2585-15

Joint ICTP-TWAS School on Coherent State Transforms, Time-Frequency and Time-Scale Analysis, Applications

2 - 20 June 2014

Coherent States and Wavelets: A Unified Approach - II

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on Coherent State Transforms, Time-Frequency
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The Abdus Salam International Centre for Theoretical Physics,
Miramare, Trieste, ITALY

June 2 – 21, 2014

Abstract

The canonical coherent states were seen to emanate from a unitary irreducible representation of the Weyl-Heisenberg group on a Hilbert space. As mentioned earlier, one way to generalize the notion of coherent states would be to look at uintary irreducible representations of other suitably chosen groups and anologous sets of vectors constructed out of them. The continuous wavelet transform arises from one such choice of a group, namely, the affine group of the real line. We now proceed to develop this idea in some detail. The resulting coherent states are called wavelets in the signal analysis literature and they form the basis of much of modern digital data analysis. Let us note that in the signal analysis literature, the canonical coherent states are also known as gaborettes or Gabor wavelets.

Contents

References

Shakespeare knew wavelets:

Like as the waves make toward the pebbled shore,
So do our minutes hasten to their end;
Each changing place with that which goes before,
In sequent toil all forwards do contend.

Sonnet LX

Wavelets as coherent states

The continuous wavelet transform, as presently used extensively in signal analysis and image processing, is a joint time frequency transform. This is in sharp contrast to the Fourier transform, which can be used either to analyze the frequency content of a signal, or its time profile, but not both at the same time.

Interestingly, the continuous wavelet transform is built out of what may be called the generalized coherent states, obtained from a representation of the one-dimensional affine group, a group of translations and dilations of the real line. This group is also one of the simplest examples of a group which has a square integrable representation.

We ought to point out, however, that in actual practice, for computational purposes, one uses a discretized version of the transform that we shall obtain here. But the advantage of working with the continuous wavelet transform is that starting with it, one can obtain many more than just one discrete transform.

Wavelets as coherent states

In signal processing, one is interested in analyzing various types of signals. For our purposes, we shall identify a signal with an element $f \in L^2(\mathbb{R}, dx)$. Its L^2-norm squared, $\|f\|^2$, will be identified with the energy of the signal.

Consequently, the wavelet transform will be built out of a single element $\psi \in L^2(\mathbb{R}, dx)$ and it will have to be an admissible vector in the sense of square-integrable representations. In the singnal analysis literature, such a vector is called a mother wavelet.

The resulting resolution of the identity will enable us to reconstruct the signal from its wavelet transform, ie., its joint time frequency transform.

The orthogonality relations then allow one to decompose an arbitrary time frequency transform into orthogonal sums of wavelet transforms, corresponding to different mother wavelets.

We shall also be able to choose a mother wavelet in a way such that the resulting wavelet transform consists of holomorphic functions in a certain Hardy space.

Transformations on signals

Let $\psi \in L^2(\mathbb{R}, dx)$ and consider the basic 1-D transformation:

$$\psi(x) \mapsto \psi_{b,a}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right), \quad b \in \mathbb{R},\ a \neq 0\ ,$$

and rewrite it in the form

$$\psi_{b,a}(x) = |a|^{-1/2}\psi\left((b,a)^{-1}x\right)\ ,$$

where we have introduced the affine transformation of the line, consisting of a dilation (or scaling) by $a \neq 0$ and a (rigid) translation by $b \in \mathbb{R}$:

$$x = (b,a)y = ay + b,$$

and its inverse

$$y = (b,a)^{-1}x = \frac{x-b}{a}\ .$$

Transformations on signals

Writing $\phi = \psi_{b,a}$ and making a second transformation on ϕ we get

$$\begin{aligned}\phi(x) \mapsto \phi_{b',a'}(x) &= |\, a' \,|^{-\frac{1}{2}} \phi((b', a')^{-1} x) \\ &= |\, aa' \,|^{-\frac{1}{2}} \psi((b', a')^{-1} (b, a)^{-1} x) \\ &= |\, aa' \,|^{-\frac{1}{2}} \psi\left(\frac{x - (b + ab')}{aa'}\right).\end{aligned}$$

Thus, the effect of two successive transformations is captured in the composition rule

$$(b, a)(b', a') = (b + ab', aa'),$$

which, if we represent these transformations by 2×2 matrices of the type

$$(b, a) \equiv \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \qquad a \neq 0, \quad b \in \mathbb{R}\,,$$

The 1-D affine group

is reproduced by ordinary matrix multiplication. The point to be noted about these matrices is that the product of two of them is again a matrix of the same type and so also is the inverse,

$$(b, a)^{-1} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{pmatrix}$$

of such a matrix. Furthermore, the 2×2 identity matrix is also in this class. In other words, this class of matrices constitute a group, called the (full) affine group and denoted G_{aff}. Note also, that if we consider only those matrices in for which $a > 0$, then this set is a subgroup of G_{aff}, denoted G^+_{aff}. From the action on signals, we observe that G_{aff} or G^+_{aff} consists precisely of the transformations we apply to a signal: translation (time-shift) by an amount b and zooming in or out by the factor a. Hence, the group G_{aff} relates to the geometry of the signals.

A representation of the group

Next let us study the effect of the transformation given by the group element (b, a) on the signal itself. Writing,

$$\psi \mapsto U(b,a)\psi \equiv \psi_{b,a} \, ,$$

we may interpret $U(b, a)$ as a linear operator on the space $L^2(\mathbb{R}, dx)$ of finite energy signals, with the explicit action,

$$(U(b,a)\psi)(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right) \, .$$

Additionally, for each (b, a), the operator $U(b, a)$ is unitary, i.e., it preserves the Hilbert space norm of the signal:

$$\|\psi_{b,a}\|^2 = \|\psi\|^2 = \int_{-\infty}^{\infty} dx \, \mid\psi(x)\mid^2 .$$

A representation of the group

More interestingly, the association, $(b,a) \mapsto U(b,a)$ is a group homomorphism, preserving all the group properties. Indeed, the following relations are easily verified:

$$\begin{aligned} &U(b,a)U(b',a') = U(b+ab', aa') \\ &U((b,a)^{-1}) = U(b,a)^{-1} = U(b,a)^{\dagger} \\ &U(e) = I, \quad \text{with} \quad e = (0,1), \text{ the unit element.} \end{aligned}$$

We say that the association $(b,a) \mapsto U(b,a)$ provides us with a unitary representation of G_{aff}. Note that we may also write,

$$U(b,a) = T_b \, D_a \ ,$$

where T_a, D_b are the well known shift and dilation operators, familiar from standard time-frequency analysis:

$$(T_b s)(x) = s(x-b) \ , \qquad (D_a s)(x) = |a|^{-\frac{1}{2}} \, s(a^{-1}x) \ .$$

A representation of the group

We shall see later that the representation $U(b, a)$ is in a sense minimal or irreducible, in that the entire Hilbert space of finite energy signals $L^2(\mathbb{R}, dx)$ is needed to realize it completely.

But let us first attend to another question which is pertinent here, namely, why is it that G_{aff} is made to act as a transformation group on $\mathbb{R}$ even without manifestly identifying $\mathbb{R}$ with any set of signal parameters?

The answer to the above question lies in realizing that this space is intrinsic to the group itself. Indeed, let us factor an element $(b, a) \in G_{\text{aff}}$ in the manner

$$(b, a) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

and note that the first matrix on the right hand side of this equation basically represents a point in $\mathbb{R}$. We also note that the set of matrices of the type appearing in the second term of the above product is a subgroup of G_{aff}.

