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Wavelet techniques in multifractal analysis

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Wavelet Techniques in multifractal analysis

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based on joint works with

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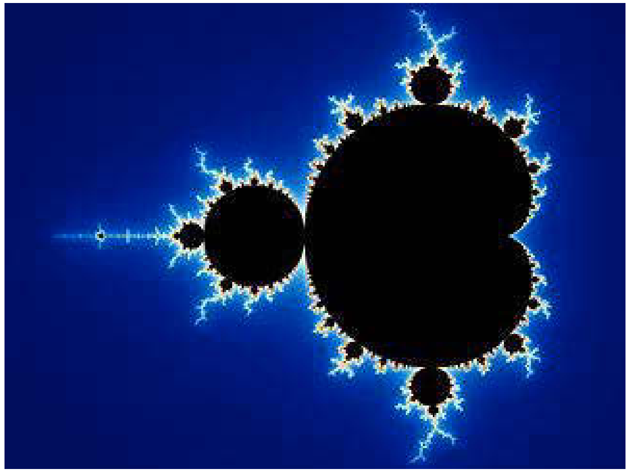
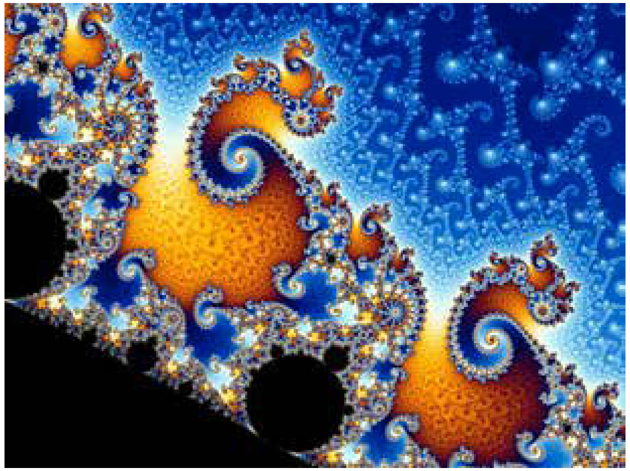
Trieste, June, 2014

What is fractal geometry ?





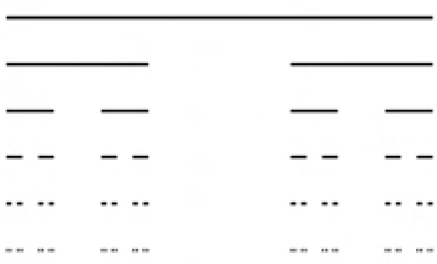
Geometry dealing with irregular sets

How can one quantify this irregularity ?

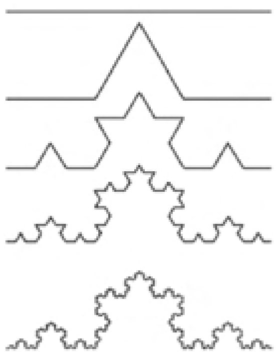
Parameters for the analysis of fractals sets

Toy examples : Selfimilarity exponent

$$\alpha = \frac{\log 2}{\log 3}$$



Triadic Cantor set



Van Koch curve

$$\alpha = \frac{\log 4}{\log 3}$$

Box Dimension

Let $N(\varepsilon)$ be the minimal number of balls of radius ε needed to cover the set A

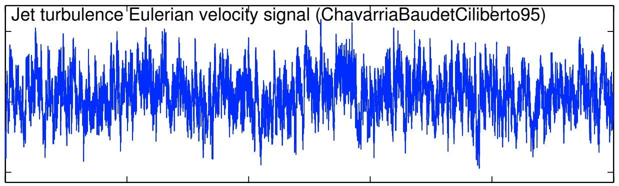
$$N(\varepsilon) \sim \varepsilon^{-dim_B(A)}$$

Advantage :

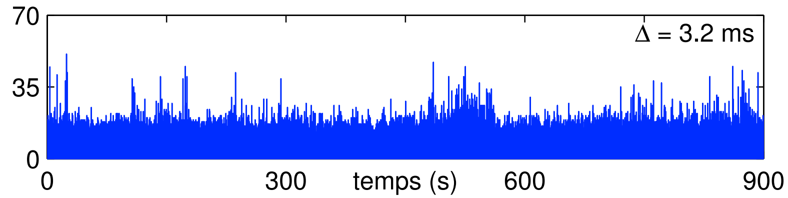
Computable on experimental data through log-log plot regressions : $\log(N(\varepsilon))$ is plotted as a function of $\log(\varepsilon)$. The slope yields the dimension

Everywhere irregular signals

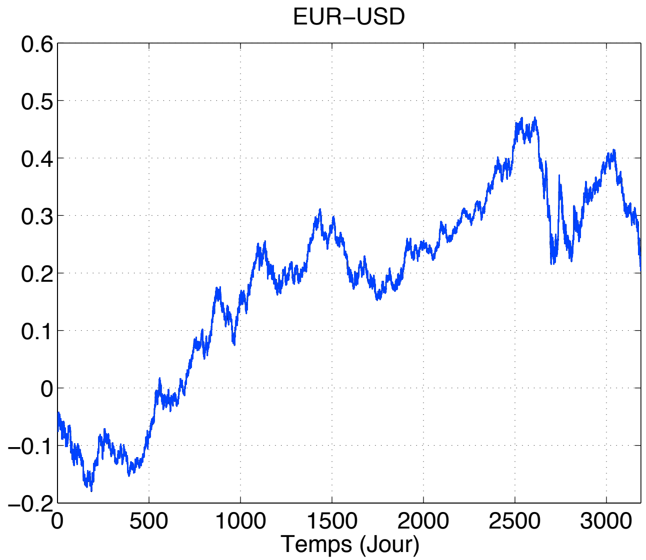
Jean Perrin, in his book, “Les atomes” (1913), insists that irregular (nowhere differentiable) functions, far from being exceptional, are the common case in natural phenomena



