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## The State of the art in shearlet coorbit space theory

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on Coherent State Transforms,

Time-Frequency and Time-Scale

Analysis, Applications

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"The State of the Art in

Shearlet Coorbit Theory"

- Extended Version -

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# 0. Motivation

Fundamental problem of applied mathematics.

- analyze, approximate, decompose.. function $f \in L_2(\Omega)$ $(\Omega = \mathbb{R}^d)$

- decomposition into suitable building blocks (Fourier transform, Gabor transform, wavelet transform....)

Wavelets: $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ (orthonormal) Basis

$\psi_{j,k} = 2^{j/2} \psi(2^j \cdot - k)$, $\psi$ „mother wavelet"

Advantages:

- $\operatorname{supp} \psi_{jk} \sim 2^{-j} \Rightarrow$

$$f = \sum_{j,k} \langle f, \psi_{jk} \rangle \psi_{jk} \quad \text{local!}$$

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- $\int_{\mathbb{R}} \psi_{jk}(x)\, x^{\gamma}\, dx = 0, \quad \gamma \leq r \Rightarrow$ compression

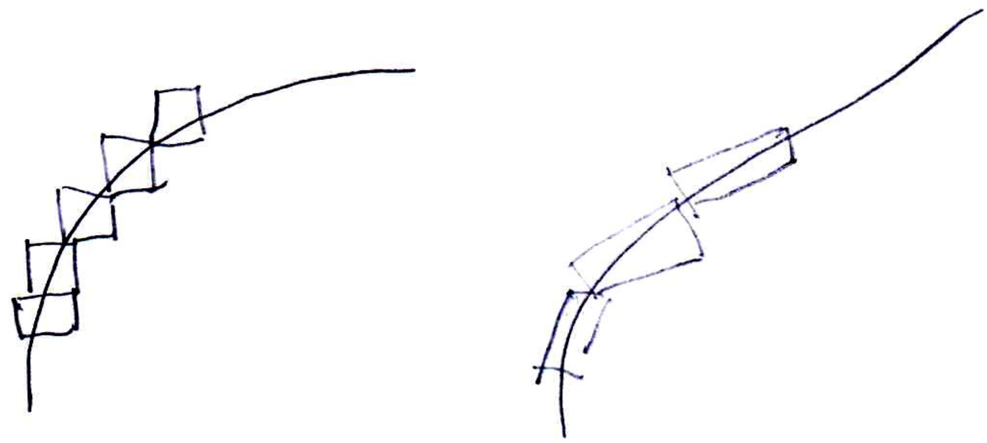
- $\|f\|_{H^s} \sim$

$$\left(\sum_{j=j_0}^{\infty} 2^{2js} \sum_{k} |\langle f, \psi_{jk} \rangle|^2\right)^{1/2} \Rightarrow$$

preconditioning, operator theory, adaptive numerical algorithms etc

Drawbacks:

- isotropic approach! directional information?



ridgelets, curvelets, contourlets..

Shearlets

in 2D:



$$\pi(a,s,t)\psi(x) = (T_t D_{S_s A_a}\psi)(x) = |a|^{-3/4}\psi(A_a^{-1}S_s^{-1}(x-t))$$

$$A_a = \begin{pmatrix} a & 0 \\ 0 & \operatorname{sign}(a)\sqrt{|a|} \end{pmatrix} \qquad S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$$SH_\psi f(a,s,t) := \langle f, \underbrace{\pi(a,s,t)\psi}_{\psi_{a,s,t}} \rangle$$

1. Multivariate Shearlet Transform

2. Coorbit Theory

3. Shearlet Coorbit Spaces

# 1. Multivariate Shearlet Transform

shear model

$$V = W \oplus W' , \quad v = w + w'$$

$$S(v) = w + (w' + M w')$$

$$S = \begin{pmatrix} I & M \\ 0 & I \end{pmatrix} \qquad (1.1)$$

$$S_s := \begin{pmatrix} 1 & s^T \\ 0_{d-1} & I_{d-1} \end{pmatrix} \qquad (1.2)$$

$$A_a := \begin{pmatrix} a & 0_{d-1}^T \\ 0_{d-1} & \operatorname{sign}(a)\, |a|^{\frac{1}{d}} I_{d-1} \end{pmatrix} \qquad (1.3)$$

Continuous shearlet transform

$$SH_\psi f(a,s,t) := \langle f, \underbrace{|a|^{\frac{1}{2d}-1} \psi(A_a^{-1} S_s^{-1} (\cdot - t))}_{=: \psi_{a,s,t}} \rangle \qquad (1.4)$$

**Lemma 1.1** $\mathbb{R}^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$ with

$$(a,s,t)\circ(a',s',t') := \left(aa', \; s+|a|^{1-1/d} s', \; t+S_s A_a t'\right) \qquad (1.5)$$

is a locally compact group, full shearlet group (FSG)

$$d\mu_l(a,s,t) = \frac{1}{|a|^{d+1}} \, da\, ds\, dt \qquad (1.6)$$

$$d\mu_r(a,s,t) = \frac{1}{|a|} \, da\, ds\, dt$$

Proof: (1.6) 

$$\int_{FSG} F((a',s',t')\circ(a,s,t))\,d\mu_\ell(a,s,t)$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} F(a'a,\ s'+|a'|^{1-\frac{1}{d}}s,\ t'+S_{s'}A_{a'}t)\,dt\,ds\,\frac{da}{|a|^{d+1}}$$