A representation of the group

Dividing out by this matrix, we get $(b,a)(0,a)^{-1} = (b,0)$, which enables us to identify the point $b \in \mathbb{R}$ with an element of the quotient space G_{aff}/H, (where H is the subgroup of matrices $(0,a),\ a \neq 0$). Next we see that, since

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ax+b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax+b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

the action of the group G_{aff} on its quotient space G_{aff}/H is exactly the same as its action on $\mathbb{R}$ as given earlier. Thus, the parameter space $\mathbb{R}$ on which the signals $\psi(x)$ are defined is a quotient space of the group and hence intrinsic to the set of signal symmetries. We shall see below that the parameter space on which the wavelet transform of ψ is defined can also be identified with a quotient space of the group. In fact this space will turn out to be a phase space, in a sense to be specified later. Let us re-emphasize that the group (of signal symmetries) is determinative of all aspects of the signal and its various transforms.

A representation of the group

We come back now to the point made earlier, that the representation $U(b, a)$ is irreducible.

We shall see that it also enjoys a second crucial property, that of being square integrable.

The group G_{aff} has a natural action on itself (by matrix multiplication from the left), according to which for a given $(b_0, a_0) \in G_{\text{aff}}$, a general point $(b, a) \in G_{\text{aff}}$ is mapped to $(b', a') = (b_0, a_0)(b, a) = (b_0 + a_0 b, a_0 a)$.

It is not hard to see that the measure

$$d\mu(b, a) = \frac{db\ da}{a^2},$$

is invariant under this action:

$$\frac{db\ da}{a^2} = \frac{db'\ da'}{a'^2}.$$

We call the measure $d\mu$ the left Haar measure of G_{aff}.

In a similar manner we could obtain a right Haar measure $d\mu_r$ (invariant under right multiplication):

$$d\mu_r(b, a) = a^{-1}\, db\, da\ .$$

It is important to realize, that while these two measures are (measure theoretically) equivalent, they are not the same measure.

The function $\mathbf{\Delta}(b, a) = a^{-1}$, for which $d\mu(b, a) = \mathbf{\Delta}(b, a)\; d\mu_r(b, a)$, is called the modular function of the group.

The square-integrability of the representation $U(b, a)$ now means that there exist signals $\psi \in L^2(\mathbb{R},\ dx)$ for which the matrix element $\langle U(b, a)\psi \mid \psi\rangle$ is square integrable as a function of the variables b, a, with respect to the left Haar measure $d\mu$, i.e.,

$$\iint_{G_{\text{aff}}} d\mu(b, a)\ |\langle U(b, a)\psi|\psi\rangle|^2 < \infty,$$

and a straightforward computation would then establish that the function is also square integrable with respect to the right Haar measure.

Also, it is a fact that the existence of one such (nonzero) vector implies the existence of an entire dense set of them. Indeed, the condition for a signal to be of this type is precisely the condition of admissibility required of mother wavelets.

To derive the admissibility condition, and also to verify our claim of irreducibility of the representation $U(a, b)$, it will be convenient to go over to the Fourier domain. For $\psi \in L^2(\mathbb{R}, dx)$, we denote its Fourier transform by $\widehat{\psi}$.

It is not hard hard to see that on the Fourier transformed space the unitary operator $U(b, a)$ transforms to $\widehat{U}(b, a)$, with explicit action,

$$\left(\widehat{U}(b, a)\widehat{\psi}\right)(\xi) = |a|^{1/2}\,\widehat{\psi}(a\xi)e^{-ib\xi} \quad (b \in \mathbb{R}, a \neq 0).$$

Square integrability, admissibility and irreducibility

The Fourier transform is a linear isometry, and we denote by $L^2(\widehat{\mathbb{R}}, d\xi)$ the image of $L^2(\mathbb{R}, dx)$ under this map.

It follows that the operators $\widehat{U}(b, a)$ are also unitary and that they again constitute a unitary representation of the group G_{aff}.

Let $\widehat{\psi} \in L^2(\widehat{\mathbb{R}}, d\xi)$ be a fixed nonzero vector in the Fourier domain. We will now show that the set of all vectors $\widehat{U}(b, a)\widehat{\psi}$ as (b, a) runs through G_{aff} is dense in $L^2(\widehat{\mathbb{R}}, d\xi)$ and this is what will constitute the mathematically precise statement of the irreducibility of $\widehat{U}$.

Indeed, let $\widehat{\chi} \in L^2(\widehat{\mathbb{R}}, d\xi)$ be a vector which is orthogonal to all the vectors $\widehat{U}(b, a)\widehat{\psi}$:

$$\langle \widehat{\chi} \mid \widehat{U}(b, a)\widehat{\psi} \rangle = 0.$$

Then,

$$\langle \widehat{\chi} | \widehat{U}(b, a)\widehat{\psi} \rangle = |a|^{1/2} \int_{-\infty}^{\infty} d\xi \, \overline{\widehat{\chi}(\xi)} \, \widehat{\psi}(a\xi) \, e^{-ib\xi} = 0.$$

By the unitarity of the Fourier transform, this yields $\overline{\widehat{\chi}(\xi)}\,\widehat{\psi}(a\xi) = 0$, almost everywhere, for all $a \neq 0$. Since $\widehat{\psi} \not\equiv 0$, this in turn implies $\widehat{\chi}(\xi) = 0$, almost everywhere.

Thus, the only subspaces of $L^2(\widehat{\mathbb{R}}, d\xi)$ which are stable under the action of all the operators $\widehat{U}(b,a)$ are $L^2(\widehat{\mathbb{R}}, d\xi)$ itself and the trivial subspace containing just the zero vector.

In other words, $L^2(\widehat{\mathbb{R}}, d\xi)$ is sort of a minimal space for the representation. The unitarity of the Fourier transform also tells us that the representations $U(b,a)$ and $\widehat{U}(b,a)$ are equivalent and since $\widehat{U}(b,a)$ is irreducible, so also is $U(b,a)$. (Note, this is also clear from the fact that the linear isometry property of the Fourier transform implies that

$$\langle \chi \mid U(b,a)\psi \rangle = \langle \widehat{\chi} \mid \widehat{U}(b,a)\widehat{\psi} \rangle \ ,$$

χ, ψ denoting the inverse Fourier transforms of $\widehat{\chi}, \widehat{\psi}$, respectively.)

Square integrability, admissibility and irreducibility

Now we address the question of square integrability. We require that,

$$\iint_{G_{\text{aff}}} \frac{da\, db}{a^2}\, |\langle \widehat{U}(b,a)\widehat{\psi}|\widehat{\psi}\rangle|^2 =$$

$$\begin{aligned} &= \iiiint d\xi\, d\xi'\, \frac{da}{|a|}\, db\, e^{ib(\xi-\xi')}\, \overline{\widehat{\psi}(a\xi)}\, \widehat{\psi}(a\xi')\widehat{\psi}(\xi)\, \overline{\widehat{\psi}(\xi')} \\ &= 2\pi \iint \frac{da}{|a|}\, d\xi\, |\widehat{\psi}(a\xi)|^2\, |\widehat{\psi}(\xi)|^2 \\ &= 2\pi\|\psi\|^2 \int_{-\infty}^{\infty} \frac{d\xi}{|\xi|}\, |\widehat{\psi}(\xi)|^2 < \infty \end{aligned}$$

(the integral over b yields a delta distribution, which can be used to perform the ξ' integration and the interchange of integrals can be justified using s distribution theoretic arguments). This means that the vector ψ is admissible in the sense of our earlier definition if and only if

$$c_\psi \equiv 2\pi \int_{-\infty}^{\infty} \frac{d\xi}{|\xi|}\, |\widehat{\psi}(\xi)|^2 < \infty.$$

Square integrability, admissibility and irreducibility

From this discussion we draw two immediate conclusions. First, there is a dense set of vectors $\widehat{\psi}$ which satisfy the admissibility condition. Second, the admissibility condition, $c_\psi < \infty$, simply expresses the square integrability of the representation U.

