] f(z).$$

Here f is an analytic function of the complex variable z. In terms of $z, \overline{z}$ we may write

$$d\nu(q,p) = \frac{dq \wedge dp}{\pi} = \frac{dz \wedge d\overline{z}}{2\pi i},$$

and let us define the new measure

$$d\mu(z, \overline{z}) = \exp[-|z|^2] \frac{dz \wedge d\overline{z}}{2\pi i}.$$

In measure theoretic terms, the quantity $idz \wedge d\overline{z}/2$ simply represents the Lebesgue measure $dxdy$, $z = x + iy$, on $\mathbb{C}$.

A complex analytic viewpoint

We see that $\mathfrak{H}_K$ can be identified with the Hilbert space of all analytic functions in z which are square-integrable with respect to $d\mu$.

Let $\mathfrak{H}_{hol}$ denote this Hilbert space. Then, the linear map

$$W_{hol} : \mathfrak{H} \to \mathfrak{H}_{hol} \ , \qquad (W_{hol}\phi)(z) = \exp\left[\frac{|z|^2}{2}\right] \langle \overline{z}|\phi\rangle,$$

is an isometry.

The vectors

$$\zeta_{\overline{z}} = W_{hol}\ \eta^s_{\sigma(q,p)} = W_{hol}\ |\overline{z}\rangle$$

are the images of the $\eta^s_{\sigma(q,p)}$ in $\mathfrak{H}_{hol}$.

They represent the analytic functions

$$\zeta_{\overline{z}}(z') = e^{z'\overline{z}} = \exp\left[\frac{1}{2}(|z|^2 + |z'|^2)\right] K(z', \overline{z}).$$

A complex analytic viewpoint

From this it is clear that the function $K_{hol} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$,

$$K_{hol}\,(z', \overline{z}) = \langle \zeta_{\overline{z}'} | \zeta_{\overline{z}} \rangle_{\mathfrak{H}_{hol}} = e^{z'\overline{z}},$$

is a reproducing kernel for $\mathfrak{H}_{hol}$. Indeed, for any $f \in \mathfrak{H}_{hol}$ and $z \in \mathbb{C}$,

$$\int_{\mathbb{C}} K_{hol}(z, \overline{z}') f(z')\, d\mu(z', \overline{z}') = f(z) = \langle \zeta_{\overline{z}} | f \rangle_{\mathfrak{H}_{hol}}.$$

Note that

$$K_{hol}(z', \overline{z}) = \exp\,[\frac{1}{2}(|z|^2 + |z'|^2)]\; K(z', \overline{z}).$$

Furthermore, the vectors $\zeta_{\overline{z}}$ satisfy the resolution of the identity relation on $\mathfrak{H}_{hol}$,

$$\int_{\mathbb{C}} |\zeta_{\overline{z}}\rangle\langle\zeta_{\overline{z}}|\, d\mu(z, \overline{z}) = I_{\mathfrak{H}_{hol}}.$$

A complex analytic viewpoint

The representation $\exp\,[-|z|^2/2]\,\zeta_{\overline{z}}$ of the CS in terms of the analytic functions $\zeta_{\overline{z}} \in \mathfrak{H}_{hol}$ is known among physicists as the Fock-Bargmann representation, and the Hilbert space $\mathfrak{H}_{hol}$ as the Bargmann space of entire analytic functions. In the mathematical literature, such spaces are generally called Bergman spaces.

The operators a, $a^\dagger$, in this representation, are given by

$$(af)(z) = \frac{\partial f}{\partial z}(z), \qquad (a^\dagger f)(z) = zf(z), \qquad f \in \mathfrak{H}_{hol}.$$

The basis vectors $|n\rangle \in \mathfrak{H}$ are mapped by W_{hol} to the vectors

$$W_{hol}\,|n\rangle = u_n, \qquad u_n(z) = \frac{z^n}{\sqrt{n!}},$$

in $\mathfrak{H}_{hol}$. Additionally,

$$K_{hol}(z', \overline{z}) = \sum_{n=0}^{\infty} u_n(z')\overline{u_n(z)}.$$

Some geometrical considerations

As already pointed out, the existence of the CS $\zeta_{\overline{z}}$ can be traced back to certain intrinsic geometrical properties of $\mathbb{C}$, considered as a one-dimensional, complex Kähler manifold. Without discussing this notion in depth here, we may still look at a few main features of this property.

