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Coherent states, POVM, quantization and measurement

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Contents

[1] H Bergeron and J.-P G. Integral quantizations with two basic examples, *Annals of Physics (NY)*, **344** 43-68 (2014) arXiv:1308.2348 [quant-ph, math-ph]

[2] S.T. Ali, J.-P Antoine, and J.P. G. *Coherent States, Wavelets and their Generalizations* 2d edition, Theoretical and Mathematical Physics, Springer, New York (2013), specially Chapter 11.

[3] H. Bergeron, E. M. F. Curado, J.P. G. and Ligia M. C. S. Rodrigues, *Quantizations from (P)OVM's*, Proceedings of the 8th Symposium on Quantum Theory and Symmetries, El Colegio Nacional, Mexico City, 5-9 August, 2013, Ed. K.B. Wolf, J. Phys.: Conf. Ser. (2014); arXiv: 1310.3304 [quant-ph, math-ph]

[4] H. Bergeron, A. Dapor, J.P. G. and P. Małkiewicz, Smooth big bounce from affine quantization, *Phys. Rev. D* **89**, 083522 (2014) (2014); arXiv:1305.0653 [gr-qc]

[5] M. Baldiotti, R. Fresneda, J.P. G. *Three examples of covariant integral quantization* Proceedings of Science (2014)

1. What is really quantization?

What is really ...?

- In digital signal processing: *quantization* maps a large set of input values to a smaller set such as rounding values to some unit of precision. Typically, a change of scale.
- In physics or mathematics, the term has a different meaning. For instance, the perplexing: " *First quantization is a mystery. It is the attempt to get from a classical description of a physical system to a quantum description of the "same" system. Now it doesn't seem to be true that God created a classical universe on the first day and then quantized it on the second day...*"[a]
- Or the following: " *We quantize things we do not really know to obtain things most of which we are unable to measure*"[b]
- The basic procedure, named "canonical", starting from a phase space or symplectic manifold

$$\mathbb{R}^2 \ni (q,p)\,, \quad \{q,p\} = 1 \mapsto \textbf{self-adjoint}\,(Q,P)\,, \quad [Q,P] = i\hbar I\,,$$
$$f(q,p) \mapsto f(Q,P) \mapsto (\mathrm{Sym} f)(Q,P)\,.$$

- Remind that $[Q,P] = i\hbar I$ holds true with self-adjoint Q, P, **only if** they have continuous spectrum $(-\infty,+\infty)$
- But then what about singular f, e.g. the angle $\arctan(p/q)$? What about barriers or other impassable boundaries? The motion on a circle? In a bounded interval? On the half-line? ....

[a] J. Baez, Categories, quantization and much more, http://math.ucr.edu/home/baez/categories.html (2006)

[b] J.P.G., Metrobus Gavea-Botafogo 04/09/2013 morning


What about integral quantization??
What about POVM??
Quantization MUST be CANONICAL !!

More mathematically precise:

- Quantization is

  (i) a linear map

$$\mathfrak{Q} : \mathcal{C}(X) \mapsto \mathcal{A}(\mathfrak{H})$$

  $\mathcal{C}(X)$: vector space of complex-valued functions $f(x)$ on a set X
  $\mathcal{A}(\mathfrak{H})$: vector space of linear operators

$$\mathfrak{Q}(f) \equiv A_f$$

  in some complex Hilbert space $\mathfrak{H}$ such that

  (ii) $f = 1 \mapsto$ identity operator I on $\mathfrak{H}$,

  (iii) real $f \mapsto$ (essentially) self-adjoint operator A_f in $\mathfrak{H}$.

- Add further requirements on X and $\mathcal{C}(X)$ (e.g., measure, topology, manifold, closure under algebraic operations, time evolution or dynamics...)

- Add physical interpretation about measurement of spectra of classical $f \in \mathcal{C}(X)$ or quantum $\mathcal{A}(\mathfrak{H})$ to which are given the status of *observables*.