Defining an operator $\widehat{C}$ on $L^2(\widehat{\mathbb{R}}, d\xi)$,

$$(\widehat{C}\widehat{\psi})(\xi) = \left[\frac{2\pi}{|\,\xi\,|}\right]^{\frac{1}{2}} \widehat{\psi}(\xi),$$

and denoting by C its inverse Fourier transform, we see that the vector ψ is admissible if and only if

$$c_\psi = \|C\psi\|^2 < \infty\ .$$

This operator, known as the Duflo-Moore operator, is positive, self-adjoint and unbounded. It also has an inverse. It is easily seen that if a vector ψ is admissible, then so also is the vector $U(b,a)\psi$, for any $(b,a) \in G_{\mathrm{aff}}$.

Square integrability, admissibility and irreducibility

A word now about the form of the representation $U(b, a)$. How does one arrive at it? In fact, given the way the group acts on $\mathbb{R}$, $x \mapsto ax + b$, the representation $U(b, a)$ is recognized as being the most natural, nontrivial way to realize a group homomorphism onto a set of unitary operators on the signal space $L^2(\mathbb{R}, dx)$. (Unitarity is required in order to ensure that the signal ψ and the transformed signal $U(b, a)\psi$ both have the same total energy).

Indeed, given any differentiable mapping $T : \mathbb{R}^n \to \mathbb{R}^n$, the operator $U(T)$, on the Hilbert space $L^2(\mathbb{R}^n, d^n\vec{x})$, defined as

$$(U(T)f)(\vec{x}) = |\det[J(T)]|^{-\frac{1}{2}} f(T^{-1}(\vec{x})),$$

where $J(T)$ is the Jacobian of the map T, is easily seen to be unitary. (Recall that

$$d(T(\vec{x})) = |\det[J(T)]| \ d\vec{x} \ .)$$

Square integrability, admissibility and irreducibility

This provides the rationale for our definition of the representation $U(b, a)$ in the way we did.

Of course, the interesting point here is that this representation turns out to be both irreducible and square integrable.

But then, why is square integrability of the representation a desirable criterion for wavelet analysis? In order to answer this question, let us take a vector ψ satisfying the admissibility condition and use it to construct the wavelet transform of the signal s:

$$S(b, a) = \langle \psi_{b,a} \mid s \rangle.$$

As we already know, the total energy of the transformed signal is given by the integral

$$E(S) = \iint_{G_{\mathrm{aff}}} d\mu(b, a) \; |S(b, a)|^2 ,$$

and we would like this to be finite, like that of the signal itself.

Square integrability, admissibility and irreducibility

An easy computation now shows that

$$E(S) = \|C\psi\|^2 \, \|s\|^2 = c_\psi \, \|s\|^2 \, ,$$

which means that the total energy of the wavelet transform will be finite if and only if the mother wavelet can be chosen from the domain of the operator C, i.e., if and only if it satisfies the square integrability condition.

However, this is not the whole story, for let us rewrite the above equation in the expanded form,

$$\begin{aligned} E(S) &= \iint_{G_{\rm aff}} d\mu(b,a) \, \langle s \mid \psi_{b,a}\rangle\langle\psi_{b,a} \mid s\rangle \\ &= \langle s \mid \left[\iint_{G_{\rm aff}} d\mu(b,a) \mid \psi_{b,a}\rangle\langle\psi_{b,a} \mid \right] s\rangle \\ &= c_\psi \, \langle s \mid Is\rangle \, , \end{aligned}$$

Using the well-known polarization identity for scalar products we infer that

Square integrability, admissibility and irreducibility

$$\frac{1}{c_\psi} \iint_{G_{\rm aff}} d\mu(b,a)\ |\psi_{b,a}\rangle\langle\psi_{b,a}| = I\ ,$$

i.e., the resolution of the identity. It is immediately clear that this is completely equivalent to the square integrability of the representation $U(b,a)$.

The resolution of the identity also incorporates within it the possibility of reconstructing the the signal $s(x)$, from its wavelet transform $S(b,a)$. To see this, let us act on the vector $s \in L^2(\mathbb{R}, dx)$ with both sides of the above identity. We get

$$\frac{1}{c_\psi} \iint_{G_{\rm aff}} d\mu(b,a)\ \psi_{b,a}\langle\psi_{b,a}\,|\,s\rangle = Is = s\ ,$$

implying

$$s(x) = \frac{1}{c_\psi} \iint_{G_{\rm aff}} d\mu(b,a)\ S(b,a)\psi_{b,a}(x)\ , \quad \text{almost everywhere,}$$

Square integrability, admissibility and irreducibility

which is the celebrated reconstruction formula we encountered before.

Summarizing, we conclude that square integrability (which is a group property) is precisely the condition which ensures, in this case, the very desirable consequences of

1. the finiteness of the energy of the wavelet transform, and
2. the validity of the reconstruction formula.

These two properties are also shared by the Fourier transform of a signal.
The resolution of the identity condition has independent mathematical interest. First of all, it implies that any vector in $L^2(\mathbb{R}, dx)$ which is orthogonal to all the wavelets $\psi_{b,a}$ is necessarily the zero vector, i.e., the linear span of the wavelets is dense in the Hilbert space of signals.

Square integrability, admissibility and irreducibility

This fact, which could also have been inferred from the irreducibility of the representation $U(b, a)$, is what enables us to use the wavelets as a basis set for expressing arbitrary signals.

In fact we have here what is also known as an overcomplete basis. Secondly, this overcomplete basis is a continuously parametrized set, meaning that this is an example of a continuous basis and a continuous frame.

As mentioned earlier, for practical implementation, one samples this continuous basis to extract a discrete set of basis vectors which forms a discrete frame.

The space of all wavelet transforms

A finite energy wavelet transform $S(b, a)$ is an element of the Hilbert space $L^2(\mathbb{H},\ d\mu)$. Here we have written $\mathbb{H} = \mathbb{R} \times \mathbb{R}^*$, ($\mathbb{R}^*$ = real line with the origin removed). Although, $\mathbb{H}$ and G_{aff} are homeomorphic as topological spaces, we prefer to denote them by different symbols, for presently we shall identify $\mathbb{H}$ with a *phase space* of G_{aff}, arising from its matrix geometry.

Computing the L^2-norm of the wavelet transform S (as an element in $L^2(\mathbb{H}, d\mu)$) and the L^2-norm of the signal s (as an element in $L^2(\mathbb{R}, dx)$), we get

$$\|S\|^2 = c_\psi \ \|s\|^2 \ ,$$

i.e., up to a constant, the wavelet transform preserves norms (or energies).

We define a map $W_\psi : L^2(\mathbb{R}, dx) \to L^2(\mathbb{H}, d\mu)$, by the relation

$$(W_\psi s)(a, b) = [c_\psi]^{-\frac{1}{2}} \langle \psi_{b,a} \mid s \rangle_{L^2(\mathbb{R}, dx)} = [c_\psi]^{-\frac{1}{2}} \ S(b, a).$$

This map is linear and, in view of the previous relation, an isometry.

The space of all wavelet transforms

The range of W_ψ, which is the set of all wavelet transforms corresponding to the mother wavelet ψ, is a closed subspace of $L^2(\mathbb{H}, d\mu)$. We denote the range by $\mathfrak{H}_\psi$:

$$\mathfrak{H}_\psi = W_\psi \left[L^2(\mathbb{R}, dx) \right] \subset L^2(\mathbb{H}, d\mu) \ .$$

From the nature of W_ψ, we infer that $\mathfrak{H}_\psi$ consists of continuous functions over $\mathbb{H}$ and hence is a proper subspace of $L^2(\mathbb{H}, d\mu)$. It is worthwhile reiterating here the fact that the condition of W_ψ being an isometry implies, not only that the wavelet transform (with respect to the mother wavelet ψ) of any signal $s \in L^2(\mathbb{R}, dx)$ is an element in $\mathfrak{H}_\psi$, but also that every element in $\mathfrak{H}_\psi$ is the wavelet transform of some signal $s \in L^2(\mathbb{R}, dx)$.
Is there some convenient, intrinsic way to characterize the subspace $\mathfrak{H}_\psi$? To answer this question we appeal to the resolution of the identity and a bit of group theory.