Fully developed turbulence



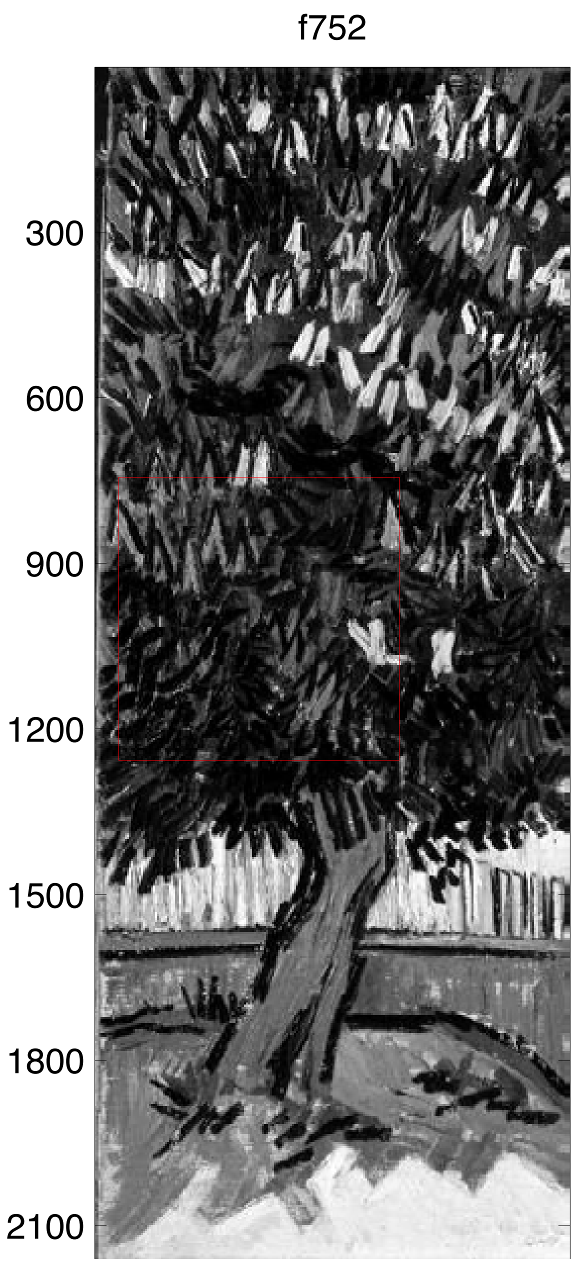
Internet Trafic

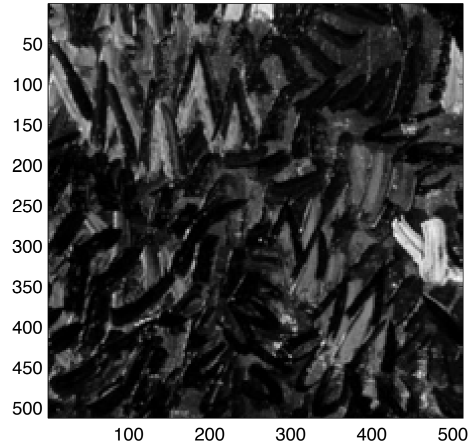


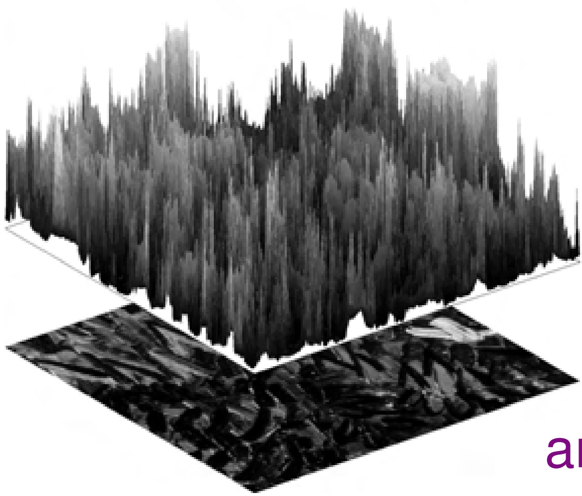
Euro vs Dollar (2001-2009)


300
600
900
1200
1500
1800
2100
300
600
900
1200
1500
50
100
150
200
250
300
350
400
450
500
100
200
300
400
500

Red channel







Nature,

Sciences,

and Arts

supply a large variety

of everywhere irregular

functions

Challenge :

Measure this irregularity

and use it for classification

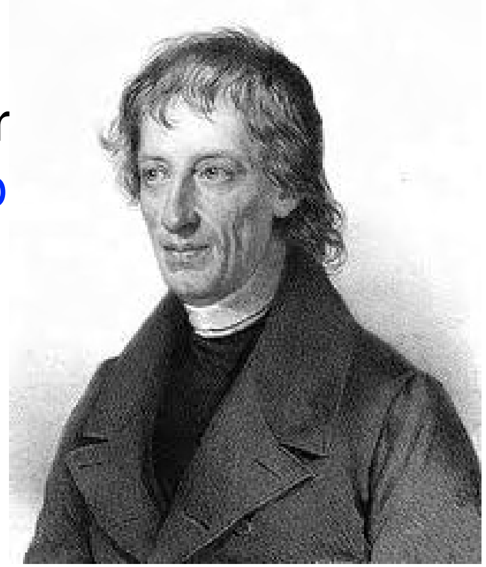
and model selection

Motivation

Multifractal analysis : Its purpose is to introduce and study classification parameters for data (functions, measures, distributions, signals, images), which are based on global and local regularity