- Add requirement of unambiguous classical limit of the quantum physical quantities, the limit operation being associated to a change of scale

2. Integral quantization

Integral quantization: general setting and POVM

- (X,ν): measure space.
- $X \ni x \mapsto \mathsf{M}(x) \in \mathcal{L}(\mathfrak{H})$: X-labelled family of bounded operators on Hilbert space $\mathfrak{H}$ resolving the identity I:

$$\int_X \mathsf{M}(x)\, \mathrm{d}\nu(x) = I\,, \quad \text{in a weak sense} \tag{1}$$

- If the $\mathsf{M}(x)$'x are positive and unit trace,

$$\mathsf{M}(x) \equiv \rho(x) \quad \text{(density matrix)}$$

- If X is space with suitable topology, the map

$$\mathcal{B}(X) \ni \Delta \mapsto \int_\Delta \rho(x)\, \mathrm{d}\nu(x)$$

may define a normalized positive operator-valued measure (POVM) on the σ-algebra $\mathcal{B}(X)$ of Borel sets.

Integral quantization: the map

- Quantization of complex-valued functions $f(x)$ on X is the linear map:

$$f \mapsto A_f = \int_X \mathsf{M}(x)\, f(x)\, \mathrm{d}\nu(x)\,, \tag{2}$$

- understood as the sesquilinear form,

$$B_f(\psi_1, \psi_2) = \int_X \langle \psi_1 | \mathsf{M}(x) | \psi_2 \rangle\, f(x)\, \mathrm{d}\nu(x)\,, \tag{3}$$

  defined on a dense subspace of $\mathfrak{H}$.

- If f is real and at least semi-bounded, the Friedrich's extension of B_f univocally defines a self-adjoint operator.

- If f is not semi-bounded, no natural choice of a self-adjoint operator associated with B_f, a subtle question[a]. We need more information on $\mathcal{H}$.

[a] see for instance H. Bergeron, JPG, P. Siegl, A. Youssef, Eur. Phys. Lett. **92** 60003 (2010); H. Bergeron, P. Siegl, A. Youssef, J. Phys. A: Math. Theor. **45** 244028 (2012)

Integral quantization: back to classical

- If $\mathsf{M}(x) = \rho(x)$ and with another (or the same) family of positive unit trace operators $X \ni x \mapsto \widetilde{\rho}(x) \in \mathcal{L}^+(\mathfrak{H})$ go back to the classical

$$A_f \mapsto \check{f}(x) := \int_X \mathrm{tr}(\widetilde{\rho}(x)\rho(x'))\, f(x')\, \mathrm{d}\nu(x')\,, \text{ "lower symbol"} \tag{4}$$

  provided the integral be defined.

- Then classical limit condition means: given a scale parameter ϵ and a distance $d(f, \check{f})$:

$$d(f, \check{f}) \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{5}$$

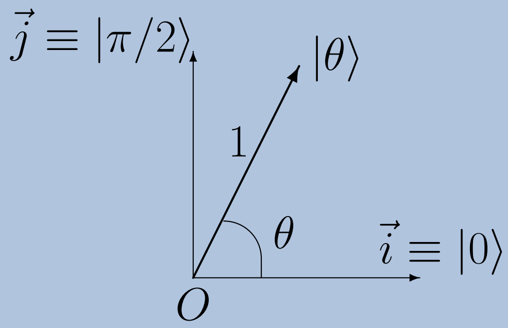
Integral quantization: comments

- Quantization issues, e.g. spectral properties of A_f, may be derived from functional properties of the *lower symbol* $\check{f}$.
- Quantizing constraints: suppose that (X, ν) is a smooth n-dim. manifold on which is defined space $\mathcal{D}'(X)$ of distributions as the topological dual of compactly supported n-forms on X. Some of these distributions, e.g. $\delta(u(x))$, express geometrical constraints. Extending the map $f \mapsto A_f$ to these objects yields the quantum version $A_{\delta(u(x))}$ of these constraints.
- Different starting point, more in Dirac's spirit[a] (e.g. see (Loop) Quantum Gravity and Quantum Cosmology) would consist in determining the kernel of the operator A_u issued from integral quantization $u \mapsto A_u$.
- Both methods are obviously not *mathematically* equivalent, except for a few cases. They are possibly *physically* equivalent.