An intrinsic characterization of the space of wavelet transforms

The final characterization will be spelled out in a theorem below. Multiplying each side of the resolution of the identity by itself we find,

$$\frac{1}{c_\psi}\iint_{\mathbb{H}\times\mathbb{H}} d\mu(b',a')\,d\mu(b,a)\,|\psi_{b,a}\rangle\, K_\psi(b,a;\ b',a')\,\langle\psi_{b',a'}| = I\ ,$$

where we have written

$$K_\psi(b,a\,;\ b',a') = \frac{1}{c_\psi}\,\langle\psi_{b,a}\,|\,\psi_{b',a'}\rangle\ .$$

Acting on the signal vector s with both sides of the first equation we obtain

$$\frac{1}{c_\psi}\int_{\mathbb{H}} d\mu(b,a)\,\left[\int_{\mathbb{H}} d\mu(b',a')\,K_\psi(b,a\,;\ b',a')S(b',a')\right]\,\psi_{b,a}(x) = s(x)\ ,$$

almost everywhere (the change in the order of integrations being easily justified by Fubini's theorem).

An intrinsic characterization of the space of wavelet transforms

Comparing the above equation with the reconstruction formula we obtain

$$\int_{\mathbb{H}} d\mu(b', a')\ K_\psi(b, a\,;\ b', a')S(b', a') = S(b, a)\ ,$$

for almost all (b, a) in $\mathbb{H}$ (with respect to the measure $d\mu$). This then is the condition which characterizes wavelet transforms coming from the mother wavelet ψ – the reproducing property of the integral kernel K_ψ.

As we know, the kernel $K_\psi : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is called a reproducing kernel. It has the easily verifiable properties:

$$K_\psi(b, a\,;\ b, a) > 0, \quad \forall (b, a) \in \mathbb{H}\ , \qquad K_\psi(b, a\,;\ b', a') = \overline{K_\psi(b', a'\,;\ b, a)}\ ,$$

$$\iint_{\mathbb{H}\times\mathbb{H}} d\mu(b'', a'')\ K_\psi(b, a\,;\ b'', a'')\ K_\psi(b'', a''\,;\ b', a') = K_\psi(b, a\,;\ b', a')\ ,$$

the last relation being again the reproducing property.The last two equations hold pointwise, $\forall (b, a),\ (b', a') \in \mathbb{H}$.

An intrinsic characterization of the space of wavelet transforms

Next let us compute the wavelet transforms of the wavelets $\psi_{b,a}$ themselves. Denoting the transforms by $S_{b,a}$ we find

$$S_{b,a}(b', a') = \langle \psi_{b',a'} \,|\, \psi_{b,a} \rangle = c_\psi \; K_\psi(b', a' \,;\, b, a) \; .$$

Since the vectors $\psi_{b,a},\ (b, a) \in \mathbb{H}$, are overcomplete in $L^2(\mathbb{R},\ dx)$, the wavelet transforms $S_{b,a}$ must also be overcomplete in $\mathfrak{H}_\psi$. One also has the easily verifiable resolution of the identity,

$$[c_\psi]^{-2} \int_{\mathbb{H}} d\mu(b, a) \; |\, S_{b,a} \rangle \langle S_{b,a} \,| \; = I_\psi \; ,$$

where we have written I_ψ for the identity operator on $\mathfrak{H}_\psi$. Thus, any vector $F \in L^2(\mathbb{H}, d\mu)$ which lies in the orthogonal complement of $\mathfrak{H}_\psi$ must satisfy

$$\int_{\mathbb{H}} d\mu(b', a') \; K_\psi(b, a \,;\, b', a') F(b', a') = 0 \; .$$

An intrinsic characterization of the space of wavelet transforms

Thus, the reproducing kernel K_ψ defines the projection operator $\mathbb{P}_\psi$ from $L^2(\mathbb{H}, d\mu)$ to $\mathfrak{H}_\psi$:

$$(\mathbb{P}_\psi F)(b, a) = \int_{\mathbb{H}} d\mu(b', a')\, K_\psi(b, a\,;\, b', a')F(b', a')\,, \qquad F \in L^2(\mathbb{H}, d\mu)\,,$$

the previous relations mirroring the conditions $\mathbb{P}_\psi = \mathbb{P}_\psi^* = \mathbb{P}_\psi^2$. Stated differently, an arbitrary vector $F \in L^2(\mathbb{H}, d\mu)$ can be uniquely written as the sum

$$F = F_\psi + F_\psi^\perp\,,$$

of a part $F_\psi \in \mathfrak{H}_\psi$ and a part $F_\psi^\perp$ orthogonal to it. The operator $\mathbb{P}_\psi$, acting on F, projects out the part F_ψ (which is a wavelet transform). It is natural to ask at this point if $F_\psi^\perp$ could also be written as the wavelet transform with respect to some other mother wavelet. As will be seen below, generally $F_\psi^\perp$ can be written as an infinite sum of orthogonal wavelet transforms, corresponding to different mother wavelets.

The space of all wavelet transforms

To proceed further, we go back to the affine group, G_{aff}, and note that there is a natural unitary representation of it on the Hilbert space $L^2(\mathbb{H}\ , d\mu)$, given by its natural action on $\mathbb{H}$. This is just $\mathbf{U}_\ell(b, a)$, the left regular representation

$$
\begin{aligned}
(\mathbf{U}_\ell(b, a)F)(b', a') &= F((b, a)^{-1}(b', a')) \\
&= F\left(\frac{b' - b}{a}, \frac{a'}{a}\right), \qquad F \in L^2(\mathbb{H}\ , d\mu)\ .
\end{aligned}
$$

The unitarity of this representation,

$$
\|\mathbf{U}_\ell(b, a)F\|^2_{L^2(\mathbb{H}, d\mu)} = \|F\|^2_{L^2(\mathbb{H}, d\mu)}\ ,
$$

is guaranteed by the invariance of the measure $d\mu$. However, the left regular representation is not irreducible, since as we shall see below, the subspace $\mathfrak{H}_\psi$ carries a subrepresentation of it. The isometry W_ψ maps the unitary operators $U(b, a)$ onto unitary operators $\mathbf{U}_\psi(b, a) = W_\psi U(b, a) W_\psi^{-1}$ on $L^2(\mathbb{H}, d\mu)$.

Computing the action of these operators, we find

$$(\mathbf{U}_\psi(b,a)F)(b',a') = F\left(\frac{b'-b}{a}, \frac{a'}{a}\right), \qquad F \in \mathfrak{H}_\psi \; .$$

This is the same action as that of the operators $\mathbf{U}_\ell(b,a)$ of the left regular representation, except that now it is expressed exclusively in terms of vectors in $\mathfrak{H}_\psi$.

This means, first of all, that the subspace $\mathfrak{H}_\psi$ is stable under the action of the operators $\mathbf{U}_\ell(b,a)$ and, secondly, that restricted to this subspace, it gives an irreducible unitary representation of G_{aff}.

We shall identify $L^2(\mathbb{H}, d\mu)$ with the space of all wavelet transforms.

Let ψ and ψ' be two different mother wavelets. This means that they are both vectors in the domain of the operator C.

How are wavelet transforms, taken with respect to these mother wavelets, related?