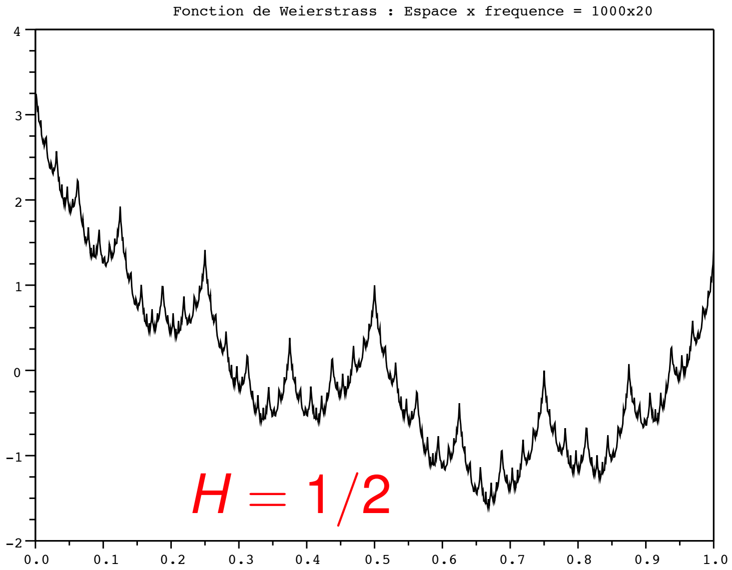
A key problem along the 19th century was to determine if a continuous function on $\mathbb{R}$ necessarily has points of differentiability

A first negative answer was obtained by B. Bolzano in 1830 but was unnoticed



A second counterexample due to K. Weierstrass settled the issue in 1872

Weierstrass functions



$$W_H(x) = \sum_{j=0}^{+\infty} 2^{-Hj} \cos(2^j x)$$

$0 < H < 1$

C. Hermite : I turn my back with fright and horror to this appalling wound : Functions that have no derivative

H. Poincaré called such functions “monsters”

Self-similarity for deterministic functions

Weierstrass-Mandelbrot function

$$W_\alpha(x) = \sum_{-\infty}^{+\infty} 2^{-\alpha j} \sin(2^j x) \qquad 0 < \alpha < 1$$

Exact selfsimilarity :

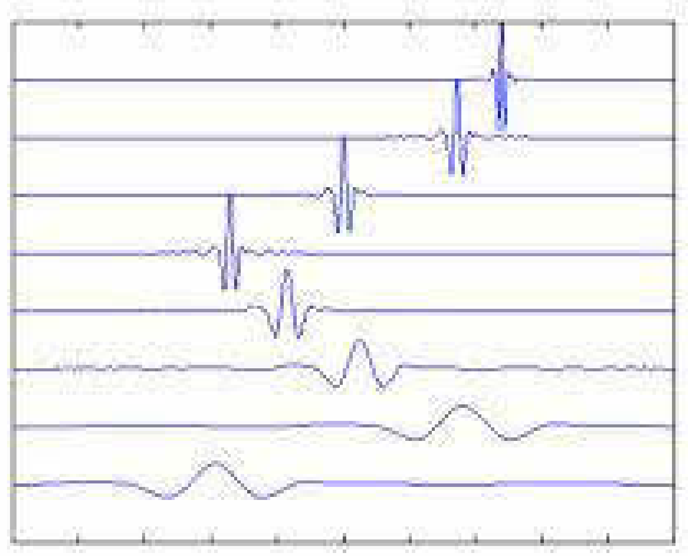
$$\begin{aligned} W_\alpha(2x) &= \sum_{-\infty}^{+\infty} 2^{-\alpha j} \sin(2^{j+1} x) \\ &= \sum_{-\infty}^{+\infty} 2^{-\alpha(l-1)} \sin(2^l x) \\ &= 2^\alpha W_\alpha(x) \end{aligned}$$

Wavelet bases : a tool fitted to selfsimilarity

A wavelet basis on $\mathbb{R}$ is generated by a smooth, well localized, oscillating function ψ such that the

$\psi(2^j x - k), \qquad j, k \in \mathbb{Z}$

form an orthogonal basis of $L^2(\mathbb{R})$



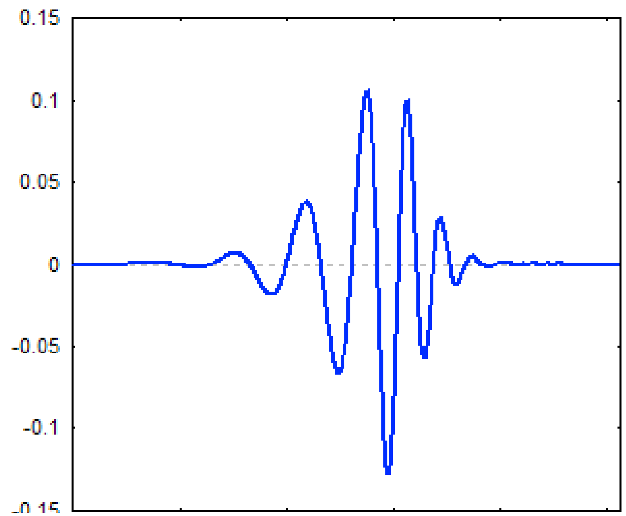
$\forall f$ such that $\int |f(x)|^2 \, dx < \infty$,

$$f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{j,k} \, \psi(2^j x - k)$$

where

$$c_{j,k} = 2^j \int f(x) \, \psi(2^j x - k) \, dx$$

Daubechies Wavelet



Orthogonal basis

Let $f : \mathbb{R} \to \mathbb{R}$. The "energy" of f is $\int |f(x)|^2 \, dx$

Wavelets form an orthogonal basis i.e.

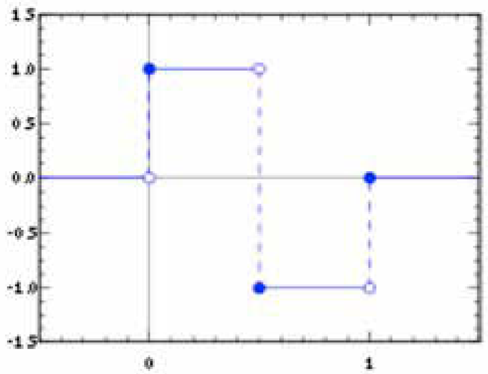
$$\int \psi(2^j x - k) \, \psi(2^{j'} x - k') dx = \begin{cases} 2^{-j} & \text{if } j = j', k = k' \\ 0 & \text{else} \end{cases}$$

$$c_{j,k} = 2^j \int f(x) \, \psi(2^j x - k) \, dx$$

The energy of f can be recovered from the coefficients

$$\int |f(x)|^2 \, dx = \sum_j \sum_k 2^j \, |c_{j,k}|^2$$

A. Haar, J. O. Strömberg, Y. Meyer, P.-G. Lemarié, S. Mallat, I. Daubechies, ...



Haar Wavelet (1909)

Why use wavelet bases ?