$$\tilde{t} = t' + S_{s'}A_{a'}t, \quad d\tilde{t} = |a'|^{2-\frac{1}{d}}dt$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} F(a'a,\ s'+|a'|^{1-\frac{1}{d}}s,\ \tilde{t})\,d\tilde{t}\,ds\,\frac{da}{|a|^{d+1}|a'|^{2-\frac{1}{d}}}$$

$$\tilde{s} = s' + |a'|^{1-\frac{1}{d}}s, \quad d\tilde{s} = |a'|^{\frac{(d-1)^2}{d}}ds$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} F(a'a,\tilde{s},\tilde{t})\,d\tilde{t}\,d\tilde{s}\,\frac{da}{|a|^{d+1}|a'|^{2-\frac{1}{d}}|a'|^{\frac{(d-1)^2}{d}}}$$

$$\hat{a} = a'a, \quad d\tilde{a} = |a'|\,da$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} F(\tilde{a},\tilde{s},\tilde{t})\,d\tilde{t}\,d\tilde{s}\,\frac{d\tilde{a}}{\frac{|\tilde{a}|^{d+1}}{|a'|^{d+1}}|a'|\,|a'|^{2-\frac{1}{d}}|a'|^{\frac{(d-1)^2}{d}}}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} F(\tilde{a},\tilde{s},\tilde{t})\,d\tilde{t}\,d\tilde{s}\,\frac{d\hat{a}}{|\hat{a}|^{d+1}}$$

## Lemma 1.2

$$\pi : FSG \longrightarrow \mathcal{U}(L_2(\mathbb{R}^d))$$
$$(a,s,t) \longmapsto \pi(a,s,t) \qquad (1.7)$$
$$\pi(a,s,t)f(x) := |a|^{\frac{1}{2d}-1} f(A_a^{-1} S_s^{-1}(x-t))$$

is a continuous unitary representation of FSG in $L_2(\mathbb{R}^d)$

## Theorem 1.3

Let $\psi \in L_2$ satisfy

$$C_\psi := \int_{\mathbb{R}^d} \frac{|\hat{\psi}(\xi)|^2}{|\xi_1|^d} \, d\xi < \infty \qquad (1.8)$$

Then:

$$\int_{FSG} |\langle \psi, \pi(a,s,t)\psi\rangle|^2 \, d\mu(a,s,t) < \infty \qquad (1.9)$$

and

$$\int_{FSG} |\langle f, \psi_{a,s,t}\rangle|^2 \, d\mu(a,s,t) = C_\psi \, \|f\|_{L_2}^2 \qquad (1.10)$$

Proof.

$$\hat{g}_{a,s,t}(\xi) = \int_{\mathbb{R}^d} g_{a,s,t}\, e^{-2\pi i x\cdot\xi}\, dx \qquad (1.11)$$

$$= |a|^{1-\frac{1}{2d}} e^{-2\pi i t\cdot\xi}\, \hat{g}(A_a^T S_s^T \xi)$$

$$= |a|^{1-\frac{1}{2a}} e^{-2\pi i t\cdot\xi}\, \hat{g}\left(\begin{matrix} a\,\xi_1 \\ \mathrm{sign}(a)\,|a|^{\frac{1}{d}}(\xi_1 s + \tilde{\xi}) \end{matrix}\right)$$

$$\xi = (\xi_1, \tilde{\xi})$$

$$g^*_{a,s,0}(x) := \overline{g_{a,s,0}(-x)} \qquad (1.12)$$

$$\int_{FSG} |\langle f, g_{a,s,t}\rangle|^2 d\mu = \int_{FSG} |f * g^*_{a,s,0}(t)|^2 dt\, ds\, \frac{da}{|a|^{d+1}}$$

$$\overset{\text{Plancherel}}{=} \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\hat{g}^*_{a,s,0}(\xi)|^2 d\xi\, ds\, \frac{da}{|a|^{d+1}}$$

$$\overset{(1.11),\, t=0}{=} \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |a|^{2-\frac{1}{d}} |\hat{g}(A_a^T S_s^T \xi)|^2 d\xi\, ds\, \frac{da}{|a|^{d+1}}$$

Fubini

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} |\hat{f}(\xi)|^2 |a|^{1-d-\frac{1}{d}} \left|\hat{g}\left(\frac{a\xi_1}{\operatorname{sign} a\, |a|^{1/d}(\tilde{\xi} + \xi_1 s)}\right)\right|^2 ds\, d\xi\, da$$

$$\tilde{\eta} := \operatorname{sign} a\, |a|^{\frac{1}{d}}(\tilde{\xi} + \xi_1 s)$$

$$\Rightarrow d\tilde{\eta} = (|a|^{\frac{1}{d}} \xi_1)^{d-1} ds \Rightarrow$$

$$\int_{FSG} |\langle f, g_{a,s,t}\rangle|^2 d\mu = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} |\hat{f}(\xi)|^2 |a|^{-d} |\xi_1|^{-(d-1)} \left|\hat{g}\left(\frac{a\xi_1}{\tilde{\eta}}\right)\right|^2 d\tilde{\eta}\, d\xi\, da$$

$\eta_1 = a\xi_1$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} |\hat{f}(\xi)|^2 \frac{|\xi_1|^d}{|\eta_1|^d |\xi_1|^d} \left|\hat{g}\left(\frac{\eta_1}{\tilde{\eta}}\right)\right|^2 d\tilde{\eta}\, d\xi\, d\eta_1$$

$$= \|f\|_{L_2}^2 \int_{\mathbb{R}^d} \frac{|\hat{g}(\eta)|^2}{|\eta_1|^d} d\eta$$

