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A survey of uncertainty principles, in continuous and discrete settings

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Uncertainty Principles, Lecture I

Variance Inequalities

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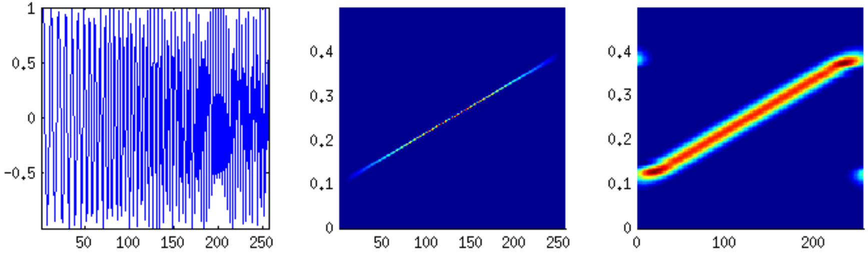
Preliminaries I

This lecture is about uncertainty principles and inequalities

- Quantum physics: impossibility to measure simultaneously *incoherent* quantities with arbitrary precision (example: position/momentum, time/frequency); how to model this problem ?
- Harmonic analysis: a function (or vector) cannot be represented in a *concentrated'* way simultaneously in two *incoherent* representations. How to describe this property ?
- Signal processing: to which extent can we define a single note of musics (time/frequency)? how to design radar waveforms in such a way that the reflected waves can be optimally detected ?

Preliminaries II

Signal analysis example: different ways of representing a linearly frequency modulated signal:

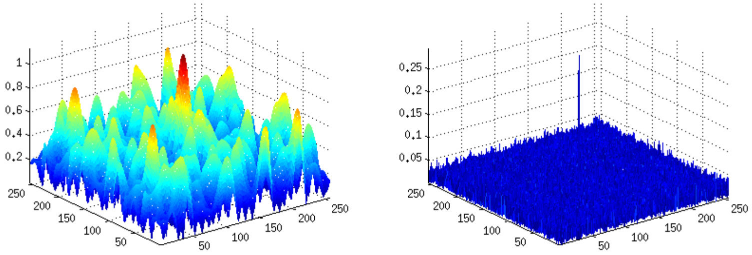


Time course (left), and time-frequency representations: Wigner function (middle) and Gabor (right)

Obviously, the Wigner function displays the information in a much neater way: frequency that varies linearly as a function of time.

Preliminaries III

radar/sonar detection problem: the location of the tallest peak gives an estimate for the parameters (location and speed) of the target



Gaussian waveform (left), and optimized waveform (right)

Obviously, detection will be easier with the optimized waveform.

Mathematics background I

- Hilbert space $\mathcal{H}$: linear space (vectors), equipped with an inner product $x, y \in \mathcal{H} \mapsto \langle x, y\rangle \in \mathbb{C}$
  - $\langle \alpha x + \alpha' x', y\rangle = \alpha\langle x, y\rangle + \alpha'\langle x', y\rangle$
  - $\langle \alpha x, \beta y + \beta' y'\rangle = \overline{\beta}\langle x, y\rangle + \overline{\beta}'\langle x, y'\rangle$
  - Norm: $\|x\| = \sqrt{\langle x, x\rangle}$.
- Examples (signal models)
  - $\mathbb{C}^N$: finite sequences. $\langle x, y\rangle = \sum_{n=0}^{N-1} x[n]\overline{y}[n]$.
  - $\ell^2(\mathbb{Z})$: finite energy infinite sequences. $\langle x, y\rangle = \sum_{-\infty}^{\infty} x[n]\overline{y}[n]$, provided $\sum |x[n]|^2$ and $\sum |y[n]|^2$ converge.
  - $L^2(\mathbb{R})$: finite energy continous time functions $\langle x, y\rangle = \int_{-\infty}^{\infty} x(t)\overline{y}(t)dt$, provided $\int |x(t)|^2 dt$ and $\int |y(t)|^2 dt$ converge.
  - $L^2([0,1])$: guess...