Decomposition of the space of all finite energy wavelet transforms

In particular, denoting by S_ψ the wavelet transform of the signal s, taken with respect to the mother wavelet ψ, and by $S'_{\psi'}$ the wavelet transform of the signal s', taken with respect to the mother wavelet ψ', we would like to evaluate the overlap

$$I(\psi, \psi';\ s', s) = \iint_{\mathbb{H}} d\mu(b,a)\ \overline{S'_{\psi'}(b,a)}\ S_\psi(b,a)$$

Let us begin by assuming that s, s' are taken from a class of smooth functions (e.g., the Schwartz class, $\mathcal{S}(\mathbb{R})$), which is dense in $L^2(\mathbb{R},\ dx)$. Then

$$\begin{aligned} I(\psi, \psi';\ s', s) &= \int_{\mathbb{R}} \int_{\mathbb{R}^*} \frac{db\ da}{a^2}\ \langle \widehat{s'} \mid \widehat{\psi'}_{b,a} \rangle \langle \widehat{\psi}_{b,a} \mid \widehat{s} \rangle \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^*} \frac{db\ da}{a^2} \left[\int_{\mathbb{R}} d\xi\ \mid a \mid^{\frac{1}{2}} e^{ib\xi}\ \overline{\widehat{s'}(\xi)}\ \widehat{\psi'}(a\xi) \right] \\ &\qquad \times \left[\int_{\mathbb{R}} d\xi'\ \mid a \mid^{\frac{1}{2}} e^{-ib\xi'}\ \widehat{s}(\xi')\ \overline{\widehat{\psi}(a\xi')} \right]. \end{aligned}$$

Decomposition of the space of all finite energy wavelet transforms

We exploit the smoothness of the functions s, s' to use the identity

$$\frac{1}{2\pi}\int_{\mathbb{R}} db\; e^{ib(\xi-\xi')} = \delta(\xi-\xi')\ ,$$

which holds in the sense of distributions, and then perform the ξ'-integration to obtain

$$I(\psi,\psi';\; s', s) = 2\pi \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{da}{a}\, d\xi\; \overline{\widehat{\psi}(a\xi)}\; \widehat{\psi}'(a\xi)\; \widehat{s}(\xi)\; \overline{\widehat{s'}(\xi)}\ .$$

Changing variables, we get

$$\begin{aligned} I(\psi,\psi';\; s', s) &= 2\pi \left[\int_{\mathbb{R}} dy\; \frac{\overline{\widehat{\psi}(y)}\; \widehat{\psi}'(y)}{|\,y\,|}\right]\cdot\left[\int_{\mathbb{R}} d\xi\; \widehat{s}(\xi)\; \overline{\widehat{s'}(\xi)}\right] \\ &= \langle \widehat{C}\widehat{\psi} \mid \widehat{C}\widehat{\psi'}\rangle\; \langle \widehat{s'} \mid \widehat{s}\rangle\ . \end{aligned}$$

Thus,

$$\iint_{\mathbb{H}} d\mu(b,a)\ \overline{S'_{\psi'}(b,a)} S_\psi(b,a) = \iint_{\mathbb{H}} d\mu(b,a)\ \langle s' \mid \psi'_{b,a}\rangle\langle \psi_{b,a} \mid s\rangle$$
$$= \langle C\psi \mid C\psi' \rangle\ \langle s' \mid s\rangle\ .$$

Using the continuity of the scalar product $\langle s' \mid s\rangle$ in s and s', we may now extend the above expression to all signals $s, s' \in L^2(\mathbb{R},\ dx)$.

The equation above is a general orthogonality relation for wavelet transforms. In particular, if $C\psi$ and $C\psi'$ are orthogonal vectors, then the corresponding wavelet transforms are also orthogonal in $L^2(\mathbb{H}, d\mu)$. We may also write this equation in the form of an operator identity on $L^2(\mathbb{R}, dx)$:

$$\iint_{\mathbb{H}} d\mu(b,a)\ |\psi'_{b,a}\rangle\langle \psi_{b,a}| = \langle C\psi \mid C\psi' \rangle\ I\ ,$$

which clearly is a generalization of the resolution of the identity.

Decomposition of the space of all finite energy wavelet transforms

Let us emphasize that the above orthogonality relation implies:

- If s and s' are signals which are orthogonal vectors in $L^2(\mathbb{R}, dx)$, then their wavelet transforms S and S', whether with respect to the same or different mother wavelets, are orthogonal as vectors in $L^2(\mathbb{H}, d\mu)$.
- Spaces of wavelet transforms, $\mathfrak{H}_\psi$, $\mathfrak{H}_{\psi'}$, corresponding to mother wavelets ψ, ψ' which satisfy the orthogonality condition $C\psi \perp C\psi'$, are orthogonal subspaces of $L^2(\mathbb{H}, d\mu)$.

Acting on a signal $s \in L^2(\mathbb{R}, dx)$ with both sides of the orthogonality relation, and assuming that $\langle C\psi \mid C\psi' \rangle \neq 0$,

$$s = \frac{1}{\langle C\psi \mid C\psi' \rangle} \iint_{\mathbb{H}} d\mu(b, a)\; S_\psi(b, a)\psi'_{b,a}, \qquad \psi'_{b,a} = U(b, a)\psi' \,,$$

where $S_\psi(b, a) = \langle \psi_{b,a} \mid s \rangle$ is the wavelet transform of s computed with respect to the mother wavelet ψ.

Decomposition of the space of all finite energy wavelet transforms

Thus, although the wavelet transform is computed with respect to the mother wavelet ψ, it can be reconstructed using the wavelets of any other mother wavelet ψ', so long as $\langle C\psi \mid C\psi' \rangle \neq 0$.

Moreover, up to a multiplicative constant, the reconstruction formula is exactly the same as that in which the same wavelet ψ is used both for analyzing and reconstructing. This indicates, that in some sense, analysis and reconstruction are independent of the mother wavelet chosen.

Let us choose a set of mother wavelets $\{\psi_n\}_{n=1}^{\infty}$ such that the vectors $\phi_n = C\psi_n$ form an orthonormal basis of $L^2(\mathbb{R}, dx)$,

$$\langle \phi_n \mid \phi_m \rangle = \langle C\psi_n \mid C\psi_m \rangle = \delta_{nm} \, , \qquad n, m = 0, 1, 2, \ldots, \infty \, .$$

Such a basis is easy to find and we shall construct one below. If $\mathfrak{H}_{\psi_n}$, $n = 0, 1, 2, \ldots$, are the corresponding spaces of wavelet transforms and K_{ψ_n} the associated reproducing kernels, then $\mathfrak{H}_{\psi_n} \perp \mathfrak{H}_{\psi_m}$, for $n \neq m$, and

Decomposition of the space of all finite energy wavelet transforms

$$\iint_{\mathbb{H}} d\mu(b'', a'')\, K_{\psi_n}(b, a\, ;\, b'', a'')\, K_{\psi_m}(b'', a''\, ;\, b', a') = \delta_{nm}\, K_{\psi_m}(b, a\, ;\, b', a')\ .$$

More interestingly, it is possible to show that the complete decomposition,

$$L^2(\mathbb{H}, d\mu) \simeq \bigoplus_{n=1}^{\infty} \mathfrak{H}_{\psi_n},$$

of the space of all finite energy signals (on the parameter space $\mathbb{H}$) into an orthogonal direct sum of spaces of wavelet transforms, holds.