- Characterization of selfsimilarity
- Images are stored by their wavelet coefficients (JPEG 2000)
- Fast decomposition algorithms
- Sparse representations
- Characterization of regularity
- Scaling invariance

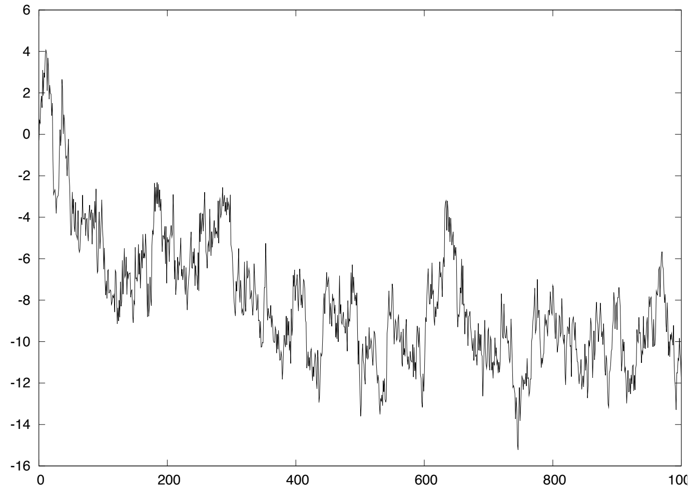
Wavelet analysis of the Weierstrass-Mandelbrot function

$$
\begin{aligned}
c_{j,k} &= \int W_\alpha(x)\, 2^j \psi(2^j x - k) dx \\
&= 2^{-\alpha} \int W_\alpha(2x)\, 2^j \psi(2^j x - k) dx \\
&= 2^{-\alpha} \int W_\alpha(u)\, 2^j \psi(2^{j-1} u - k) \frac{du}{2} \\
&= 2^{-\alpha} \int W_\alpha(u)\, 2^{j-1} \psi(2^{j-1} u - k) du \\
&= 2^{-\alpha}\, c_{j-1,k}
\end{aligned}
$$

This scaling invariance of wavelet coefficients justifies the nonstandard normalization used for wavelet coefficients

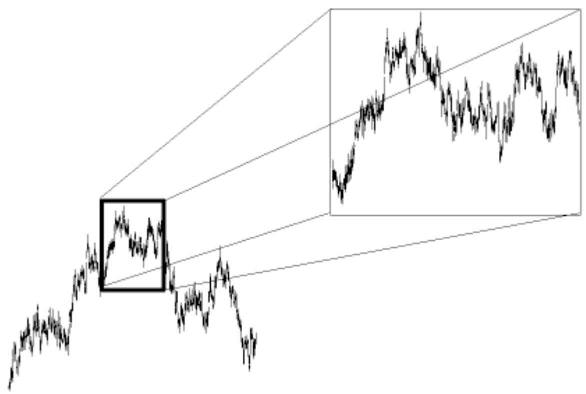
Self-similarity : Stochastic processes

Fractional Brownian Motion ($0 < H < 1$)



FBM of exponent

$H = 1/3$



Statistical selfsimilarity

$$\forall a > 0, \quad B_H(a\,\cdot) \overset{\mathcal{L}}{=} a^H B_H(\cdot)$$

Wavelet counterpart of probabilistic properties : $c_{2j,k} \overset{\mathcal{L}}{=} 2^{-Hj} c_{j,k}$

Problem : An equality in law among random quantities cannot be checked on a sample path

Fractional Brownian Motions

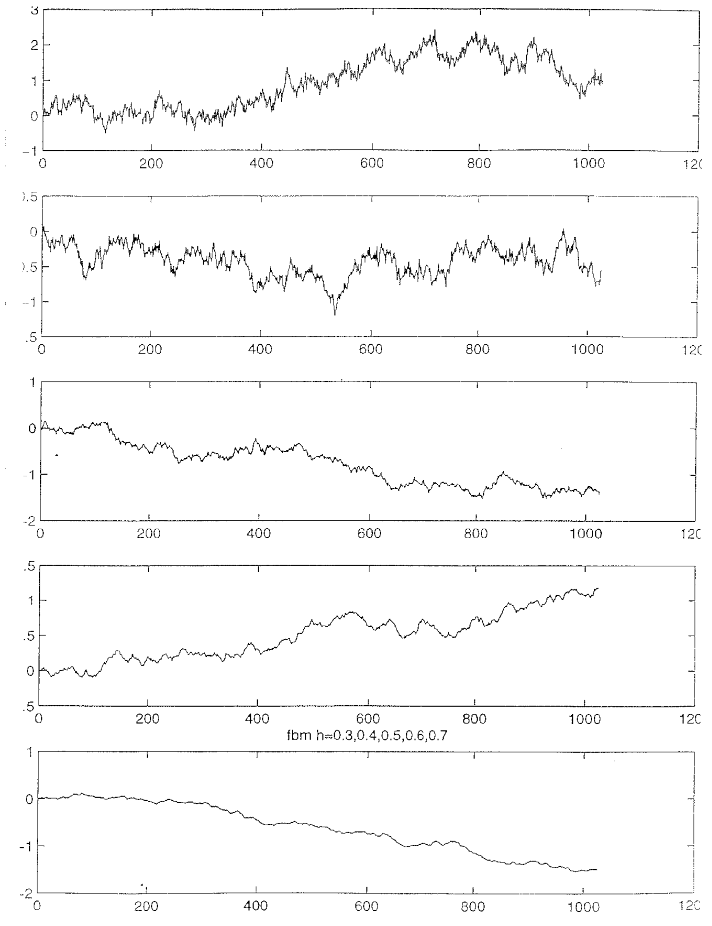
$H = 0.3$

$H = 0.4$

$H = 0.5$

$H = 0.6$

$H = 0.7$



Gaussian

processes

with

stationary

increments

Challenge : Find a numerically stable way to decide if a real-life signal can be modeled by FBM