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Mathematics background II

- Other norms: measure different properties of vectors/sequences/functions. In particular L^p (quasi)norms

$$\|x\|_p = \left(\sum_n |x[n]|^p\right)^{1/p} .$$

  For $p < 2$, measure spreading (or dispersion); for $p > 2$, measure concentration (or sparsity).

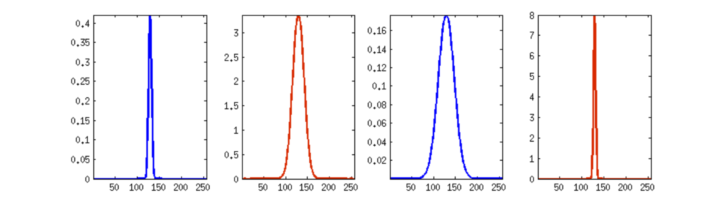
- Fourier transforms (all sorts of...), Mellin transform,... defined on the fly

Uncertainty principle: a long story: I

Generalities

Uncertainty inequalities: impossibility for a function (or a vector) to be simultaneously *sharply concentrated* in two different representations, provided the latter are *incoherent enough*.

Example: time representation and frequency representation.



Gaussian functions at different scales (blue) and their Fourier transform (red)

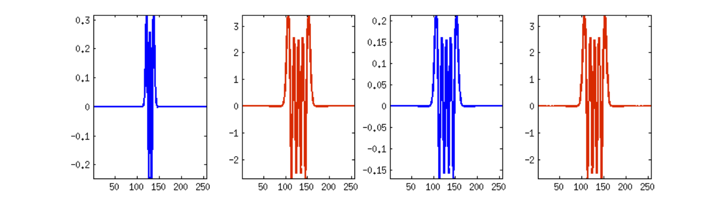
.

Uncertainty principle: a long story: II

Generalities

Uncertainty inequalities: impossibility for a function (or a vector) to be simultaneously *sharply concentrated* in two different representations, provided the latter are *incoherent enough*.

Example: time representation and frequency representation.



Hermite functions (order 4) at different scales (blue) and their Fourier transform (red)

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Uncertainty principle: a long story: III

Generalities

Uncertainty inequalities: impossibility for a function (or a vector) to be simultaneously *sharply concentrated* in two different representations, provided the latter are *incoherent enough*. Such a loose definition can be made concrete by further specifying the following main ingredients:

- A global setting, generally a couple of Hilbert spaces (of functions or vectors) providing two representations for the objects of interest (e.g. time and frequency, or more general phase space variables).
- An invertible linear transform (operator, matrix) mapping the initial representation to the other one, without information loss.
- A concentration measure for the elements of the two representation spaces: variance, entropy, L^p norms,...

Uncertainty principle: a long story: IV

- W. Heisenberg (1927): *One can never know with perfect accuracy both of those two important factors which determine the movement of one of the smallest particles-its position and its velocity. It is impossible to determine accurately both the position and the direction and speed of a particle at the same instant*.
- Heisenberg: *non-commutativity implies uncertainty*, for position and momentum operators. Uncertainty is measured by variance.
- Robertson (1929) and Schrödinger (1930): extend variance inequality to arbitrary pairs of non-commuting self-adjoint operators.
- Robertson-Schrödinger inequality criticized because the bound depends on the state. Entropic inequality proposed by Everett and Hirschman (1957) proven later by Babenko and Beckner (1975) and Lieb independently (1978).
- Recently, generalizations in various contexts (Maassen-Uffink (1988), Ozaydin-Przebinda (2004), Rastegin (2010) ...)