[a] P.A.M. Dirac, Lectures on Quantum Mechanics, Dover, New York, 2001

3. A toy example: Sea star algebra

Prologue: Euclidean plane with physicist notations



Orthonormal basis (or frame) of the Euclidean plane $\mathbb{R}^2$ defined by the two vectors (in Dirac ket notations) $|0\rangle$ and $\left|\frac{\pi}{2}\right\rangle$, where $|\theta\rangle$ denotes the unit vector with polar angle $\theta \in [0, 2\pi)$. This frame is such that

$$\langle 0|0\rangle = 1 = \left\langle \frac{\pi}{2} \middle| \frac{\pi}{2} \right\rangle , \quad \left\langle 0 \middle| \frac{\pi}{2} \right\rangle = 0 ,$$

and such that the sum of their corresponding orthogonal projectors *resolves the unity*

$$I = |0\rangle\langle 0| + \left|\frac{\pi}{2}\right\rangle \left\langle\frac{\pi}{2}\right| ,$$

i.e. a trivial reinterpretation of the matrix identity:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

How marine bottom is 5-fold orientationally explored by starfish (sea star)[a]

Possibly, in a noncommutative way through the pentagonal set of unit vectors (the "arms")

$$\left| \frac{2n\pi}{5} \right\rangle = \mathcal{R}\left(-\frac{2n\pi}{5}\right)|0\rangle \quad \equiv \text{ "coherent" state (CS)}^{bc} \quad n = 0, 1, 2, 3, 4 \bmod(5)$$

with $\mathcal{R}(\theta) := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

[a] A marine echinoderm with five radiating arms. The undersides of the arms bear tube feet for locomotion and, in predatory species, for opening the shells of mollusks. On the end of each arm or ray there is a microscopic eye which allows the sea star to see, although it only allows it to see light and dark, which is useful to see movement.

[b] J-P. G., *Coherent States in Quantum Physics* (Wiley-VCH, Berlin, 2009)

[c] S. T. Ali, J.-P. Antoine, and J.-P. G., *Coherent States, Wavelets and their Generalizations* (Graduate Texts in Mathematics, Springer, New York, 2000). Second edition just appeared, November 2013

The 5-fold frame

- To the unit vector $|\theta\rangle = \cos\theta|0\rangle + \sin\theta\left|\frac{\pi}{2}\right\rangle$, corresponds the orthogonal projector P_θ given by:

$$P_\theta = |\theta\rangle\langle\theta| = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix} = \begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}$$
$$= \mathcal{R}(-\theta)|0\rangle\langle 0|\mathcal{R}(\theta)$$

- Sea star resolution of the unity:

$$\frac{2}{5}\sum_{n=0}^{4}\left|\frac{2\pi n}{5}\right\rangle\left\langle\frac{2\pi n}{5}\right| = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} \equiv I \tag{6}$$

- Here $X = \{0,1,2,3,4\} \equiv$ is the set of orientations $\equiv$ angles $2\pi n/5$ explored by the starfish. It is equipped with discrete measure with uniform weight $2/5$. The operator

$$\mathsf{M}(n) = \rho(n) = \left|\frac{2\pi n}{5}\right\rangle\left\langle\frac{2\pi n}{5}\right| \quad \text{acts on} \quad \mathfrak{H} = \mathbb{C}^2$$

What about N-fold frame? The unit circle?

- Actually resolution of unity holds for any regular N-fold polygon in the plane.

$$\frac{2}{N}\sum_{n=0}^{N}\left|\frac{2\pi n}{N}\right\rangle\left\langle\frac{2\pi n}{N}\right| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

- And even in the continuous case:

$$\frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\theta\ |\theta\rangle\langle\theta| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

- Is thus obtained a continuous frame for the plane, that is to say, the continuous set of unit vectors forming the unit circle, for describing, with an extreme redundancy, the euclidean plane.