From self-similarity to scaling exponents

$B_H(t)$ is the unique centered Gaussian process such that

$$\forall x, y \geq 0, \qquad \mathbb{E}(|B_H(x) - B_H(y)|^2) = |x - y|^{2H}$$

It follows that $$|B_H(x+\delta) - B_H(x)| \sim |\delta|^H$$

$$\int |B_H(x+\delta) - B_H(x)|^p dx \sim |\delta|^{Hp}$$

Scale invariance

Kolmogorov scaling function : $$\int |f(x+\delta) - f(x)|^p dx \sim |\delta|^{\zeta_f(p)}$$

Advantages :

- Computable on experimental data : log-log plot regressions
- deterministic quantity that does not depend on the sample path

$\Longrightarrow$ Turbulence at small scale cannot be modeled by FBM (1950s)

Mathematical interpretations

The loose definition of the Kolmogorov scaling function can be given two mathematical interpretations :

- One which makes sense for any (bounded) function

$$\zeta_f(p) = \liminf_{\delta \to 0} \frac{\log \left(\int |f(x+\delta) - f(x)|^p dx \right)}{\log(|\delta|)}$$

- One, which is more precise, under the assumption that f satisfies an average scale invariance

$$\int |f(x+\delta) - f(x)|^p dx = |\delta|^{\zeta_f(p)+o(1)}$$

and expresses the fact that the liminf actually is a limit

Function space interpretation

Lipschitz spaces :

Let $s \in (0,1)$, and $p \in (1,\infty)$; $f \in Lip(s, L^p(\mathbb{R}^d))$ if $f \in L^p$ and $\exists C > 0$ such that

$$\forall \delta > 0, \qquad \| f(x+\delta) - f(x) \|_p \leq C \cdot |\delta|^s$$

$$\Longleftrightarrow$$

$$\forall \delta > 0, \qquad \int |f(x+\delta) - f(x)|^p dx \leq C \cdot |\delta|^{sp}$$

Therefore, if $\zeta_f(p) < p$,

$$\zeta_f(p) = p \cdot \sup\{s : f \in Lip(s, L^p)\}$$

Wavelets and function spaces : The Sobolev space H

$f \in H^1(\mathbb{R})$ if and only if $f \in L^2$ and $f' \in L^2$. We define

$$\| f \|^2 = \int |f(x)|^2 dx + \int |f'(x)|^2 dx$$

Fourier :

$$f(x) = \sum_{n\in\mathbb{Z}} c_n \, e^{2i\pi nx}, \qquad f'(x) = \sum_{n\in\mathbb{Z}} c_n \, 2i\pi n \, e^{2i\pi nx}$$

$$\| f \|^2 = \sum |c_n|^2 + 4\pi^2 \sum n^2 |c_n|^2$$

Wavelets :

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k} \psi(2^j x - k)$$

$$f'(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k} 2^j \psi'(2^j x - k)$$

$$\| f \|^2 \sim \sum_{i\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} (1 + 2^{2j}) 2^{-j} |c_{j,k}|^2$$

Wavelets and function spaces :

Sobolev spaces of fractional order

Fractional derivative :

the Fourier transform of f' is $i\xi\hat{f}$

In the Fourier domain, the fractional derivative of order s of f is the operator $(-\Delta)^{s/2}f$ is defined by

$$\widehat{(-\Delta)^{s/2}}f(\xi) = |\xi|^s\hat{f}(\xi)$$

Definition : Sobolev space $L^{p,s}$

$f \in L^{p,s}(\mathbb{R})$ if $f \in L^p(\mathbb{R})$ and $(-\Delta)^{s/2}f \in L^p(\mathbb{R})$

Wavelet estimation of the $L^{p,s}$ norm

$$f = \sum_j f_j \quad \text{with} \quad f_j(x) = \sum_k c_{j,k}\, \psi(2^j x - k)$$

In order to estimate $\| f \|_{p,s}$ we consider each “scale” block f_j

$$f_j^{(s)}(x) = \sum_k c_{j,k}\, 2^{sj} \psi^{(s)}(2^j x - k)$$

Because of the spatial localization of the wavelets,

$$\int \left| f_j^{(s)}(x) \right|^p dx \sim \sum_k |c_{j,k}|^p\, 2^{spj} \int |\psi^{(s)}(2^j x - k)|^p\, dx$$

$$\sim \sum_k |c_{j,k}|^p\, 2^{(sp-1)j}$$

Therefore the condition $f \in L^{p,s}$ is close to

$$\exists C, \forall j : \qquad 2^{-j} \sum_k |c_{j,k}|^p \leq C \cdot 2^{-spj}$$

The wavelet scaling function

$$\forall p > 0, \qquad 2^{-j} \sum_k |c_{j,k}|^p \sim 2^{-\zeta_f(p) j}$$

Because of the embeddings between Lipschitz and Sobolev spaces, if $p \geq 1$, this definition coincides with Kolmogorov's scaling function

Advantages :

- Effectively computable on experimental data through log-log plot regressions with respect to the scale parameter
- Extends the Kolmogorov scaling function to $p \in (0, 1)$
- Independent of the (smooth enough) wavelet basis
- deformation invariant
- deterministic for large classes of stochastic processes