Uncertainty principle: a long story: V

Uncertainty also showed up in other fields, for example:

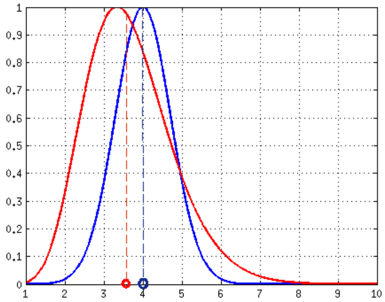
- *Information theory:* Dembo rediscovers the result by Maassen and Uffink (1991), together with other inequalities. These results are apparently useful for quantum cryptography problems.
- *Signal processing:* Lieb proves an uncertainty relation for radar ambiguity functions (1990), extended further by P. Flandrin for affine ambiguity functions.
- *"Sparsistics":* Recent works by Donoho-Stark (1989) in the continuous case, and Donoho-Huo (1999), Elad-Bruckstein (2002), Tao (2005), Meshulam (2006), Krahmer *et al* (2008), Ghobbers-Jaming (2010)... more recently in discrete settings.

A main aspect in the recent developments is the exploitation of uncertainty relations as a constructive result, compressed sensing is a good illustration.

Today: Variance as a measure of dispersion

First concepts: mean for localization, variance for dispersion.

$$e_x = \frac{1}{\|x\|^2} \int t|x(t)|^2\, dt \qquad v_x = \frac{1}{\|x\|^2} \int (t - e_x)^2 |x(t)|^2\, dt$$



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Heisenberg-Weyl inequality

Define the (unitary) Fourier transformation $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as

$$\hat{x}(\nu) = \int_{-\infty}^{\infty} x(t) e^{-2i\pi\nu t}\, dt \ .$$

Theorem (Heisenberg-Weyl)

For all $x \in L^2(\mathbb{R})$ and $t_0, \nu_0 \in \mathbb{R}$

$$\int_{-\infty}^{\infty} (t - t_0)^2 |x(t)|^2\, dt \ \times \ \int_{-\infty}^{\infty} (\nu - \nu_0)^2 |\hat{x}(\nu)|^2\, d\nu \geq \frac{\|x\|^4}{16\pi^2} \ ,$$

with equality if and only if x is a Gaussian function (suitably translated, modulated, rescaled).

$$g(t) = a e^{-(t-t_0)^2/2\sigma^2} e^{2i\pi\nu_0 t}$$

with $a \in \mathbb{C}$ and $\sigma \in \mathbb{R}$.

Sketch of the proof I

Let $x \in L^2(\mathbb{R})$. Without loss of generality, assume that all quantities are finite (otherwise the inequality is trivially satisfied). Then $\int \nu^2 |\hat{x}(\nu)|^2 d\nu < \infty$ implies that x is continuous and $x' \in L^2(\mathbb{R})$.
Assume first $t_0 = \nu_0 = 0$, and integrate by parts

$$\int_u^v t \frac{d}{dt} |x(t)|^2 dt = \left[t|x(t)|^2 \right]_u^v - \int_u^v |x(t)|^2 dt \ .$$

As $x, x' \in L^2(\mathbb{R})$, as well $t \to tx(t)$, the above integrals have finite limits as $u \to -\infty, v \to \infty$, as well as $u|x(u)|^2$ and $v|x(v)|^2$. As $x \in L^2(\mathbb{R})$, those limits necessarily vanish.
Take absolute values and develop:

$$||x||^2 \leq \left| \int_{-\infty}^{\infty} tx(t)\overline{x'}(t)dt \right| + \left| \int_{-\infty}^{\infty} tx'(t)\overline{x}(t)dt \right| \ .$$

Sketch of the proof II

The Cauchy-Schwarz inequality yields

$$||x||^2 \leq 2\sqrt{\int_{-\infty}^{\infty} t^2|x(t)|^2 dt \int_{-\infty}^{\infty} |x'(t)|^2 dt} .$$

Since the Fourier transform of x' is the function $\nu \to 2i\pi\nu\hat{x}(\nu)$, Plancherel's formula gives the desired result

$$\int_{-\infty}^{\infty} |x'(t)|^2 dt = 4\pi^2 \int_{-\infty}^{\infty} \nu^2|\hat{x}(\nu)|^2 d\nu ,$$

Cauchy-Schwarz inequality is an equality iff the two functions involved are proportional, i.e. $x'(t) = Kx(t)$, leading to centered gaussian functions.