To begin with, $\mathbb{C}$ may be thought of as being either a one-dimensional complex manifold or a two-dimensional real manifold $\mathbb{R}^2$, equipped with a complex structure.

In the first case, one works with the holomorphic coordinate z (or the antiholomorphic coordinate $\overline{z}$). In the second case, one uses the real coordinates q, p.

Considered as a real manifold, $\mathbb{R}^2$ is symplectic, i.e., it comes equipped with a closed, non-degenerate two-form

$$\Omega = dq \wedge dp = \frac{1}{i} dz \wedge d\overline{z},$$

Some geometrical considerations

while considered as a complex manifold, $\mathbb{C}$ admits the Kähler potential function:

$$\Phi(z', \overline{z}) = z'\overline{z},$$

from which the two-form emerges upon differentiation:

$$\Omega = \frac{1}{i}\frac{\partial^2 \Phi(z, \overline{z})}{\partial z \partial \overline{z}}\, dz \wedge d\overline{z}.$$

Similarly, the Kähler potential also determines the reproducing kernel:

$$K_{hol}(z', \overline{z}) = \exp\left[\Phi(z', \overline{z})\right],$$

while the measure $d\mu$, defining the Hilbert space $\mathfrak{H}_{hol}$ of holomorphic functions, is given in terms of it by

$$d\mu(z, \overline{z}) = \exp\left[-\Phi(z, \overline{z})\right] \frac{dz \wedge d\overline{z}}{2\pi i}.$$

Some geometrical considerations

Continuing, if we define the complex one-form

$$\Theta = -i\partial_{\overline{z}}\Phi(z, \overline{z}) = -iz d\overline{z},$$

we get

$$\Omega = \partial_z \Theta,$$

where $\partial_z, \partial_{\overline{z}}$ denote (exterior) differentiation with respect to z and $\overline{z}$, respectively.
It appears therefore, that it is the Kähler structure of $\mathbb{C}$, (or the fact that it comes equipped with the Kähler potential Φ) which leads to the existence of the Hilbert space $\mathfrak{H}_{hol}$ of holomorphic functions and consequently, the CS $\zeta_{\overline{z}}$ (the appearance of these latter being a consequence of the continuity of the evaluation map.
Once again, this situation is generic to all Kähler manifolds.

Some geometrical considerations

Let $\mathbb{P}(z)$ be the one dimensional projection operator onto the vector subspace of $\mathfrak{H}_{hol}$ generated by the vector $\zeta_{\overline{z}}$, and denote this subspace by $\mathfrak{H}_{hol}(z)$.

The collection of all these one-dimensional subspaces, as z ranges over $\mathbb{C}$, defines a (holomorphic) line bundle over the manifold $\mathbb{C}$ – a structure which is intimately related to the existence of a geometric prequantization of $\mathbb{C}$.

However, while a complex Kähler structure is in some sense ideally suited to the existence of a geometric prequantization, a family of CS may define a geometric prequantization even in the absence of such a structure.

A quantization problem

As an example of an application of the canonical CS we look at a quantization problem of a simple classical system.

In classical mechanics, observables are real valued functions on phase space and they form an algebra with respect to a product defined by the Poisson bracket. The observables of quantum mechanics are self-adjoint operators on a Hilbert space, forming an algebra with respect to the commutator bracket, divided by $i\hbar$.