Wavelets in several variables

In 2D, wavelets used are of tensor product type :

$$\psi^1(x,y) = \psi(x)\varphi(y)$$

$$\psi^2(x,y) = \varphi(x)\psi(y)$$

$$\psi^3(x,y) = \psi(x)\psi(y)$$

Notations

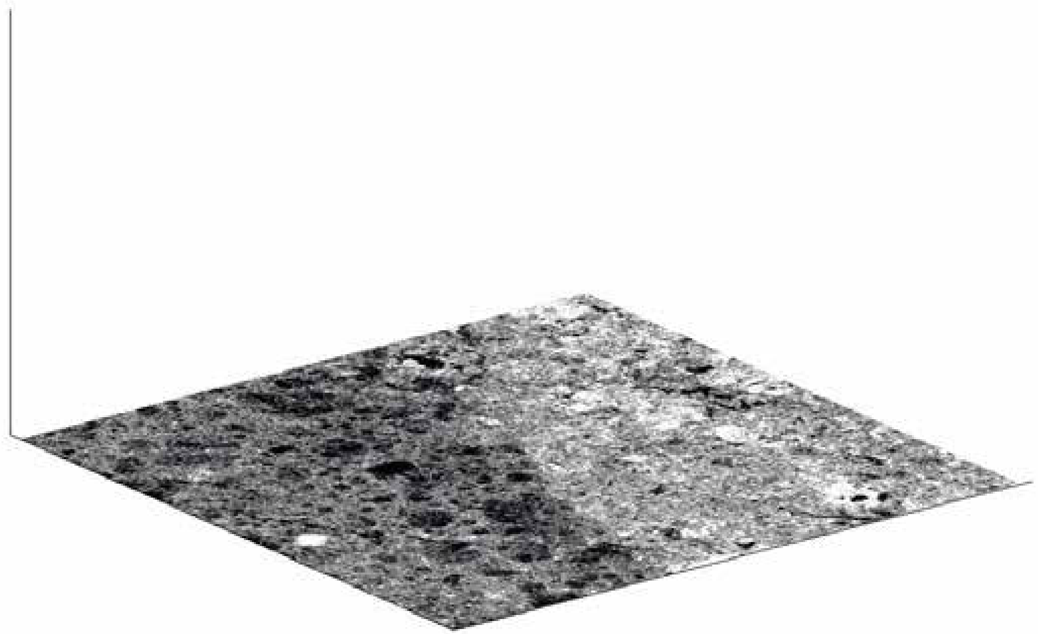
Dyadic squares : $\lambda = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right) \times \left[\frac{l}{2^j}, \frac{l+1}{2^j}\right)$

Wavelet coefficients

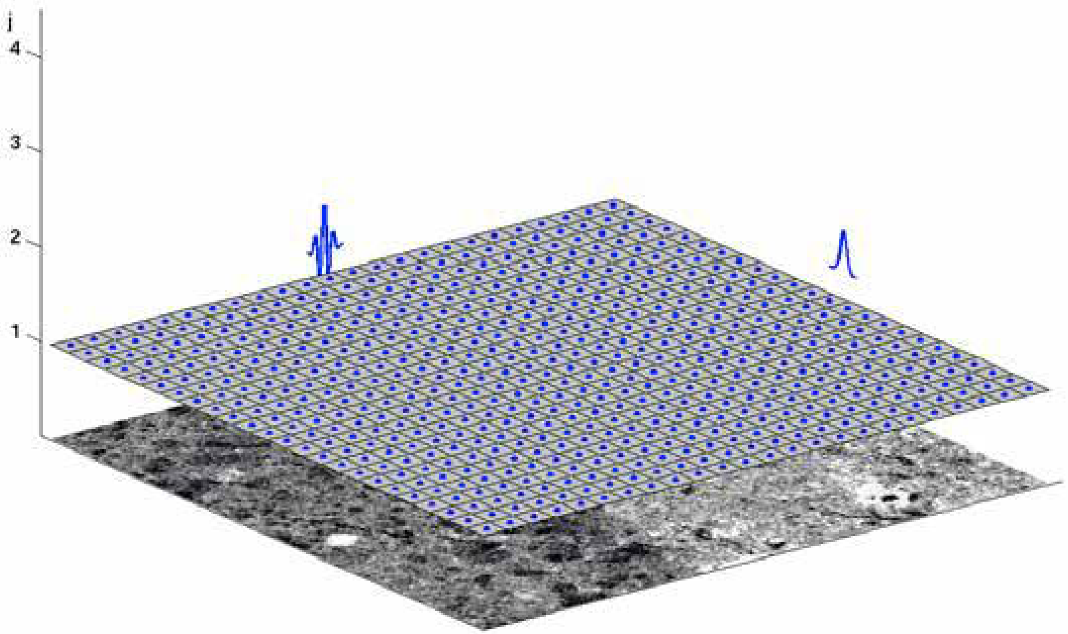
$$c_\lambda = 2^{2j} \int\int f(x,y)\, \psi^i\left(2^j(x,y) - (k,l)\right)\, dx\, dy$$