As for the general case, let y be defined by

$$y(t) = e^{-2i\pi\nu_0 t}x(t + t_0) .$$

Clearly $\|y\| = \|x\|$, and the result above applied to y yields directly the desired bound.

Variances: Robertson-Schrödinger inequality

In a Hilbert space setting, associate with a symmetric operator A its mean and variance in state $x \in \mathcal{H}$ (for simplicity we assume $\|x\| = 1$) by

$$e_x(A) = \langle Ax, x\rangle \ , \qquad v_x(A) = e_x(A^2 - e_x(A)^2) = e_x(A^2) - e_x(A)^2 \ .$$

Theorem (Robertson/Schrödinger)

Let $x \in \mathcal{H}$, with $\|x\| = 1$. Let A, B be symmetric continuous operators on $\mathcal{H}$. Denote by $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$ the commutator and the anti-commutator of A and B. Then for all $x \in \mathcal{D}([A, B])$

$$v_x(A)v_x(B) \geq \frac{1}{4}\left[e_x([A, B])^2 + e_x(\{A - e_x(A), B - e_x(B)\})^2\right] \ .$$

The proof is below.

Proof of the Robertson-Schrödinger inequality I

Let A, B be symmetric operators on $\mathcal{H}$, and set as before $v_x(A) = \|(A - e_x(A))x\|^2 = e_x(A^2) - e_x(A)^2$.
Set

$$f = (A - e_x(A))x \qquad \text{and} \qquad g = (B - e_x(B))x \ ,$$

so that $v_x(A) = \|f\|^2$ and $v_x(B) = \|g\|^2$.

Therefore

$$v_x(A)v_x(B) = \|f\|^2 \|g\|^2 \geq |\langle f, g\rangle|^2 \ .$$

Now

$$|\langle f, g\rangle|^2 = \left(\frac{\langle f, g\rangle + \langle g, f\rangle}{2}\right)^2 + \left(\frac{\langle f, g\rangle - \langle g, f\rangle}{2i}\right)^2 \ ,$$

and

$$\langle f, g\rangle = \langle (A - e_x(A))x, (B - e_x(B))x\rangle \ .$$

Proof of the Robertson-Schrödinger inequality II

Now since A and B are symmetric,

$$\langle f, g\rangle + \langle g, f\rangle = \langle\{A - e_x(A), B - e_x(B)\}x, x\rangle$$

and

$$\langle f, g\rangle - \langle g, f\rangle = \langle[A - e_x(A), B - e_x(B)]x, x\rangle = \langle[A, B]x, x\rangle$$

Then

$$\begin{aligned} v_x(A)v_x(B) &\geq \frac{1}{4}\left[\langle\{A - e_x(A), B - e_x(B)\}x, x\rangle^2 + |\langle[A, B]x, x\rangle|^2\right] \\ &\geq \frac{1}{4}\left[e_x\left(\{A - e_x(A), B - e_x(B)\}\right)^2 + |e_x[A, B]|^2\right] \\ &\geq \frac{1}{4}\left[\left(e_x(\{A, B\}) - e_x(A)e_x(B)\right)^2 + |e_x([A, B])|^2\right] \ , \end{aligned}$$

which is the desired result.

Proof of the Robertson-Schrödinger inequality III

Remark

- Ignoring the anticommutator term leads to the standard (Robertson) inequality

$$v_x(A)v_x(B) \geq \frac{1}{4}\left|e_x\left([A,B]\right)\right|^2 .$$

- The proof above assumes that Bx belongs to the domain of A, which is not necessarily the case. In some situations, this may lead to delicate problems.