Measure set X explored by the starfish

- The set of angles $2\pi n/5$, or, equivalently, of orientations n, is finite : $X = \{0, 1, 2, 3, 4\}$, and equipped with discrete measure allowing to define a scalar product:

$$\int_X f(x)\, \mathrm{d}\mu(x) \stackrel{\text{def}}{=} \frac{2}{5}\sum_{n=0}^{4} f(n)\,, \quad \langle\phi|\phi'\rangle = \frac{2}{5}\sum_{n=0}^{5}\bar{\phi}(n)\,\phi'(n)\,.$$

- Choose 2 orthonormal elements, $\phi_0(n) = \cos(2\pi n/5)$ and $\phi_1(n) = \sin(2\pi n/5)$, in "Hilbert space" $L^2(X, \mu)$ and build the 5 unit vectors (the sea star CS's!) in the Euclidean plane with usual orthonormal basis $|0\rangle$, $\left|\frac{\pi}{2}\right\rangle$:

$$X \ni n \mapsto |n\rangle \equiv \left|\frac{2n\pi}{5}\right\rangle = \phi_0(n)|0\rangle + \phi_1(n)\left|\frac{\pi}{2}\right\rangle\,.$$

- Then this set of 5 unit vectors or *coherent states* resolve the identity in the Euclidean plane $\mathbb{R}^2$.

$$\int_X |x\rangle\langle x|\, \mathrm{d}\mu(x) = I$$

The quantum world of the starfish

- The resolution of the identity by the 5 "coherent" arms of the starfish opens the door to its quantum world through the quantization

$$f(n) \mapsto \int_X f(x)\, |x\rangle\langle x|\; \mathrm{d}\mu(x) = \frac{2}{5}\sum_{n=0}^{4} f(n) \left|\frac{2n\pi}{5}\right\rangle \left\langle\frac{2n\pi}{5}\right| \equiv A_f$$

  more precisely through spectral values and CS mean values of the 2×2 matrix A_f, e.g. quantum angle is yielded with $f(n) = 2\pi n/5$

- If instead one had chosen as a finite frame the orthonormal basis $|0\rangle$, $|\pi/2\rangle$, in $\mathbb{R}^2$, we would have obtained the trivial commutative quantization:

$$(f(0), f(1)) \mapsto A_f = \mathrm{diag}(f(0),\, f(1))$$

- Similarly, quantum version of $f(\theta) \mapsto A_f = \frac{1}{\pi}\int_0^\pi f(\theta)|\theta\rangle\langle\theta|\,\mathrm{d}\theta$.

“CS Quantization” as a particular “Integral Quantization”

- Start from a set X equipped with a measure μ and the Hilbert space $L^2(X,\mu)$. Then pick an orthonormal set $\mathcal{O}$ of $\phi_n(x)'s \in L^2(X,\mu)$ satisfying $0 < \mathcal{N}(x) = \sum_n |\phi_n(x)|^2 < \infty$ (a.e.)
- Pick a Hilbert space $\mathcal{H}$ (the space of “quantum states”) with orthonormal basis $\{|e_n\rangle\}$ in one-to-one correspondence $\{|e_n\rangle \leftrightarrow \phi_n\}$ with the elements of $\mathcal{O}$.
- There results a family $\mathcal{C}$ of unit vectors $|x\rangle$ (the “coherent states”) in $\mathcal{H}$, which are labelled by elements of X and which resolve the unity operator in $\mathcal{H}$:

$$X \ni x \mapsto |x\rangle = \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \overline{\phi_n(x)} |e_n\rangle\,, \quad \int_X \mathcal{N}(x)\, |x\rangle\langle x|\ \mathrm{d}\mu(dx) = I\,.$$