Natural normalization for scaling invariance

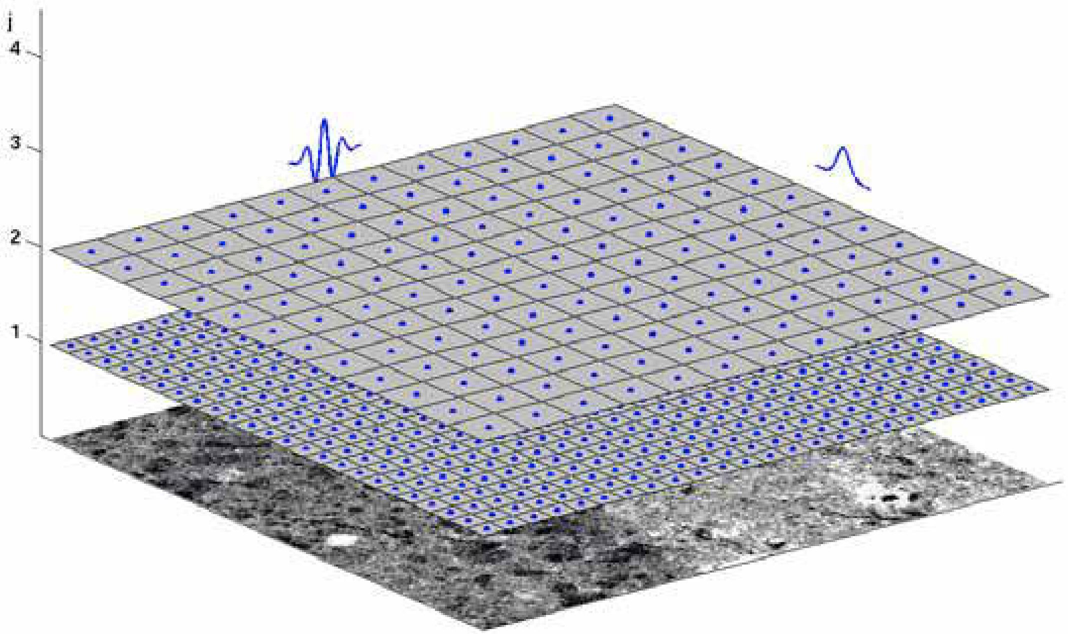
Computation of 2D wavelet coefficients



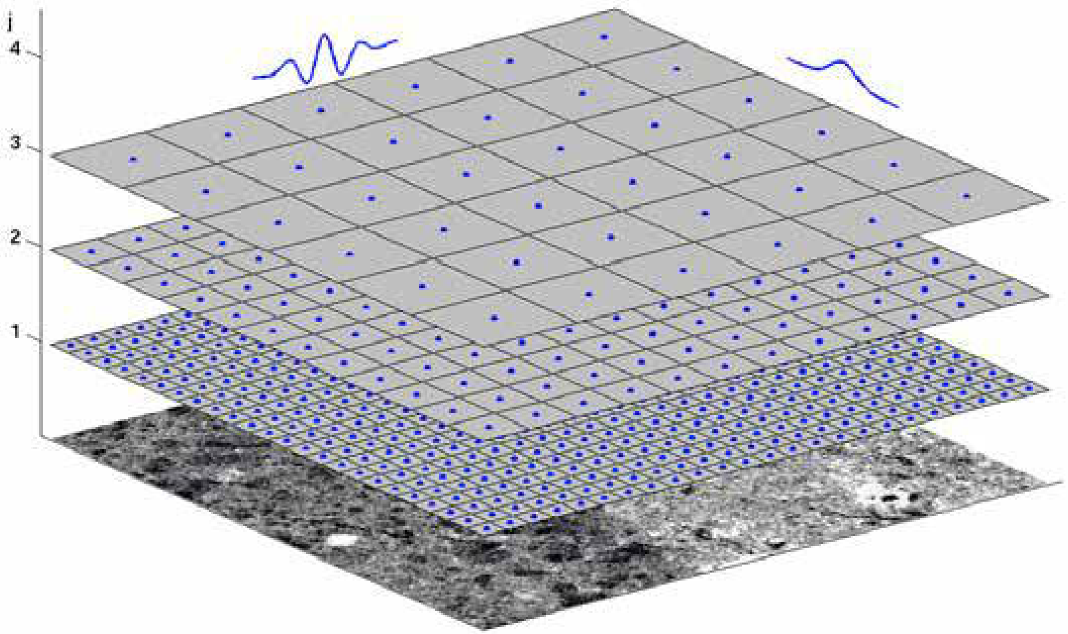
Computation of 2D wavelet coefficients