Back to Heisenberg-Weyl inequality

The Heisenberg-Weyl inequality appears as a particular case. Set $\mathcal{H} = L^2(\mathbb{R})$, and take for $A = T$ and $B = F$ the infinitesimal generators of modulations and translations

$$[Tx](t) = tx(t) \;, \qquad [Fx](t) = \frac{i}{2\pi} x'(t) \;, \qquad [T, F] = -\frac{i}{2\pi}\ \mathrm{I}_{L^2(\mathbb{R})}$$

yields the standard Heisenberg-Weyl inequality

$$v_x(T) v_x(F) \geq \frac{1}{16\pi^2} \;,$$

that coincides with the inequality above.

Generalizations to higher dimensions are straightforward.

Back to Heisenberg-Weyl inequality (2)

Taking into account the anticommutator term: introduce the *time-frequency covariance*

$$c(x) = \int_{-\infty}^{\infty} t|x(t)|^2 \frac{d\arg(x(t))}{dt}\, dt\ ,$$

The corresponding generalized Heisenberg-Weyl inequality becomes

Corollary

Let $x \in L^2(\mathbb{R})$ *be such that* $\|x\| = 1$. *For all* $t_0, \nu_0 \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} (t - t_0)^2 |x(t)|^2\, dt \ \times\ \int_{-\infty}^{\infty} (\nu - \nu_0)^2 |\hat{x}(\nu)|^2\, d\nu \geq \frac{1}{16\pi^2}\ \left(1 + c(x)^2\right)\ ,$$

with equality if and only if x *is a linear Gaussian chirp function (Gaussian function with linear frequency modulation).*

Back to Heisenberg-Weyl inequality (2)

Example: chirps

- for linear chirps, $x(t) = e^{-\pi t^2} e^{2i\pi\alpha t^2/2}$, the time-frequency covariance equals the linear modulation rate $c(x) = \alpha$.
- for cubic chirps, $x(t) = e^{-\pi t^2} e^{2i\pi(\alpha t^2/2+\beta t^3/3+\gamma t^4/4)}$, the time-frequency covariance equals $c(x) = \alpha + 3\gamma/4\pi^2$.
- ...

Remark

- The bound $\sqrt{1 + c(x)^2}$ obviously depends on the function x.
- Uncertainty equalizers are not necessarily uncertaity minimizers...

Time-scale and other generalizations

- The previous inequality addresses time-frequency uncertainty, and rests on translation and modulation operators
- There are other representation domains of interest
  - Time-scale (translation and dilation operators)
  - Higher dimensional position-momentum (straightforward extension of the previous inequality)
  - Higher dimensional position, momentum, scale, shear,...

Time-scale: Klauder's inequality I

Let $A = D$ and $B = F$ be infinitesimal generator of dilations and translations

$$[Dx](t) = \frac{i}{2\pi}\left(\frac{1}{2}x(t) + tx(t)\right) \ , \quad [Fx](t) = \frac{i}{2\pi}x'(t) \ , \quad [D,F] = -\frac{i}{2\pi}F \ .$$

Introduce the Fourier-Mellin transformation on the Hardy space

$$H^2(\mathbb{R}) = \left\{x \in L^2(\mathbb{R}),\ \hat{x} = 0 \text{ on } \mathbb{R}_-\right\} \ .$$

defined by logarithmic warping followed by Fourier transformation

$$\underline{x}(s) = \int_0^\infty \hat{x}(\nu)\nu^{-2i\pi s}\frac{d\nu}{\sqrt{\nu}} = \int_0^\infty \hat{x}(\nu)e^{-2i\pi \ln(\nu)s}\frac{d\nu}{\sqrt{\nu}} \ .$$

The Fourier-Mellin transform is unitary: for all $x \in H^2(\mathbb{R})$,

$$\|\underline{x}\|_{L^2(\mathbb{R})} = \|x\|_{H^2(\mathbb{R})} \ .$$

Time-scale: Klauder's inequality II

Corollary (Klauder's inequality)