- It is the departure point for analysing the original set X and functions living on it from the point of view of the frame (in its true sense) $\mathcal{C}$:

$$f(x) \mapsto A_f \overset{\text{def}}{=} \int_X \mathcal{N}(x)\, f(x)\, |x\rangle\langle x|\ \mathrm{d}\mu(dx) \qquad \text{(CS quantization)}$$

- We end in general with a non-commutative algebra of operators in $\mathcal{H}$. In turn, considering the properties of $\check{f}(x) \overset{\text{def}}{=} \langle x|A_f|x\rangle$ in comparison with the original $f(x)$ allows to decide if the procedure does or does not make sense mathematically.
- Changing the frame family $\mathcal{C}$ produces another quantization, possibly mathematically and/or physically equivalent to the previous one, possibly not.

Quantum angle for starfish?

- Quantization of "classical observable":

$$\mathbb{C}^5 \ni f = (a_0, a_1, a_2, a_3, a_4) \to \frac{2}{5} \sum_{n=0}^{4} a_n \left| \frac{2n\pi}{5} \right\rangle \left\langle \frac{2n\pi}{5} \right| = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv A_f \in M(2, \mathbb{C})$$

- Quantum version of the classical angle function $n \mapsto \mathcal{A}(n) = 2\pi n/5 \mod 5$:

$$\mathcal{A}(n) \mapsto A_{\mathcal{A}} = \frac{2}{5} \sum_{n=0}^{4} \frac{2n\pi}{5} \left| \frac{2n\pi}{5} \right\rangle \left\langle \frac{2n\pi}{5} \right| = \frac{\pi}{5} \begin{pmatrix} 3 & \beta \\ \beta & 5 \end{pmatrix}, \; \beta = -\frac{1}{5}(3-\tau)^{3/2} \approx -0 \cdot 325,$$

where $\tau = 2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2}$ is the golden mean.

- Spectral values of the starfish quantum angle:

$$\lambda_{\pm} = (4 \pm \sqrt{1+\beta^2}) \frac{\pi}{5} \approx \begin{cases} 1 \cdot 01\, \pi \\ \dfrac{2 \cdot 95}{5}\, \pi \end{cases} \qquad \frac{\lambda_+ + \lambda_-}{2} = \frac{4\pi}{5}.$$

- Eigenvectors: $|2.25\pi/5 = 9\pi/20\rangle$, $|19\pi/20\rangle$

Covariance

- Given $f = (a_0, a_1, a_2, a_3, a_4)$, i.e. $f(n) = a_n$ and extended periodically mod(5), the operator A_f is covariant with respect to rotations $\mathcal{R}(2\pi n'/5)$, $n' = 0,\ 1,\ 2,\ 3,\ 4$ mod(5):

$$\mathcal{R}(-2\pi n'/5) A_f \mathcal{R}(2\pi n'/5) = A_{\mathcal{R}(-2\pi n'/5) f}$$

  where

$$\mathcal{R}(2\pi n'/5) f(n) = f(n - n' \bmod (5))\,.$$

- In particular for the quantum angle,

$$\mathcal{R}(-2\pi n'/5) A_{\mathcal{A}} \mathcal{R}(2\pi n'/5) = A_{\mathcal{A}} + \frac{2n'\pi}{5} I\,.$$

Back to the classical world

- Lower symbol of the quantum angle: $\check{f} \equiv (\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4)$, $\bar{a}_n \equiv \langle \frac{2n\pi}{5} \vert A_f \vert \frac{2n\pi}{5} \rangle$, is more regular.
- It is the following function on X:

$$\check{A}(0) = \frac{3\pi}{5}\,,\ \check{A}(1) = \frac{4.31\pi}{5}\,,\ \check{A}(2) = \frac{3.39\pi}{5}\,,\ \check{A}(3) = \frac{2.46\pi}{5}\,,\ \check{A}(4) = \frac{4.14\pi}{5}\,,$$

- We observe that its values oscillate around $4\pi/5$ which is the mean value of the two eigenvalues and which is also the average of the original angle function in the following classical sense:

$$\langle f \rangle_{\text{class}} \stackrel{\text{def}}{=} \frac{1}{5}\sum_{n=0}^{4} f(n) = \frac{1}{2}\text{tr}(A_f)\,.$$

- It is proved[a] that $\langle f \rangle_{\text{class}}$ is the limit which is reached after infinitely repeated maps $f \mapsto \check{f} \mapsto \check{\check{f}} \mapsto \dots$.
- Meaning of all that in Biomechanics?