$g = \psi$ satisfying $(1.8) \Rightarrow$

$$\int_{FSG} |\langle f, \psi_{a,s,t}\rangle|^2 = C_\psi \|f\|_{L_2}^2 \qquad \blacksquare$$

Remark 1.4 $\pi$ is

- multiple of an isometry
- irreducible
- square integrable

Theorem 1.5

i) $$f = SH_\psi^* SH_\psi(f) = C_\psi^{-1} \int_{FSG} \langle f, \psi_{a,s,t} \rangle \psi_{a,s,t} \, d\mu \qquad (1.13)$$

ii) $C_\psi = 1 \Rightarrow$

$$SH_\psi(f) * SH_\psi(\psi) = SH_\psi(f) \qquad (1.14)$$

Proof i)

$$\tilde{f} = C_\psi^{-1} \int_{FSG} \langle f, \psi_{a,s,t} \rangle \psi_{a,s,t} \, d\mu \Rightarrow$$

$$\langle \tilde{f}, \eta \rangle = C_\psi \int_{FSG} \langle f, \psi_{a,s,t} \rangle \langle \psi_{a,s,t}, \eta \rangle \, d\mu$$

$$= C_\psi \int_{FSG} SH_\psi(f) \cdot \overline{SH_\psi(\eta)} \, d\mu$$

$$= C_\psi^{-1} \langle SH_\psi f, SH_\psi \psi \rangle_{L_2(FSG, d\mu)}$$

$$\overset{(1.10)}{=} \langle f, \eta \rangle_{L_2}$$

ii) $(a, s, t) =: g, \quad (a', s', t') =: h$

$$(SH_\psi(f) * (SH_\psi(\psi))(g)$$

$$= \int_{FSG} \langle f, \psi_h \rangle \langle \psi, \psi_{h^{-1} \circ g} \rangle \, d\mu(h)$$

$$= \int_{FSG} \langle f, \psi_h \rangle \langle \psi, \pi(h^{-1} \circ g) \psi \rangle \, d\mu(h)$$

$$= \int_{FSG} \langle f, \psi_h \rangle \langle \pi(h) \psi, \pi(g) \psi \rangle \, d\mu(h)$$

$$= \int_{FSG} (SH_\psi f)(h) \, \overline{SH_\psi(\pi(g)\psi)(h)} \, d\mu(h)$$

$$\overset{(1.10)}{=} \langle f, \pi(g)\psi \rangle = SH_\psi f(g)$$

## 2. Coorbit Theory

in practice: discretisation necessary:
we need bases, frames...

Coorbit theory provides.

- new smoothness spaces, decay of the voice transform
- (Banach) frames

## General setting

- $G$ locally compact topological group
  $dg$ left Haar measure
- $\pi$ unitary, square integrable representation of $G$ in Hilbert space $\mathcal{H}$, $c_\psi = 1$
- weight function $\omega : G \longrightarrow \mathbb{R}^+$

$$\omega(g \circ h) \leq \omega(g)\,\omega(h) \quad , \omega \geq 1 \qquad (2.1)$$

$$\omega(g) = \omega(g^{-1})\,\Delta(g^{-1}) \qquad (2.2)$$

- $\psi$ admissible vector

$$V_\psi : \mathcal{H} \longrightarrow L_2(G, dg) \qquad (2.3)$$

$$f \longmapsto \langle f, \pi(g)\psi \rangle_{\mathcal{H}}$$

voice transform

## 2.1. Coorbit spaces

$$L_{p,\omega}(G) := \left\{ F : \left( \int_G |F(g)|^p \omega(g)^p \, dg \right)^{1/p} < \infty \right\} \qquad (2.4)$$

Assumption 2.1 The set

$$\mathcal{A}_\omega := \left\{ \psi \,\middle|\, \int_G |V_\psi(\psi)(g)| \, \omega(g) \, dg < \infty \right\} \qquad (2.5)$$

$\psi$ admissible, is non-trivial

$$(V_\psi \psi)(g) =: K(g) \qquad (2.6)$$

$$\mathcal{H}_{1,\omega} := \left\{ f \in \mathcal{H} \,\middle|\, V_\psi(f) = \langle f, \pi(\cdot)\psi \rangle_{\mathcal{H}} \in L_{1,\omega} \right\} \qquad (2.7)$$

$$\| f \|_{\mathcal{H}_{1,\omega}} := \| V_\psi f \|_{L_{1,\omega}}$$

## Proposition 2.2

i.) $\mathcal{H}_{1,\omega}$ is a Banach space

ii.) Gelfand - Triple

$$\mathcal{H}_{1,\omega} \hookrightarrow \mathcal{H} \hookrightarrow \tilde{\mathcal{H}}_{1,\omega} \qquad (2.8)$$

$\tilde{\mathcal{H}}_{1,\omega}$ dual space

iii.) $f \in \mathcal{H}_{1,\omega} \Longrightarrow \pi(h) f \in \mathcal{H}_{1,\omega} \quad \forall h \in G$

iv.) $$V_\psi(\pi(g) f) = L_g V_\psi f \qquad (2.9)$$

Proof. ii)

$$\|f\|^2_{\mathcal{H}} = \|V_\psi f\|^2_{L_2} = \int_G |V_\psi f(g)| \, |V_\psi f(g)| \, dg$$

$$= \int_G |\langle f, \pi(g)\psi\rangle| \, |V_\psi f(g)| \, dg$$

(2.1)

$$\leq \int_G \|f\|_{\mathcal{H}} \|\pi(g)\psi\|_{\mathcal{H}} |V_\psi f(g)| \, \omega(g) \, dg$$

$$\leq \|f\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} \|V_\psi f\|_{L_{1,\omega}}$$

iii.)