A quantization of a classical system is a linear map $f \mapsto O_f$ of the classical observables f to self-adjoint operators O_f in a way such that the Poisson bracket $\{f, g\}$ of two classical observables is mapped to $\frac{1}{i\hbar}[O_f, O_g]$.

One also tries to ensure, in the process, that some particular subalgebra of the quantized observables, chosen for physical reasons, be irreducibly realized on the Hilbert space.

A quantization problem

Consider a classical particle of mass m, having a single degree of freedom, moving on the configuration space $\mathbb{R}$ and having the phase space $\mathbb{R}^2$.
On the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}, dx)$ take the set of CS

$$\eta_{\sigma(q,p)} = \exp[\frac{i}{\hbar}(x - \frac{q}{2})p]\eta(x - q),$$

and corresponding to a function f of the variables (q, p), define the formal operator:

$$O_f = \int_{\mathbb{R}^2} f(q,p)|\eta_{\sigma(q,p)}\rangle\langle\eta_{\sigma(q,p)}| \, d\nu(q,p), \qquad d\nu(q,p) = \frac{dqdp}{2\pi\hbar}$$

In general, the operator O_f defined in this way will be unbounded and technical questions involving domains have to be addressed.
However, assuming that O_f can be defined on a dense set, its action on a vector ϕ, taken from this set is given by the integral operator relation:

A quantization problem

$$(O_f\phi)(x) = \frac{1}{h}\int_{\mathbb{R}^2} dqdp\; f(q,p)\;[\int_{\mathbb{R}} dx'\; e^{-\frac{i}{\hbar}(x'-x)p}\eta(x'-q)\phi(x')]\;\eta(x-q).$$

From this it follows that if $f(q,p) = f(q)$ is a function of q alone, then O_f is the operator of multiplication by the function $|\eta|^2 * f$ (the asterisk denotes a convolution):

$$|\eta|^2 * f(x) = \int_{\mathbb{R}} |\eta(x-q)|^2 f(q)\; dq.$$

Similarly, if $f(q,p) = f(p)$ is a function of p alone then O_f is the (in general pseudo-) differential operator

$$O_f = f(-i\hbar\frac{\partial}{\partial x}),$$

(formally, if $f(q)$ is written as a power series in q, then O_f is obtained by replacing q by $-i\hbar\dfrac{\partial}{\partial x}$).

A quantization problem

In particular, taking $f(q) = q$ and $f(p) = p$, we get:

$$(O_q\phi)(x) = x\phi(x), \qquad (O_p\phi)(x) = -i\hbar\frac{\partial\phi}{\partial x}(x),$$

while if $f = H$, the harmonic oscillator Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{m^2\omega^2}{2}q^2,$$

then

$$O_H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m^2\omega^2}{2}x^2 + C,$$

where C is the constant

$$C = \frac{\sqrt{\pi}}{4}\hbar^{\frac{3}{2}}m^2\omega^2.$$

which simply changes the ground state energy.

A quantization problem

We see in this example, that this method of quantization yields the expected result, in that the Poisson bracket $\{q, p\}$ is properly mapped to the commutator bracket $\dfrac{1}{i\hbar}[O_q, O_p]$, and the algebra generated by O_q, O_p and I is irreducibly represented on $\mathfrak{H}$.
Thus, formally at least, the use of the canonical CS leads to the same quantization result as ordinarily obtained by making the substitutions, $q \to$ "multiplication by x" and $p \to -i\hbar \dfrac{\partial}{\partial x}$.
The method is quite general and can be applied to a large number of physical situations.

Outlook

We have quickly gleaned through a number of illustrative properties of the canonical coherent states. Each one of these properties can be taken as the starting point for a generalization of the notion of a CS.

From a purely physical point of view, for example, it could be useful to look for generalizations which preserve the minimal uncertainty property. In doing so, it is useful to exploit some of the group theoretical properties as well.

Mathematical generalizations could be based on group theoretical, analytic or related geometrical properties.

What we shall study next is the relationship between such a group theoretical generalization and the continuous wavelet transform.