[a] *Finite tight frames and some applications*, N. Cotfas and JPG (topical review) J. Phys. A: Math. Theor. **43**, 193001-27 (2010).

How deep marine bottom is 7-fold orientationally explored by starfish?

Very recently seven-fold sea stars have been observed in Antarctic deep-sea hydrothermal vents!

Might hand fingers form a quantum frame?

What about the continuous frame?

- Quantization of "classical observable":

$$f(\theta) \to \frac{1}{\pi} \int_0^{2\pi} d\theta\, f(\theta)\, |\theta\rangle\langle\theta| \equiv A_f \in M(2, \mathbb{C})$$

- Quantum version of the classical angle function:

$$\theta \mapsto A_\theta = \begin{pmatrix} \pi & -\frac{1}{2} \\ -\frac{1}{2} & \pi \end{pmatrix} .$$

- Spectral values:

$$\lambda = \pi \pm 1/2$$

- Lower symbol

$$\langle\theta|A_\theta|\theta\rangle = \pi - \frac{\sin 2\theta}{2}$$

is a sort of rough regularization of the angle function which varies between the two eigenvalues of A_θ.

Probabilistic aspects I

- Behind the resolution of the identity lies an interpretation in terms of geometrical probability which could reveal interesting for understanding the starfish!
- Consider a subset $\Delta \subset X$ (finite case) or a Borel $\Delta \subset [0, 2\pi)$ and the restrictions

$$a(\Delta) = \frac{2}{N} \sum_{n \in \Delta} \left| \frac{2\pi n}{N} \right\rangle \left\langle \frac{2\pi n}{N} \right| \text{ or } = \frac{1}{\pi} \int_{\Delta} d\theta |\theta\rangle\langle\theta| \,.$$

- One easily checks:

$$a(\emptyset) = 0\,, \quad a(X \text{ or } [0, 2\pi)) = I_d,$$
$$a(\cup_{i \in J} \Delta_i) = \sum_{i \in J} a(\Delta_i)\,, \quad \text{if } \Delta_i \cap \Delta_j = \emptyset \text{ for all } i \neq j\,.$$

Map $\Delta \mapsto a(\Delta)$ defines a normalized measure on the set of subsets of X or on the σ-algebra of the Borel sets in the interval $[0, 2\pi)$, assuming its values in the set of positive linear operators on the Euclidean plane: it is a POVM.

Probabilistic aspects II (continuous case)

- For $|\phi\rangle$ a unit vector the application

$$\Delta \mapsto \langle\phi|a(\Delta)|\phi\rangle = \frac{1}{\pi}\int_{\Delta} \cos^2(\theta - \phi)\, d\theta$$

  is a probability measure. It is positive, of total mass 1, and it inherits σ-additivity from $a(\Delta)$.

- The quantity $\langle\phi|a(\Delta)|\phi\rangle$ means that direction $|\phi\rangle$ is examined from the point of view of the family of vectors $\{|\theta\rangle,\ \theta \in \Delta\}$. As a matter of fact, it has a *geometrical probability* interpretation in the plane. With no loss of generality let us choose $\phi = 0$.