For all $x \in \mathcal{D}(F)$ *with* $\|x\| = 1$

$$v_x(D)v_x(F) \geq \frac{1}{16\pi^2} e_x(F)^2 \ .$$

More concretely, for all $s_0 \in \mathbb{R}$ *and* $\nu_0 \in \mathbb{R}_+$,

$$\sqrt{\int_0^\infty (\nu - \nu_0)^2 |\hat{x}(\nu)|^2 \, d\nu} \times \sqrt{\int (s - s_0)^2 |\underline{x}(s)|^2 \, ds} \geq \frac{1}{4\pi} \int_0^\infty \nu |\hat{x}(\nu)|^2 \, d\nu \ ,$$

Equality is reached for specific waveforms (Klauder waveforms).

$$\hat{x}(\nu) = K \exp\left\{a \ln(\nu) - b\nu + i(c \ln(\nu) + d)\right\} \ , \qquad \nu \in \mathbb{R}^+$$

Time-scale: Klauder's inequality III

Remark

the bound depends explicitely (and strongly) on x (Maass et al 2010 [17]); uncertainty equalizers do not coincide with uncertainty minimizers.

Example (Maass & al 2010) [17]

The family of functions

$$x_n(t) = \sqrt{\frac{3}{2\sqrt{n}}} \begin{cases} 1 - \frac{|t|}{n} & \text{for} \quad |t| \leq n \\ 0 & \text{otherwise} \end{cases}$$

satisfy $\|x_n\| = 1$, $e_{x_n}(D) = e_{x_n}(F) = 0$, $v_{x_n}(D) = 3/4$ and $v_{x_n}(F) = 3/n^2$ for all n.

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Altes inequality I

Alternative: (Altes 1979, Flandrin 2001, Ricaud & BT 2013) geometric mean: for $x \in H^2(\mathbb{R})$, set

$$\tilde{e}_{\hat{x}} = \exp\left\{\frac{1}{\|\hat{x}\|^2}\int_0^\infty |\hat{x}(\nu)|^2 \ln(\nu) d\nu\right\} \ , \qquad .$$

In this new setting, one obtains a more familiar inequality, for $x \in H^2(\mathbb{R})$ such that $\|x\| = 1$,

$$\int_0^\infty \left[\ln(\nu/\tilde{e}_{\hat{x}})\right]^2 |\hat{x}(\nu)|^2 d\nu \times \int (s-s_0)^2 |\underline{x}(s)|^2 \, ds \geq \frac{1}{16\pi^2}$$

with equality if and only if x takes the form of an Altes waveform,

$$\hat{x}(\nu) = K \exp\left\{-\frac{1}{2}\ln(\nu) - a\ln^2(\nu/b) + i(c\ln(\nu) + d)\right\} \ , \qquad \nu \in \mathbb{R}^+ \ .$$

Altes inequality II

Connection with Heisenberg-Weyl:

Let $U : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the (unitary) warping operator defined by

$$\widehat{Ux}(\nu) = e^{\nu/2}\hat{x}(e^{\nu}) \ , \nu \in \mathbb{R}_+ \ , \qquad U^* x(s) = x(\ln(s))/\sqrt{s} \ , s \in \mathbb{R}_+$$

Consider now the linear operators $\tilde{T}$ and $\tilde{F}$ on $H^2(\mathbb{R})$ defined by

$$\tilde{T} = U^* TU \ , \qquad \tilde{F} = U^* FU \ .$$

Altes inequality III

Lemma (Warping)

1. *The warped operators* $\tilde{T}$ *and* $\tilde{F}$ *relate to dilation and frequency as*

$$\tilde{T} = \frac{1}{2\pi} D \;, \qquad \tilde{F} = 2\pi \ln\left(\frac{F}{2\pi}\right) \;.$$

2. $\tilde{T}$ *and* $\tilde{F}$ *satisfy the canonical commutation relations on* $H^2(\mathbb{R})$*:*

$$[D, \ln(F)] = [D, \ln(F/2\pi)] = [\tilde{T}, \tilde{F}] = U^* [T, F] U = -\frac{i}{2\pi} \mathrm{I}_{H^2(\mathbb{R})} \;.$$