$$\int_G |V_\psi(\pi(h)f)(g)|\,\omega(g)\,dg$$

$$= \int_G |\langle \pi(h) f, \pi(g)\psi\rangle|\,\omega(g)\,dg$$

$$= \int_G |\langle f, \pi(h^{-1}\circ g)\psi\rangle|\,\omega(g)\,dg$$

$$= \int_G |\langle f, \pi(g)\psi\rangle|\,\omega(h\circ g)\,dg$$

$$\overset{(2.1)}{\leq} \int_G |\langle f, \pi(g)\psi\rangle|\,\omega(h)\,\omega(g)\,dg$$

$$\leq \omega(h) \int_G |V_\psi f(g)|\,\omega(g)\,dg$$

Proposition 2.2. iii) $\Rightarrow$

$$V_\psi(f)(g) := \langle f, \pi(g)\psi\rangle_{\tilde{\mathcal{H}}_{1,\omega} \times \mathcal{H}_{1,\omega}} \qquad (2.10)$$

$f \in \tilde{\mathcal{H}}_{1,\omega}$ is well-defined!

## Proposition 2.3

i) $V_\psi f * K = V_\psi f \quad , f \in \tilde{\mathcal{H}}_{1,\omega}$ (2.11)

ii) $V_\psi : \tilde{\mathcal{H}}_{1,\omega} \longrightarrow L_{\infty,\frac{1}{\omega}}(G)$ is one-to-one

Proof: i) similar to Theorem 1.5.

$$\pi(g)\psi = \int_G \langle \pi(g)\psi, \pi(h)\psi \rangle \pi(h)\psi \, dh \quad \text{in } \mathcal{H}_{1,\omega}$$

$$(V_\psi f)(g) = \langle f, \pi(g)\psi \rangle_{\tilde{\mathcal{H}}_{1,\omega} \times \mathcal{H}_{1,\omega}} =$$

$$= \langle f, \int_G \langle \pi(g)\psi, \pi(h)\psi \rangle \pi(h)\psi \, dh \rangle$$

$$= \int_G \overline{\langle \pi(g)\psi, \pi(h)\psi \rangle} \langle f, \pi(h)\psi \rangle \, dh$$

$$= \int_G \langle \psi, \pi(h^{-1} \circ g)\psi \rangle \langle f, \pi(h)\psi \rangle \, dh$$

$$\equiv \int_G K(h^{-1} \circ g) V_\psi(f)(h) \, dh$$

$$= (V_\psi f * K)$$

iii) $|(V_\psi f)(g)| = |\langle f, \pi(g)\psi\rangle_{\mathcal{H}^\sim_{1,\omega} \times \mathcal{H}_{1,\omega}}|$

$$\leq \|f\|_{\mathcal{H}^\sim_{1,\omega}} \|\pi(g)\psi\|_{\mathcal{H}_{1,\omega}}$$

Proof of Prop 22

$$= \omega(g)\, \|\psi\|_{\mathcal{H}_{1,\omega}} \|f\|_{\mathcal{H}^\sim_{1,\omega}}$$

**Def. 2.4** Let $m$ be a moderate function,

$$m(g \circ h \circ k) \leq w(g)\, m(h)\, w(k) \qquad (2.12)$$

The coorbit space $\mathcal{H}_{p,m}$ is the Banach space

$$\mathcal{H}_{p,m} := \{ f \in \mathcal{H}^{\sim}_{1,w} : V_\psi(f) \in L_{p,m}(G) \} \qquad (2.13)$$

$$\| f \|_{\mathcal{H}_{p,m}} := \| V_\psi(f) \|_{L_{p,m}}$$

**Remarks 2.5** i.) $\mathcal{H}_{p,m}$ independent of $\psi$

ii.) (2.12) $\Longrightarrow$

$$L_{p,m} * L_{1,w} \subseteq L_{p,m}$$
$$L_{1,w} * L_{p,m} \subseteq L_{p,m} \qquad (2.14)$$

iii.) in practice: start with $m$, find $w$

$$\mathcal{M}_{p,m} := \{ F \in L_{p,m},\ F * K = F \} \quad (2.15)$$

<u>Theorem 2.6</u> (Correspondence Principle)

$V_\gamma$ induces isomorphism

$$V_\gamma : \mathcal{H}_{p,m} \longleftrightarrow \mathcal{M}_{p,m} \quad (2.16)$$

<u>Proof</u>. $V_\gamma(\mathcal{H}_{p,m}) \subset \mathcal{M}_{p,m}$

$(2.11) \Rightarrow V_\gamma f * K = V_\gamma f$ in $\tilde{\mathcal{H}}_{1,\omega}$

$\Rightarrow V_\gamma f * K = V_\gamma f$ in $\mathcal{H}_{p,m}$ $\blacksquare$

Examples 2.7

i) Wavelet transform, affine group

$\mathcal{H} = L_2(\mathbb{R})$, $\quad m(a,t) = |a|^{-s} \Rightarrow$

$\mathcal{H}_{p,m} = \dot{B}^{s+1/2-1/p}_{p,p}(\mathbb{R})$ homogeneous Besov space

ii) Gabor transform, Weyl-Heisenberg group

$\mathcal{H} = L_2(\mathbb{R})$, $\quad m(t,\omega) = (1+|\omega|)^{2s}$

$\mathcal{H}_{p,m} = M^{s}_{pp}$ modulation space

## 2.2. Discretisation

Idea: find $X = (g_\lambda)_{\lambda \in \Lambda}$ in $G$

$$f = \sum_{\lambda \in \Lambda} c_\lambda(f) \pi(g_\lambda)\psi \qquad (2.17)$$

$X$ $\underline{\mathcal{U}\text{-dense}}$ $\Leftrightarrow \bigcup_{\lambda \in \Lambda} g_\lambda \mathcal{U} = G$, $e \in \mathcal{U}$ (2.18)