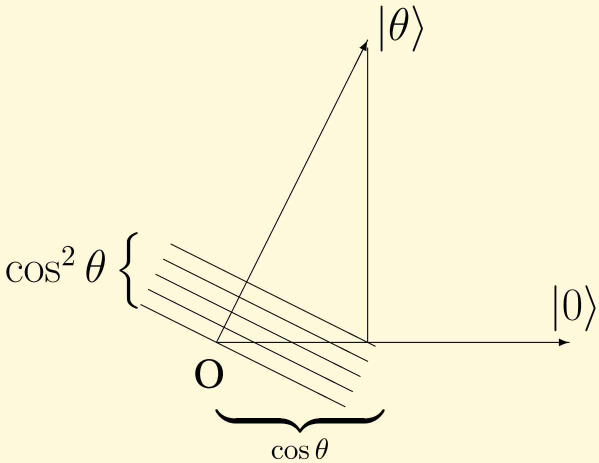
- Recall the canonical equation describing a straight line $D_{\theta,p}$ in the plane :

$$\langle\theta|u\rangle \equiv \cos\theta\, x + \sin\theta\, y = p\,,$$

  where $|\theta\,\rangle$ is the direction normal to $D_{\theta,p}$ and the parameter p is equal to the distance of $D_{\theta,p}$ to the origin.

- There follows that $dp\, d\theta$ is the (non-normalized) probability measure element on the set $\left\{D_{\theta,p}\right\}$ of the lines randomly chosen in the plane.

- Picking a certain θ, consider the set $\{D_{\theta,p}\}$ of the lines normal to $|\theta\rangle$ that intersect the segment with origin O and length $|\cos\theta|$ equal to the projection of $|\theta\rangle$ onto $|0\rangle$ as shown in Figure of next slide.



Set $\{D_{\theta,p}\}$ of straight lines normal to $|\theta\rangle$ that intersect the segment with origin O and length $|\cos\theta|$ equal to the projection of $|\theta\rangle$ onto $|0\rangle$.

Probabilistic aspects III (continuous case)

- The measure of this set is equal to :

$$\left(\int_0^{\cos^2 \theta} dp \right) d\theta = \cos^2 \theta \, d\theta \,. \tag{7}$$

  Integrating (7) over all directions $|\theta\rangle$ gives the area of the unit circle.

- Hence $\langle \phi | a(\Delta) | \phi \rangle$ is the probability for a straight line in the plane to belong to the set of secants of segments that are projections $\langle \phi | \theta \rangle$ of the unit vectors $|\theta\rangle$, $\theta \in \Delta$ onto the unit vector $|\phi\rangle$.

- One could think in terms of *polarizer* $\langle \theta |$ and *analyzer* $|\theta\rangle$ "sandwiching" the directional signal $|\phi\rangle$.

- The discrete case can be considered in the same way: maybe something of interest here for analyzing the perception of orientations by the sea star...

Unit circle: quantization with more general POVM

- Just replace $P_0 = |0\rangle\langle 0|$ in

$$I = \frac{1}{\pi}\int_0^{2\pi} P_\theta \, \mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi} \mathcal{R}(-\theta) P_0 \mathcal{R}(\theta)\, \mathrm{d}\theta\,,\ ,$$

by a 2×2 symmetric matrix $M = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$.

- We still have the resolution of the unity and the resultant quantization:

$$A_f = \frac{1}{\pi(a+d)}\int_0^{2\pi} f(\theta)\, \mathcal{R}(-\theta) M \mathcal{R}(\theta)\, \mathrm{d}\theta\,.$$

- In particular, with a density matrix $M \equiv \rho = \begin{pmatrix} a & b \\ b & 1-a \end{pmatrix}$, $0 \leqslant a \leqslant 1$ and $\det \rho = a - (a^2 + b^2) \geqslant 0$, i.e. $|b| \leqslant \sqrt{a(1-a)}$, we obtain a POVM.

- The corresponding quantum angle reads as $A^\rho_\theta = \begin{pmatrix} \pi - b & \frac{1}{2} - a \\ \frac{1}{2} - a & \pi - b \end{pmatrix}$ with eigenvalues: $\pi - b \pm \left(\frac{1}{2} - a\right)$ and eigenvectors $|\pi/4\rangle$, $|3\pi/4\rangle$.