Thus, in an L^2-sense, any element $S \in L^2(\mathbb{H}, d\mu)$ has the orthogonal decomposition, into orthogonal wavelet transforms S_n, with respect to a basis of mother wavelets:

$$S(b, a) = \bigoplus_{n=1}^{\infty} S_n(b, a)\ , \quad \text{almost everywhere}\ ,$$

Decomposition of the space of all finite energy wavelet transforms

This result, which can be proved by direct computation in the present case, is a particular example of a much more general result on the decomposition of the left regular representation of a group into irreducibles, as we have already seen. The components $S_n(b, a)$ have the form

$$S_n(b, a) = \langle U(b, a)\psi_n \mid s_n \rangle \ , \qquad n = 0, 1, 2, , \ldots$$

for some signal vectors $s_n \in L^2(\mathbb{R}, dx)$, which, in general, are different for different n. We also have the relations

$$\iint_{\mathbb{H}} d\mu(b', a') \ K_{\psi_n}(b, a \, ; \, b', a') S_m(b', a') = \delta_{nm} \ S_m(b, a) \ .$$

Finally, we construct an explicit example of a basis set of mother wavelets satisfying the orthogonality relations mentioned above.. Let $H_n(\xi)$, $n = 0, 1, 2, \ldots, \infty$, be the Hermite polynomials, normalized as

Decomposition of the space of all finite energy wavelet transforms

$$\int_{\mathbb{R}} d\xi \; e^{-\xi^2} \; H_m(\xi) \; H_n(\xi) = \begin{cases} 0, & \text{if} \;\; m \neq n, \\ 2^n n! \sqrt{\pi}, & \text{if} \;\; m = n. \end{cases}$$

The first few are:

$$H_0(\xi) = 1, \qquad H_1(\xi) = 2\xi,$$
$$H_2(\xi) = 4\xi^2 - 2, \qquad H_3(\xi) = 8\xi^3 - 12\xi, \quad \text{etc.}$$

In the Fourier domain, define

$$\widehat{\psi}_n(\xi) = \frac{1}{\pi^{\frac{3}{4}} \; 2^{\frac{n+1}{2}} \; \sqrt{n!}} \; |\, \xi \,|^{\frac{1}{2}} \; e^{-\frac{\xi^2}{2}} \; H_n(\xi) \; .$$

Then, it is easily verified that

$$\|\widehat{\psi}_n\|^2_{L^2(\widehat{\mathbb{R}}, d\xi)} < \infty \; ,$$

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and

$$\langle \widehat{C}\widehat{\psi}_m \mid \widehat{C}\widehat{\psi}_n\rangle = 2\pi \int_{\mathbb{R}} \frac{d\xi}{|\,\xi\,|}\ \overline{\widehat{\psi}_m(\xi)}\ \widehat{\psi}_n(\xi) = \delta_{mn}\ .$$

Thus, in the inverse Fourier domain, the vectors ψ_n are in the domain of the operator C, satisfying the condition for being mother wavelets, while from the well-known properties of Hermite polynomials, the vectors ϕ_n constitute an orthonormal basis of $L^2(\mathbb{R}, dx)$.
More generally, since the range of C is dense in $L^2(\mathbb{R}, dx)$, we can take any orthonormal basis, $\{\phi_n\}_{n=1}^{\infty}$ of $L^2(\mathbb{R}, dx)$, chosen from vectors in this range and then $\{\psi_n = C^{-1}\phi_n\}_{n=1}^{\infty}$ will be the desired wavelet basis.
We collect the above results into a theorem:

Theorem

The wavelet transform of the space of signals $L^2(\mathbb{R}, dx)$, with respect to the mother wavelet ψ, is a closed subspace of $L^2(\mathbb{H}, d\mu)$. This subspace has a reproducing kernel K_ψ, which is the integral kernel of the projection operator,

Decomposition into orthogonal channels

Theorem (Contd.)

$\mathbb{P}_\psi : L^2(\mathbb{H},\ d\mu) \to \mathfrak{H}_\psi$. *The Hilbert space* $L^2(\mathbb{H},\ d\mu)$ *can be completely decomposed into an orthogonal direct sum of an infinite number of subspaces* $\mathfrak{H}_{\psi_n}$, *each a space of wavelet transforms with respect to a mother wavelet* ψ_n. *The vectors* ψ_n *are constructed by taking an orthonormal basis* $\{\phi_n\}_{n=1}^{\infty}$ *of* $L^2(\mathbb{R}, dx)$, *chosen from the range of the Duflo-Moore operator* C, *and writing* $\psi_n = C^{-1}\phi_n$.

The above theorem can be used to analyze a given wavelet transform into orthogonal channels. Since the mother wavelets $\{\psi_n\}_{n=1}^{\infty}$ form a complete, linearly independent set, any mother wavelet ψ can be written as a linear combination,

$$\psi = \sum_{n=1}^{\infty} a_n \psi_n \ , \qquad a_n = \langle \phi_n \mid C\psi \rangle = \langle \psi_n \mid C^2 \psi \rangle \ .$$

Decomposition into orthogonal channels

Hence if $s \in L^2(\mathbb{R}, dx)$ is any signal vector and $S_\psi(b, a)$ its wavelet transform with respect to ψ, then clearly

$$S_\psi(b,a) = \sum_{n=1}^{\infty} S_n(b,a) , \quad \text{where} \quad S_n(b,a) = a_n \, \langle U(b,a)\psi_n \mid s \rangle \, .$$

In this way, the wavelet transform of the signal $S_\psi(b, a)$ has been decomposed into a a set of mutually orthogonal wavelet transforms $S_n(b, a)$ (of this same signal). We call this a decomposition into orthogonal channels. Note that although the wavelet transforms are orthogonal in $L^2(\mathbb{H}, d\mu)$, the mother wavelets ψ_n are not orthogonal in $L^2(\mathbb{R}, dx)$.

Localization operators

Let ψ be a mother wavelet and $\mathfrak{H}_\psi$ the corresponding space of wavelet transforms. Let $\Delta \subset \mathbb{H}$ be a measurable set (with respect to the measure $d\mu$).
We associate to this set an integral kernel $a_\psi^\Delta : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$:

$$a_\psi^\Delta(b, a;\ b', a') = \iint_\Delta d\mu(b'', a'')\ K_\psi(b, a\,;\ b'', a'')K_\psi(b'', a''\,;\ b', a'),$$

and an operator $a_\psi(\Delta)$ on $\mathfrak{H}_\psi$ acting via this kernel:

$$(a_\psi(\Delta)S)(b, a) = \iint_{\mathbb{H}} d\mu(b', a')\ a_\psi^\Delta(b, a;\ b', a')S(b', a')\ , \qquad S \in \mathfrak{H}_\psi\ .$$

If we compute the matrix element of this operator for the wavelet S, using the properties of the reproducing kernel, we easily obtain,

$$\langle S \mid a_\psi(\Delta)S\rangle = \iint_\Delta d\mu(b, a)\ \mid S(b, a) \mid^2.$$

which shows that this operator is bounded, positive and self-adjoint.

Localization operators

We call $a_\psi(\Delta)$ a localization operator, since in view of the above relation, the quantity

$$p_S(\Delta) = \frac{\langle S \mid a_\psi(\Delta) S\rangle}{\|S\|^2}$$

is the fraction of the wavelet transform which is localized in the region Δ of phase space. If $S(b,a) = \langle \psi_{b,a} \mid s\rangle$, then we shall also write $p_s(\Delta)$ for the above probability, for indeed, it measures the concentration of the phase space content (e.g., time-frequency content) of the signal s in the region Δ.