$\underline{\mathcal{U}\text{-oscillation}}$.

$$\operatorname{osc}_{\mathcal{U}}(g) := \sup_{u \in \mathcal{U}} |K(ug) - K(g)| \qquad (2.19)$$

$$B_\omega := \{\psi \in L_2 \mid K \in W(C_0, L_{1,\omega})\} \qquad (2.20)$$

$$W(C_0, L_{1,\omega}) := \{F \mid \|(L_g \chi_Q) F\|_{L_\infty} \in L_{1,\omega}\}$$

Theorem 2.8 $\psi \in B_\omega$, $e \subset U$ sufficiently small, $X$ $U$-dense

i.) Atomic decomposition $f \in \mathcal{H}_{p,m} \Rightarrow$

$$f = \sum_{\lambda \in \Lambda} c_\lambda(f)\, \pi(g_\lambda)\psi$$

$$\| (c_\lambda(f))_{\lambda \in \Lambda} \|_{\ell_{p,m}} \lesssim \| f \|_{\mathcal{H}_{p,m}} \tag{2.21}$$

$$(c_\lambda(f))_{\lambda \in \Lambda} \in \ell_{p,m} \Rightarrow$$

$$f = \sum_{\lambda \in \Lambda} c_\lambda(f)\, \pi(g_\lambda)\psi \in \mathcal{H}_{p,m} \text{ and}$$

$$\| f \|_{\mathcal{H}_{p,m}} \lesssim \| (c_\lambda(f))_{\lambda \in \Lambda} \|_{\ell_{p,m}} \tag{2.22}$$

ii.) Banach frame

- $$\| f \|_{\mathcal{H}_{p,m}} \sim \| \langle f, \pi(g_\lambda)\psi \rangle_{\tilde{\mathcal{H}}_{1,w} \times \mathcal{H}_{1,w}} \|_{\ell_{p,m}} \tag{2.23}$$
- $\exists$ reconstruction operator $\mathcal{R}$

$$\mathcal{R}\left( \left( \langle f, \pi(g_\lambda)\psi \rangle_{\tilde{\mathcal{H}}_{1,w} \times \mathcal{H}_{1,w}} \right)_{\lambda \in \Lambda} \right) = f \tag{2.24}$$

i.)

Idea of proof.

(21)

Approximate

$$T : \mathcal{M}_{p,m} \longrightarrow \mathcal{M}_{p,m}$$

$$F \longmapsto F * K$$

by

$$T_\varphi F = \sum_{\lambda \in \Lambda} \langle F, \varphi_\lambda \rangle L_{g_\lambda} K \qquad (2.25)$$

$\{\varphi_\lambda\}_{\lambda \in \Lambda}$ partition of unity,

Theorem 2.6 $\Longrightarrow$ result

$$|(F - T_\varphi F)(g)| =$$

$$\left| \int_G F(h)\, K(h^{-1} \circ g)\, dh - \sum_{\lambda \in \Lambda} \int_G \varphi_\lambda(h)\, F(h)\, K(g_\lambda^{-1} \circ g)\, dh \right|$$

$$= \left| \int_G F(h) \Big( \sum_{\lambda \in \Lambda} \varphi_\lambda(h) \big( K(h^{-1} \circ g) - K(g_\lambda^{-1} \circ g) \big) \Big) dh \right|$$

$h \in g_\lambda \mathcal{U} \iff g_\lambda^{-1} = u \circ h^{-1} \implies$

$$|(F - T_\varphi F)g| \leq \sum_{\lambda \in \Lambda} \int_G |F(h)| \, \varphi_\lambda(h) \sup_{u \in \mathcal{U}} |K(h^{-1} \circ g) - K(u \circ h^{-1} g)| \, dh$$

$$= |F| * \operatorname{osc}_u \implies$$

$$\| F - T_\varphi F \|_{L_{p,m}} = \| \, |F| * \operatorname{osc}_u \|_{L_{p,m}}$$

(2.14)

$$\leq \| F \|_{L_{p,m}} \| \operatorname{osc}_u \|_{L_{1,a}}$$

$\psi \in \mathcal{B}_a \implies \forall \varepsilon > 0 \; \exists \; \mathcal{U}_\varepsilon$ s.t

$\| \operatorname{osc}_{\mathcal{U}_\varepsilon} \|_{L_{1,a}} < \varepsilon \implies$

$T_\varphi$ invertible $\implies$

$$F = T_\varphi T_\varphi^{-1} F = \sum_{\lambda \in \Lambda} \langle T_\varphi^{-1} F, \varphi_\lambda \rangle L_{g_\lambda} K \qquad (2.21)$$

$f \in \mathcal{H}_{p,m} \Longrightarrow$

$f = V_\psi^{-1} \circ V_\psi$

Theorem 2.6, (2.27)

$$= \sum_{\lambda \in \Lambda} \langle T_\varphi^{-1} V_\psi f, e_\lambda \rangle \, V_\psi^{-1}(L_{g_\lambda} K)$$

$$= \sum_{\lambda \in \Lambda} (\cdot, \cdot) \, V_\psi^{-1}(L_{g_\lambda} V_\psi \psi)$$

$$\overset{(2.9)}{=} \sum_{\lambda \in \Lambda} (\cdot, \cdot) \, V_\psi^{-1} V_\psi \, \pi(g_\lambda)\psi$$

$$= \sum_{\lambda \in \Lambda} \underbrace{\langle T_\varphi^{-1} V_\psi f, e_\lambda \rangle}_{c_\lambda(f)} \pi(g_\lambda)\psi$$