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Altes inequality IV

Given any self adjoint operator A on $H^2(\mathbb{R})$, and for any $x \in H^2(\mathbb{R})$, set $y = Ux$; then

$$e_x(A) = \langle x, Ax \rangle = \langle U^* y, AU^* y \rangle = e_y(UAU^*) \ , \quad \text{and similarly}$$

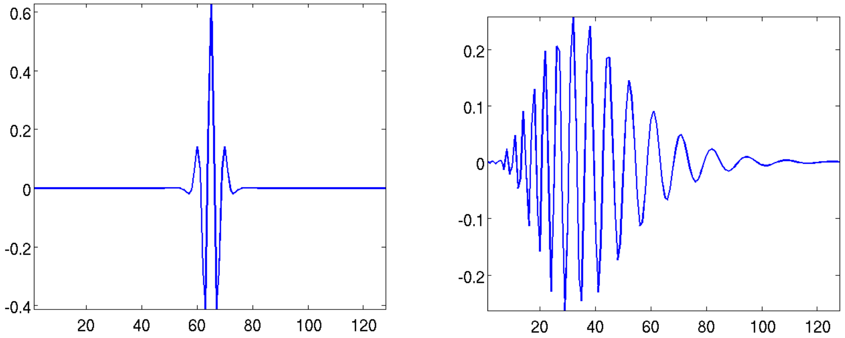
$$v_x(A) = e_x(A^2) - e_x(A)^2 = e_y((UAU^*)^2) - e_y(UAU^*)^2 = v_y(UAU^*)$$

Therefore,

$$v_x(D).v_x(\ln(F)) = v_y(T).v_y(F) \geq \frac{1}{16\pi^2} \ ,$$

with equality if and only if y is a Gaussian function, i.e. x is an Altes wavelet.

Klauder and Altes wavelets



Breitenberger's inequality I

Breitenberger's inequality addresses the case of continuous-time, periodic functions (or equivalently, by Fourier duality, discrete-time, infinite sequences).

Fourier transformation: given $x \in L^2([0,1])$, define its Fourier (coefficient) transform $\hat{x}$ by

$$\hat{x}[n] = \int_0^1 x(t) e^{-2i\pi nt}\, dt\ , \qquad x(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n] e^{2i\pi nt}\ ,$$

in the "standard" L^2 convergence sense:

$$\lim_{N\to\infty} \left\| x - \sum_{n=-N}^{N} \hat{x}[n] e^{2i\pi n\cdot} \right\| = 0\ .$$

Breitenberger's inequality II

On $L^2([-\pi,\pi])$, define the operators A and B

$$Ax(t) = e^{2i\pi t}x(t)\ , \qquad Bx(t) = -\frac{i}{2\pi}x'(t)\ ; \qquad \text{then} \quad [A,B] = \frac{i}{2\pi}A\ .$$

Remark

B is symmetric, but A is not (A is unitary, thus normal). A theorem by M. Erb [10] states that the standard uncertainty inequality holds true when B is symmetric and A is normal:

$$v_x(A)v_x(B) \geq \frac{1}{16\pi^2}|\langle [A,B]x,x\rangle|^2\ ,$$

for all $x \in \mathcal{D}([A,B])$ such that $\|x\| = 1$.

Breitenberger's inequality III

Introduce now means and variances

Definition

Angular mean and variance of x (with $\|x\| = 1$):

$$e_x(A) = \int_0^1 e^{2i\pi t}|x(t)|^2\,dt\,, \quad v_x(A) = \|(A - e_x(A))x\|^2 = 1 - |e_x(A)|^2$$

Mean and variance in frequency domain:

$$e_x(B) = \sum_n n|\hat{x}_n|^2\,, \quad v_x(B) = \|(B - e_x(B))x\|^2 = \left(\|x'\|^2 - |\langle x, x'\rangle|^2\right)$$