As a set function, $p_S(\Delta)$ has the properties of a probability measure:

$$p_S(\mathbb{H}) = 1\ , \qquad p_S(\varnothing) = 0\ , \quad \varnothing \ = \ \text{null set}\ ,$$

$$p_S(\bigcup_{i\in J} \Delta_i) = \sum_{i\in J} p_S(\Delta_i)\ , \qquad \text{if}\ \ \Delta_i \bigcap \Delta_j = \varnothing\ , \ \ \text{whenever}\ \ i \neq j\ ,$$

J being some index set. Since this holds for all $S \in \mathfrak{H}_\psi$, we say that the operators $a_\psi(\Delta)$ themselves constitute a positive operator-valued measure or POV-measure, satisfying the properties:

$$a_\psi(\mathbb{H}) = I_\psi , \qquad a_\psi(\varnothing) = 0 ,$$

$$a_\psi(\bigcup_{i\in J} \Delta_i) = \sum_{i\in J} a_\psi(\Delta_i) , \qquad \text{if } \Delta_i \bigcap \Delta_j = \varnothing , \quad \text{whenever } \; i \neq j ,$$

where the sum above has to be understood in the weak sense.
It is instructive to see how $p_S(\Delta)$ changes if the set Δ gets transformed under the action of the group G_{aff}. Since, for any $(b_0, a_0) \in G_{\text{aff}}$,

$$\langle U(b_0, a_0)\psi_{b,a} \mid \psi_{b',a'} \rangle_{L^2(\mathbb{R},dx)} = \langle \psi_{b,a} \mid U(b_0, a_0)^* \psi_{b',a'} \rangle_{L^2(\mathbb{R},dx)} ,$$

using the group properties and the definition of the reproducing kernel, we find that it satisfies the following covariance property:

Localization operators

$$K_\psi(b_0 + a_0 b, a_0 a\,;\, b', a') = K_\psi(b, a\,;\, \frac{b' - b_0}{a_0}, \frac{a}{a_0})\ ,$$

i.e.,

$$K_\psi((b_0, a_0)(b, a)\,;\, b', a') = K_\psi(b, a\,;\, (b_0, a_0)^{-1}(b', a'))\ .$$

Let $(b_0, a_0)\Delta$ denote the translate of the set Δ by (b_0, a_0):

$$(b_0, a_0)\Delta = \{(b_0 + a_0 b, a_0 a) \in \mathbb{H} \,|\, (b, a) \in \Delta\}\ .$$

Then, taking note of the action of the left regular representation of G_{aff} on wavelet transforms and exploiting the invariance of the measure $d\mu$, we easily find that

$$\langle \mathbf{U}_\psi(b, a)S \,|\, a_\psi(\Delta)\mathbf{U}_\psi(b, a)S' \rangle = \langle S \,|\, a_\psi((b, a)^{-1}\Delta)S' \rangle\ , \qquad S, S' \in \mathfrak{H}_\psi\ ,$$

Localization operators

i.e., we have the operator identity

$$\mathbf{U}_\psi(b,a) a_\psi(\Delta) \mathbf{U}_\psi(b,a)^* = a_\psi((b,a)\Delta) \ .$$

This is a group covariance condition satisfied by the localization operators $a_\psi(\Delta)$, and is generally known as an imprimitivity relation. For the probability measure $p_S(\Delta)$, this condition implies the transformation property

$$p_S((b,a)\Delta) = p_{\mathbf{U}_\psi(b,a)^{-1}S}(\Delta) \ , \quad \text{or}, \quad p_s((b,a)\Delta) = p_{U(b,a)^{-1}s}(\Delta) \ .$$

Physically, the above relation means that the fraction of the signal s, localized in the transformed set $(b,a)\Delta$, is the same as the fraction of the transformed signal $U(b,a)^{-1}s$ localized in the original set Δ.

Restriction to G_{aff}^{+}

We turn our attention to a different way of understanding the wavelet transform, namely, as a function on a phase space.

First let us restrict the representation $\widehat{U}(b, a)$ of the full affine group to the connected affine group G_{aff}^{+}, characterized by $a > 0$. We immediately see that this representation is no longer irreducible for this smaller group. Indeed, consider the two subspaces

$$
\begin{aligned}
\widehat{\mathfrak{H}}_{+}(\widehat{\mathbb{R}}) &= \{\widehat{f} \in L^2(\widehat{\mathbb{R}}, d\xi) \mid \widehat{f}(\xi) = 0, \ \forall \xi \leq 0\}, \\
\widehat{\mathfrak{H}}_{-}(\widehat{\mathbb{R}}) &= \{\widehat{f} \in L^2(\widehat{\mathbb{R}}, d\xi) \mid \widehat{f}(\xi) = 0, \ \forall \xi \geq 0\},
\end{aligned}
$$

of the carrier space $L^2(\widehat{\mathbb{R}}, d\xi)$ of the representation $\widehat{U}(b, a)$.

It is evident that vectors in any one of these subspaces are mapped to vectors in the same subspace under the action of the operators $\widehat{U}(b, a)$, for $(b, a) \in G_{\text{aff}}^{+}$.

This means that that each one of these subspaces carries a unitary representation of this smaller group.

Restriction to G_{aff}^{+}

We can show that both these representations, which we denote by $\widehat{U}_{+}(b, a)$ and $\widehat{U}_{-}(b, a)$, respectively, are irreducible but unitarily inequivalent.
In fact, these are the only two nontrivial, unitary irreducible representations of G_{aff}^{+}.
Moreover, the Hilbert space $L^2(\widehat{\mathbb{R}}, d\xi)$ is the orthogonal direct sum of these two subspaces:

$$L^2(\widehat{\mathbb{R}}, d\xi) = \widehat{\mathfrak{H}}_{+}(\widehat{\mathbb{R}}) \oplus \widehat{\mathfrak{H}}_{-}(\widehat{\mathbb{R}}).$$

In other words, we have here a complete decomposition of the representation $\widehat{U}(b, a)$ of the connected affine group G_{aff}^{+} and we write

$$\widehat{U}(b, a) = \widehat{U}_{+}(b, a) \oplus \widehat{U}_{-}(b, a) \ .$$

In the inverse Fourier domain, the representation $U(b, a)$ similarly breaks up (because of the unitarity of the Fourier transform) into two irreducible representations, $U_{+}(b, a)$ and $U_{-}(b, a)$, on the two subspaces

Holomorphic wavelet transforms

$$\begin{aligned} \mathfrak{H}_+(\mathbb{R}) &= \{f \in L^2(\mathbb{R}, dx) \mid \widehat{f}(\xi) = 0, \ \forall \xi \leq 0\}, \\ \mathfrak{H}_-(\mathbb{R}) &= \{f \in L^2(\mathbb{R}, dx) \mid \widehat{f}(\xi) = 0, \ \forall \xi \geq 0\}, \end{aligned}$$

of $L^2(\mathbb{R}, dx)$, respectively. These spaces are known as Hardy spaces.
Elements of $\mathfrak{H}_+(\mathbb{R})$ (respectively, $\mathfrak{H}_-(\mathbb{R})$) extend to functions analytic in the upper (respectively, lower) complex half-plane, and accordingly they are called upper (respectively, lower) analytic signals.
It is an interesting fact that for appropriate choices of mother wavelets, the wavelet transforms of signals in the spaces $\mathfrak{H}_\pm(\mathbb{R})$ can (up to a factor) become holomorphic functions. Indeed, for any $\nu \geq 0$, consider the mother wavelet $\widehat{\psi} \in \widehat{\mathfrak{H}}_+(\widehat{\mathbb{R}})$,

$$\widehat{\psi}(\xi) = \begin{cases} \left[\frac{2^\nu}{\pi \, \Gamma(\nu+1)}\right]^{\frac{1}{2}} \xi^{\frac{\nu+1}{2}} \, e^{-\xi} \, , & \xi \in (0, \infty) \, , \\ 0 \, , & \text{otherwise} \, , \end{cases} \qquad \|\widehat{C}\widehat{\psi}\|^2 = 1 \, ,$$

$\Gamma(\nu + 1)$ being the usual Gamma function.