# 3. Shearlet Coorbit Spaces

## 3.1 Existence

we need $\mathcal{A}_\omega$ and $\mathcal{B}_\omega$ nonempty

Theorem 3.1 $\psi$ Schwartz function,

$\operatorname{supp} \hat{\psi} \subseteq [-a_1, -a_0] \cup [a_0, a_1] \times [-b_1, b_2] \times \dots \times [-b_d, b_b]$

$\omega(a, s, t) = \omega(a, s) \Rightarrow$

$$\int_{FSG} |SH_\psi(\psi)(g)| \, \omega(g) \, dg < \infty \qquad (3.1)$$

$\Rightarrow \mathcal{A}_\omega$ nontrivial $\Rightarrow$

$$SC_{p,m} := \{ f \in \tilde{\mathcal{H}}_{1,\omega} : SH_\psi(f) \in L_{p,m} \} \qquad (3.2)$$

well-defined !

Theorem 3.2 $\psi \in L_2(\mathbb{R}^d)$, $\operatorname{supp}\psi \subset Q_D = [-D, D]^d$, $\omega(a,s,t) = \omega(a) \leq |a|^{-\rho_1} + |a|^{\rho_2}$

$$|\hat{\psi}(\xi_1, \tilde{\xi})| \lesssim \frac{|\xi_1|^n}{(1+|\xi_1|)^r} \prod_{k=2}^{d} \frac{1}{(1+|\xi_k|)^r} \qquad (3.3)$$

$n, r$ sufficiently large $\Rightarrow \psi \in \mathcal{B}_\omega$!

Remark 3.3 i) There exist compactly supported shearlets!

ii) (3.3) $\Rightarrow$ smoothness + vanishing moments $\rightarrow$ admissible shearlet.

## 3.2. Discretisation

Lemma 3.4 $e \in \mathcal{U} \subset FSG$, $\alpha > 1$, $\beta, \tau > 0$

$$\mathcal{U} \supset [\alpha^{d-1}, \alpha^d] \times [-\tfrac{\beta}{2}, \tfrac{\beta}{2})^{d-1} \times [-\tfrac{\tau}{2}, \tfrac{\tau}{2})^d \qquad (3.4)$$

Then

$$X := \{(\varepsilon\alpha^{-j}, \beta\alpha^{-j(1-\frac{1}{d})}k, S_{\beta\alpha^{-j(1-\frac{1}{d})}k} A_{\alpha^{-j}}\tau l), \; j \in \mathbb{Z}, \; k \in \mathbb{Z}^{d-1}, \; l \in \mathbb{Z}^d, \; \varepsilon \in \{-1, 1\}\} \qquad (3.5)$$

is $\mathcal{U}$-dense

Lemma 3.4, Theorem 3.2, Theorem 2.8

$\Rightarrow$ atomic decomposition + Banach frames

for $SC_{p,m}$ exist!

## 3.3. Density

Theorem 3.5 The set

$$\hat{\mathcal{S}}_0(\mathbb{R}^d) := \{ f \in \mathcal{S}(\mathbb{R}^d) \mid |\hat{f}(\xi)| \leq \frac{\xi_1^{2\alpha}}{(1+|\xi|^2)^{2\alpha}} \; \forall \alpha > 0 \}$$

is dense in $\mathcal{H}_{m,p}$, (3.6)

$$m(a,s,t) = m(a,s) = |a|^r \left( \frac{1}{|a|} + |a| + |s| \right)^n, \quad r \in \mathbb{R}, \; n \geq 0 \quad (3.7)$$

Proof. Choose $\psi$ as in Theorem 3.1, $f \in \mathcal{S}_0 \Rightarrow$

$$|SH_\psi(f)| \lesssim |a|^{\left(\frac{1}{2d}-1\right)} \frac{\max\{1, \delta^{\frac{d}{2}}\} |a|^\alpha}{\left(1 + \| \frac{t}{\max\{1,\delta\}} \|^2\right)^{\alpha - \frac{d}{4}} (1+a^2)^\alpha (1+|s|)^\alpha}$$

$$\delta^2 := \max\{a^2, |a|^{\frac{2}{d}}\}(|s|^2 + d)$$

$\Rightarrow \mathcal{S}_0 \in \mathcal{H}_{m,p}$. take $\psi \in \mathcal{S}_0 \Rightarrow$

$\pi(g_\lambda)\psi \in \mathcal{S}_0 \; \forall \lambda \in X$, $\{\pi(g_\lambda)\psi\}_{\lambda \in \Lambda}$

is dense in $\mathcal{H}_{m,p}$ ∎

## 3.4. Embeddings

### 3.4.1. Embeddings into curvelet spaces

$$m(a,s,t) = m(a) = |a|^{-r}, \quad SC_{p,m} =: SC_{p,r} \qquad (3.8)$$

Theorem 3.6 $1 \le p_1 \le p_2 \le \infty$

i) $SC_{p_1,r} \subset SC_{p_2,r}$

ii) $\mathcal{G}_p^r := SC_{p,\, r + d(\frac{1}{2} - \frac{1}{p})}$

$$\mathcal{G}_{p_1}^{r_1} \subset \mathcal{G}_{p_2}^{r_2} \qquad r_1 - \frac{d}{p_1} = r_2 - \frac{d}{p_2}$$