Erb's inequality yields

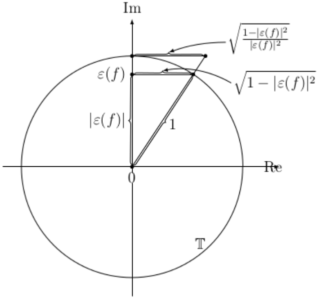
Breitenberger's inequality IV

Corollary (Breitenberger 83 [3])

For all absolutely continuous x *such that* $x' \in L^2([-\pi, \pi])$, $\|x\| = 1$,

$$\left[1 - |e_x(A)|^2\right] \times \left[\|x'\|^2 - |\langle x, x'\rangle|^2\right] \geq \frac{1}{16\pi^2}|e_x(A)|^2$$

with equality if and only if $x(t) = e^{2i\pi kt}$ *(*$k \in \mathbb{Z}$*): then* $e_x(A) = 0$.



Geometric interpretation: circular variance

$$\tilde{v}_x(A) = \frac{1 - |e_x(A)|^2}{|e_x(A)|^2}$$

$$v_x(B) = \sum n^2|\hat{x}[n]|^2 - \left(\sum n|\hat{x}[n]|^2\right)^2$$

(standard definition)

Breitenberger's inequality V

Corollary

For all a.c. x *such that* $x' \in L^2([-\pi,\pi])$, $\|x\| = 1$, *and* $e_x(A) \neq 0$,

$$\frac{1 - |e_x(A)|^2}{|e_x(A)|^2} \times \left[\|x'\|^2 - |\langle x, x'\rangle|^2\right] > \frac{1}{16\pi^2}$$

Remark

- Equality is not possible: there is room for improvement
- Phenomenon analyzed in details by Parhizkar et al [24]: improved bounds, and numerical and analytic solutions for optimizers (Mathieu sines and cosines).

Continuous limit

Consider the case of $x \in \mathbb{C}^N$, with periodic boundary conditions.

Problem: the notion of mean value is not clear, as x and $\hat{x}$ are both periodic sequences. To overcome this issue, assume that x and $\hat{x}$ are centered at 0, and defined on a set

$$\mathcal{D}(N) = \left\{ -\frac{N}{2\sqrt{N}}, -\frac{N-1}{2\sqrt{N}}, \dots \frac{N-1}{2\sqrt{N}} \right\}$$

Define the corresponding DFT of x as

$$\hat{x}[\nu] = \frac{1}{\sqrt{N}} \sum_{t \in \mathcal{D}(N)} x[t] e^{-2i\pi t\nu} \ , \quad \nu \in \mathcal{D}(N)$$

and the concentration/variance as

$$v_x = \sum_{t \in \mathcal{D}(N)} t^2 |x[t]|^2 \ .$$

Continuous limit

To suitable functions $y \in L^2(\mathbb{R})$, associate the vector x_y defined by

$$x_y[t] = \sum_{s \in \mathbb{Z}\sqrt{N}} y(t+s) \ , \qquad t \in \mathcal{D}(N)$$

Theorem (Nam 2013)

Assume there exists a constant ϵ such that

$$\max\{|y(t)|, |y'(t)|, |\hat{y}(t)|, |\hat{y}'(t)|\} \leq \frac{\epsilon}{t^2} \ , \qquad |t| > \sqrt{N}/2 \ .$$

Then

$$(1+\sqrt{\epsilon})v_y v_{\hat{y}} \geq v_{x_y} v_{\hat{x}_y} \geq (1-\sqrt{\epsilon})v_y v_{\hat{y}} \geq \frac{(1-\sqrt{\epsilon})}{16\pi^2}$$

Pending question: how close are the optimizers to sampled periodized Gaussians ?

Conclusions

- Variance is the most common way to define concentration for functions: concentration around some reference location (the mean).
- Variance is associated to a translation structure... which may not be adequate, or even available, in some situations (think of graphs for example).
- In the Robertson inequality, the lower bound for $v_x(A)v_x(B)$ generally depends on the state x, which may be inadequate if one seeks variance minimizers.
- It makes sense to study other types of concentration properties.

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