The wavelets for this vector have the form

$$\widehat{\psi}_{b,a}(\xi) = \left[\frac{2^{\nu}}{\pi\,\Gamma(\nu+1)}\right]^{\frac{1}{2}} a^{1+\frac{\nu}{2}}\ \xi^{\frac{\nu+1}{2}}\ e^{-i\xi\overline{z}}\ , \qquad \text{where,}\ \ z = b + ia\ ,$$

and the reproducing kernel is

$$K_{\psi}(b, a\,;b', a') = \langle \widehat{\psi}_{b,a} \,|\, \widehat{\psi}_{b',a'} \rangle = \frac{2^{\nu}\ (\nu+1)}{i^{2+\nu}\ \pi}\ \frac{(aa')^{1+\frac{\nu}{2}}}{(\overline{z}' - z)^{2+\nu}}\ .$$

Computing the wavelet transform of a signal $\widehat{s} \in \mathfrak{H}_{+}(\mathbb{R})$,

$$S(b, a) = \langle \widehat{\psi}_{b,a} \,|\, \widehat{s} \rangle = \left[\frac{2^{\nu}}{\pi\,\Gamma(\nu+1)}\right]^{\frac{1}{2}} a^{1+\frac{\nu}{2}} \int_{0}^{\infty} d\xi\ \xi^{\frac{1+\nu}{2}}\ e^{i\xi z}\ \widehat{s}(\xi)\ .$$

This, apart from the factor $a^{1+\frac{\nu}{2}}$, is a holomorphic function of $z = b + ia$.

Holomorphic wavelet transforms

Indeed, writing this function as

$$F(z) = \left[\frac{2^{\nu}}{\pi\,\Gamma(\nu+1)}\right]^{-\frac{1}{2}} a^{-(1+\frac{\nu}{2})} S(b,a)$$

$$= \int_0^{\infty} d\xi\; e^{i\xi z}\, \xi^{\frac{1+\nu}{2}}\, \widehat{s}(\xi)\,, \qquad z = b + ia\,,$$

we have,

$$|\,F(z)\,|^2 = \left|\int_0^{\infty} d\xi\; \xi^{\frac{1+\nu}{2}}\; e^{i\xi z}\, \widehat{s}(\xi)\right|^2 = \left|\int_0^{\infty} d\xi\; \xi^{\frac{1+\nu}{2}}\; e^{-\xi a} e^{i\xi b}\, \widehat{s}(\xi)\right|^2$$

$$\leq \|\widehat{s}\|^2 \int_0^{\infty} d\xi\; \xi^{1+\nu}\; e^{-2\xi a}\,, \quad \text{by the Cauchy-Schwarz inequality}$$

$$= \frac{\|\widehat{s}\|^2\, \Gamma(\nu+2)}{(2a)^{2+\nu}}\,.$$

Thus, the convergence of the integral representing $F(z)$ is uniform in any bounded open set containing z and differentiation with respect to it, under the integral sign, is permissible, implying that F is holomorphic in z, on the complex, upper-half plane which we identify with $\mathbb{H}_+ = \mathbb{R} \times \mathbb{R}_+^*$ (where, $\mathbb{R}_+^* = (0, \infty)$). We shall call $F(z)$ a holomorphic wavelet transform.

Furthermore, since

$$\iint_{\mathbb{H}_+} \frac{db\, da}{a^2} \,|\, S(b,a)\,|^2 = \iint_{\mathbb{H}_+} d\mu_\nu(z, \overline{z}) \,|\, F(z)\,|^2 \,,$$

where we have introduced the measure,

$$d\mu_\nu(z, \overline{z}) = \frac{(2a)^\nu}{\pi\, \Gamma(\nu + 1)}\, db\, da \,,$$

the set of all holomorphic wavelet transforms constitute a closed subspace of of the Hilbert space $L^2(\mathbb{H}_+,\ d\mu_\nu)$, of functions supported on the upper half plane. We denote this subspace by $\mathfrak{H}_{\text{hol}}^\nu$.

Holomorphic wavelet transforms

Note that $\mathfrak{H}^{\nu}_{\text{hol}}$ is also a reproducing kernel Hilbert space, with kernel

$$K^{\nu}_{\text{hol}}(z,\ \overline{z}') = \frac{\Gamma(\nu+2)}{i^{2+\nu}}\ \frac{1}{(z-\overline{z}')^{2+\nu}}\ ,$$

such that

$$\iint_{\mathbb{H}_+} d\mu_\nu(z,\overline{z})\ K^{\nu}_{\text{hol}}(z,\ \overline{z}')\ F(z')$$
$$= \frac{2^{\nu}\ (\nu+1)}{i^{2+\nu}} \iint_{\mathbb{H}_+} a^{\nu}\ da\ db\ \frac{F(z')}{(z-\overline{z}')^{2+\nu}} = F(z)\ .$$

The vectors $\eta_{\overline{z}},\quad z\in\mathbb{H}_+$, with

$$\eta_{\overline{z}}(z') = K^{\nu}_{\text{hol}}(z',\ \overline{z})\ ,$$

which are the holomorphic wavelets, are again overcomplete in $\mathfrak{H}^{\nu}_{\text{hol}}$ and satisfy the resolution of the identity:

Holomorphic wavelet transforms

$$\iint_{\mathbb{H}_+} d\mu_\nu(z, \overline{z}) \ |\eta_{\overline{z}}\rangle\langle\eta_{\overline{z}}| = I^\nu_{\text{hol}} \quad (= \text{ identity operator of } \mathfrak{H}^\nu_{\text{hol}}) \, .$$

There is also the holomorphic representation of G^+_{aff} on $\mathfrak{H}^\nu_{\text{hol}}$, unitarily equivalent to $U_+(b, a)$. Denoting this by $U^\nu_{\text{hol}}(b, a)$, we easily compute its action:

$$(U^\nu_{\text{hol}}(b, a)F)(z) = a^{-(1+\frac{\nu}{2})} F\left(\frac{z - b}{a}\right) \, .$$

The appearance of the holomorphic Hilbert spaces of wavelet transforms is remarkable in many ways.

First of all, their existence is related to a geometrical property of the half plane $\mathbb{H}_+$, which is a differential manifold with a complex Kähler structure.

This means, from a physical point of view, that it has all the properties of being a phase space of a classical mechanical system and additionally, that this phase space can be given a complex structure (consistent with its geometry).

Complex structure of phase space

In particular, it has a metric and a preferred differential two-form, which gives rise to the invariant measure $d\mu$ and using which classical mechanical quantities, such as Poisson brackets, may be defined.

We will not go into the details of this here, but only point out the existence of a potential function in this context. Consider the function,

$$\Phi(z, \overline{z}') = -\log[-(z - \overline{z}')^2] .$$

This function is called a Kähler potential for the space $\mathbb{H}_+$ and it generates all the interesting quantities characterizing its geometry, such as the invariant two-form and the invariant measure. Indeed, we immediately verify that,

$$e^{(1+\frac{\nu}{2})\ \Phi(z,\overline{z}')} = \frac{i^{2-\nu}}{\Gamma(\nu+2)}\ K^{\nu}_{\text{hol}}(z, \overline{z}') .$$

Next we define,

$$\Omega(z, \overline{z}) := \frac{1}{i}\ \frac{\partial^2 \Phi(z, \overline{z})}{\partial z\ \partial \overline{z}}\ dz \wedge d\overline{z} = \frac{db \wedge da}{a^2} ,$$

which is the invariant two-form (under the action of G_{aff}^+).
This gives the invariant measure $d\mu$ of the group and furthermore,

$$e^{-(1+\frac{\nu}{2})\ \Phi(z,\bar{z})}\ \Omega = 4(-1)^\nu\ (2a)^\nu\ db \wedge da\ ,$$

from which follows the measure with respect to which the holomorphic functions $F(z)$ are square integrable and form a Hilbert space. It ought to be emphasized here that $\mathfrak{H}_{\text{hol}}^\nu$ contains all holomorphic functions in $L^2(\mathbb{H}_+,\ d\mu_\nu)$. Note also that any such function can be obtained by computing the Fourier transform of a function $f(\xi) = \xi^{\frac{1+\nu}{2}}\ \widehat{s}(\xi)$ and then analytically continuing it to the upper half plane, where $\widehat{s}$ is a signal in the Fourier domain, with support in $(0, \infty)$.
Additionally, it ought to be noted that, for each $\nu > 0$ we get a family of holomorphic wavelet transforms, so that depending on the value of ν, the same signal s can be represented by different holomorphic functions on phase space.