Proof. Theorem 2.8 $\Rightarrow$

$$\|f\|_{SC_{p_2,r}} \lesssim \Big( \sum_{j \in \mathbb{Z}} \alpha^{j r p_2} \sum_{k,e,\varepsilon} |c_\varepsilon(j,k,e)|^{p_2} \Big)^{1/p_2}$$

$$\overset{\ell_{p_1} \subset \ell_{p_2}}{\lesssim} \Big( \sum_{j \in \mathbb{Z}} \alpha^{j r p_2} \Big( \sum |c_\varepsilon(j,k,e)|^{p_1} \Big)^{\frac{p_2}{p_1}} \Big)^{1/p_2}$$

$$\lesssim \Big( \sum_{j \in \mathbb{Z}} \alpha^{j r p_1} \Big( \sum |c_\varepsilon(j,k,e)|^{p_1} \Big)^{1/p_1}$$

$$\lesssim \|f\|_{SC_{p_1,r}} \qquad \blacksquare$$

## 3.4.2 Embedding into Besov spaces

for simplicity: $d = 2$ !

**Def. 3.7** The space

$$SCC_{p,r} := \{ f \in SC_{p,r} \, / \tag{3.9}$$
$$f(x) = \sum_{j,k,\ell} c(j,k,\ell) \psi_{j,k,\ell}(x) \mid c(j,k,\ell) = 0, \; |k| > \alpha^{\frac{2j}{3}} \}$$

is called generalized cone-adapted shearlet coorbit space.

**Def. 3.8** $\alpha > 1$, $D > 1$, $M \in \mathbb{N}_0$

$\Phi_{j,m} \in C^{M}(\mathbb{R}^d)$ is a family of K-atoms if

i) $\operatorname{supp} \Phi_{j,m} \subset D Q_{j,m}$

$D Q_{j,m}$ centred at $\alpha^{-j} m$, side length $2 \alpha^{-j} D$

ii) $|D^{\gamma} \Phi_{j,m}| \lesssim \alpha^{|\gamma| j}$, $|\gamma| \leq M$.

Theorem 3.9 $K \geq 1 + \lfloor \sigma \rfloor$, $1 \leq p \leq \infty$

$f \in \dot{B}^{\sigma}_{p,q} \Leftrightarrow$

$$f(x) = \sum_{j \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}^d} \lambda(j,\ell)\, \Phi_{j,\ell}(x) \qquad (3.10)$$

$$\|f\|_{\dot{B}^{\sigma}_{p,q}} \sim \inf \Big( \sum_{j \in \mathbb{Z}} 2^{j(\sigma - \frac{d}{p})q} \Big( \sum_{\ell} |\lambda(j,\ell)|^p \Big)^{q/p} \Big)^{\frac{1}{q}}$$

Proof: Frazier / Jawerth, Decomposition of Besov spaces, Indiana Univ. Math. J. 34 (1985), 777 - 799

Theorem 3.10 $SCC_{p,1} \subset \dot{B}^{\sigma_1}_{p,p}(\mathbb{R}^2) + \dot{B}^{\sigma_2}_{p,p}(\mathbb{R}^2)$

$$\sigma_1 + \lfloor \sigma_1 \rfloor \leq 2r - \frac{9}{2} + \frac{4}{p} \qquad (3.11)$$

$$\sigma_2 - \frac{\lfloor \sigma_2 \rfloor}{2} = r + \frac{3}{2p} + \frac{1}{4} \qquad (3.12)$$

Proof: $\psi$ as in Theorem 3.2 $\Rightarrow$

$$f(x) = \sum_{j} \sum_{|k| \leq 2^{j/2}} \sum_{\ell} c(j,k,\ell)\, 2^{\frac{3}{4}j} \cdot \psi\left(2^j x_1 - 2^{j/2} k x_2 - \ell_1,\, 2^{j/2} x_2 - \ell_2\right)$$

# 3.5 Traces

- Question, what is $SC_{pr}\big|_M$ ?
- $d = 3$, $M$ = coordinate plane
- $Tr\big|_{\partial\Omega} : B^{s}_{pp}(\Omega) \longrightarrow B^{s-1/p}_{p,p}(\partial\Omega)$, $\Omega \subset \mathbb{R}^d$ smooth

$$SCC^{(\eta)}_{p,r} := \{ f \in SC_{p,r} \mid \tag{3.13}$$

$$f(x) = \sum_{j,k,e} c(j,k,e)\, \psi_{j,k,e},\ c(j,k,e) = 0\ |k_i| > 2^{\frac{2}{3}j'},\ \eta_i = 1,\ \eta \in \{0,1\} \}$$

Theorem 3.11

$Tr_{x_1} f$ restriction to $(x_2, x_3)$-plane

$$Tr_{x_1}(SCC^{(1,1)}_{p,r}(\mathbb{R}^3)) \subset \dot{B}^{\sigma_1}_{p,p}(\mathbb{R}^2) + \dot{B}^{\sigma_2}_{p,p}(\mathbb{R}^2) \tag{3.14}$$

$$\sigma_1 + 2\lfloor \sigma_1 \rfloor = 3r - \frac{21}{2} + \frac{8}{p} \tag{3.15}$$

$$\sigma_2 = 3r - 3 + \frac{2}{p} \tag{3.16}$$

Theorem 3.12

$$Tr_{x_3}(SCC^{(0,1)}_{p,r}(\mathbb{R}^3)) \subset SC_{p,r_1}(\mathbb{R}^2) + SC_{p,r_2}(\mathbb{R}^2) \quad (3.17)$$

$$r_1 = r - \frac{5}{6} + \frac{2}{3p} \quad (3.18)$$

$$r_2 = r - \frac{1}{6} \quad (3.19)$$

idea of proof:

Def 3.13 $\{\phi_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}$ is called a set of molecules, if $\exists\, H \in W(L_\infty, L^1_v)$ such that

$$|V_\psi(\phi_\lambda)| \leq L_\lambda H, \quad \lambda \in \Lambda \quad (3.20)$$

Gröchenig / Piotrowski $\Rightarrow$

- Atomic decomposition / Banach frames as in Theorem 2.7
- characterisation by molecular expansions

Strategy: show that (linear combinations) of traces of shearlet atoms form shearlet / Besov molecules!

$$= \sum_{j \geq 0} (\cdot/.) + \sum_{j \leq 0} (\cdot/.)$$

$$=: f_1 + f_2$$

$$f_1(x) = \sum_{\substack{j \geq 0 \\ |k| \leq d^{j/2} \\ \ell_2 \in \mathbb{Z}}} \sum_{n_1 \in \mathbb{Z}} \sum_{r_1 \in I(j,n_1)} c(j,k, r_1 - k\ell_2, \ell_2)\, d^{\frac{3}{4}j} \cdot \psi(d^j x_1 - d^{j/2} k x_2 - r_1 + k\ell_2, d^{j/2} x_2 - \ell_2)$$

$$I(j,n_1) = \{ r \in \mathbb{Z} \mid d^{j/2}(n_1 - 1) < r \leq d^{j/2} n_1 \}$$

$$= \sum_{j \geq 0} \sum_{\ell_2, n_1 \in \mathbb{Z}} \lambda(j, n_1, \ell_2)\, \Phi_{j,n_1,\ell_2}(x)$$

$$\lambda(j, n_1, \ell_2) := d^{\frac{3+2M_1}{4} j} \sum_{|k| \leq d^{j/2}} \sum_{r_1 \in I(j,n_1)} |c(j,k,r_1 - k\ell_2, \ell_2)|$$

$$M_1 := 1 + \lfloor \sigma_1 \rfloor$$

$$\Phi_{j,n_1,\ell_2}(x) := \lambda(j,n_1,\ell_2)^{-1} d^{\frac{3+2M_1}{4} j} \sum_{|k| \leq d^{j/2}} \sum_{r_1 \in I(j,n_1)} c(j,k,r_1 - k\ell_2, \ell_2)\, d^{-M_1 j/2} \psi(\cdot/.)$$

$\Phi_{j,n_1,\ell_2}$ $M_1$-atoms! (w.r.t $\sqrt{\alpha}$) (31)

$$\|f_1\|^p_{\dot{B}^{\sigma_1}_{p,p}} \overset{(3.10)}{\lesssim} \sum_{j\in\mathbb{Z}} \alpha^{j/2(\sigma_1-\frac{2}{p})p} \sum_{\ell_2,n_1} |\lambda(j,n_1,\ell_2)|^p$$

$$= \sum_j \alpha^{j/2(\sigma_1-\frac{2}{p})p}\, \alpha^{j/2(\frac{3+2M_1}{2})p}$$

$$\cdot \sum_{\ell_2,n_1} \Big| \sum_{|k|\le \alpha^{j/2}} \sum_{r_1\in I(j,n_1)} |c(j,k,r_1-k\ell_2,\ell_2)| \Big|^p$$

$$\Big(\sum_{i=1}^N |z_i|\Big)^p \le N^{p-1} \sum_{i=1}^N |z_i|^p \quad \overset{N\sim \alpha^j}{\Longrightarrow}$$

$$\|f_1\|^p_{\dot{B}^{\sigma_1}_{p,p}} \lesssim \sum_j \alpha^{j/2(p(\sigma_1+\frac{7}{2}+M_1-\frac{4}{p}))} \cdot \sum |c(j,k,r_1-k\ell_2,\ell_2)|^p$$

$$= \sum_j \alpha^{j/2\, p(\sigma_1+\lfloor\sigma_1\rfloor+\frac{9}{2}-\frac{4}{p})} \sum |c(j,k,\ell_1,\ell_2)|^p$$

$$\lesssim \|f_1\|^p_{SC_{p,r}}$$

Estimate of $\|f_2\|^p_{\dot{B}^{\sigma_2}_{p,p}}$ similar ∎

# Notation.

Convolution of functions,

$$f * g(x) = \int_G f(y) g(y^{-1} \circ x)\, dy$$

modular function.

$\mu$ left Haar measure

$\mu_x(A) = \mu(A \cdot x)$

$\mu_x(yA) = \mu(yAx) = \mu(Ax) = \mu_x(A)$

$\Rightarrow$ $\mu_x$ is also a Haar measure!

$\Rightarrow$ $\mu_x = \Delta(x)\mu$

Besov space.

$$|f|_{B^s_q(L_p(\Omega))} := \left( \int_0^\infty [t^{-s} \omega_r(f,t)_{L_p}]^q \frac{dt}{t} \right)^{1/q}$$

$$\omega_r(f,t)_{L_p(\Omega)} := \sup_{|h| \le t} \| \Delta_h^r f \|_{L_p}$$

Modulation spaces

$$\|f\|^p_{M^s_{p,p}} := \int_{\mathbb{R}}\int_{\mathbb{R}} |V_\eta f(x,\omega)|^p (1+|\omega|)^{2s}\, dx\, d\omega$$

$$(V_\eta f)(x,\omega) := \int_{\mathbb{R}} f(t)\overline{\eta(t-x)}\, e^{-2\pi i t\cdot\omega}\, dt$$