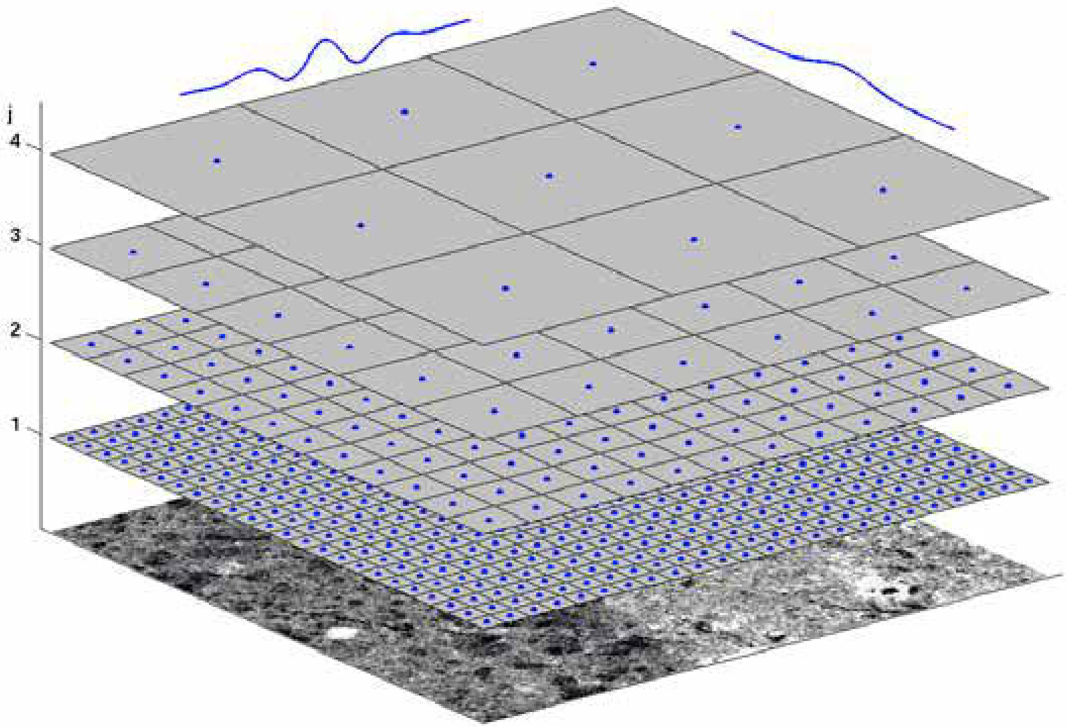
Computation of 2D wavelet coefficients



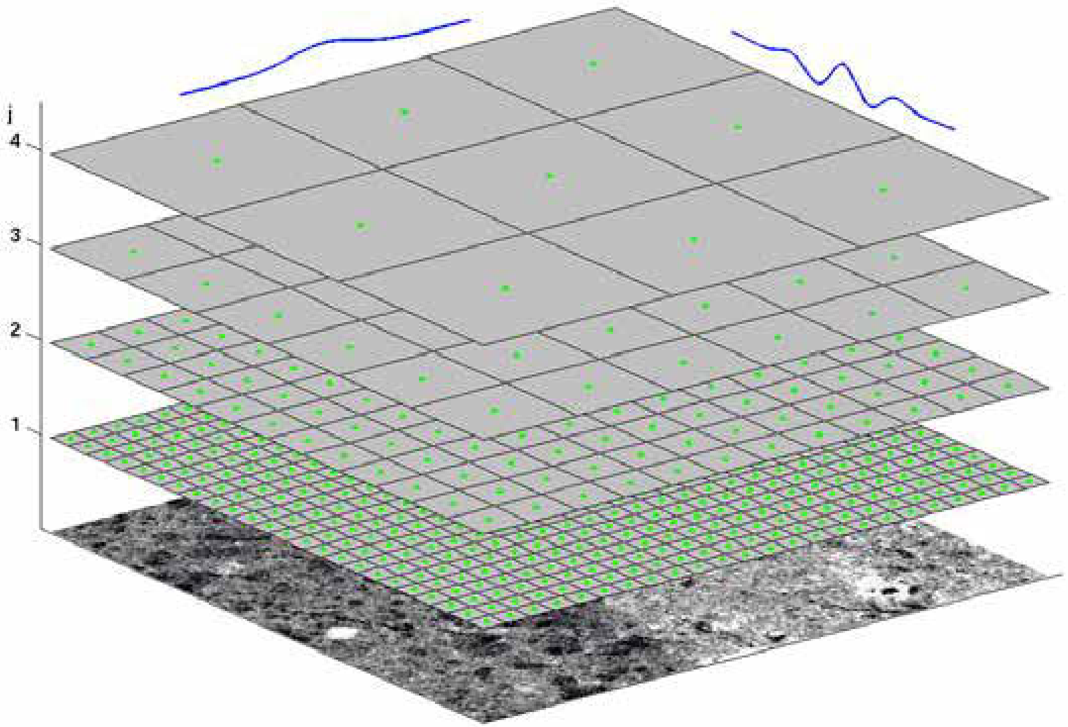
Computation of 2D wavelet coefficients



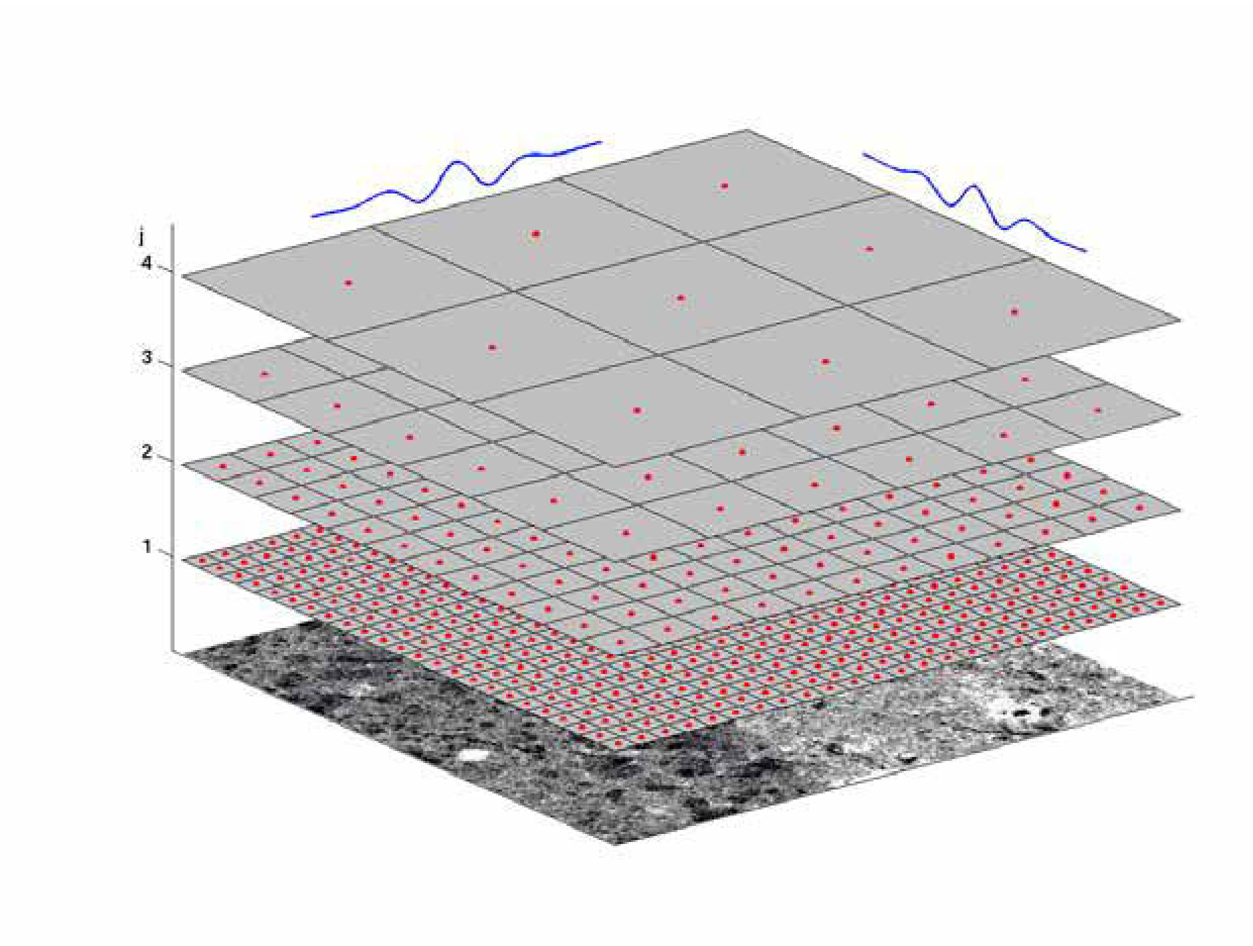
Computation of 2D wavelet coefficients



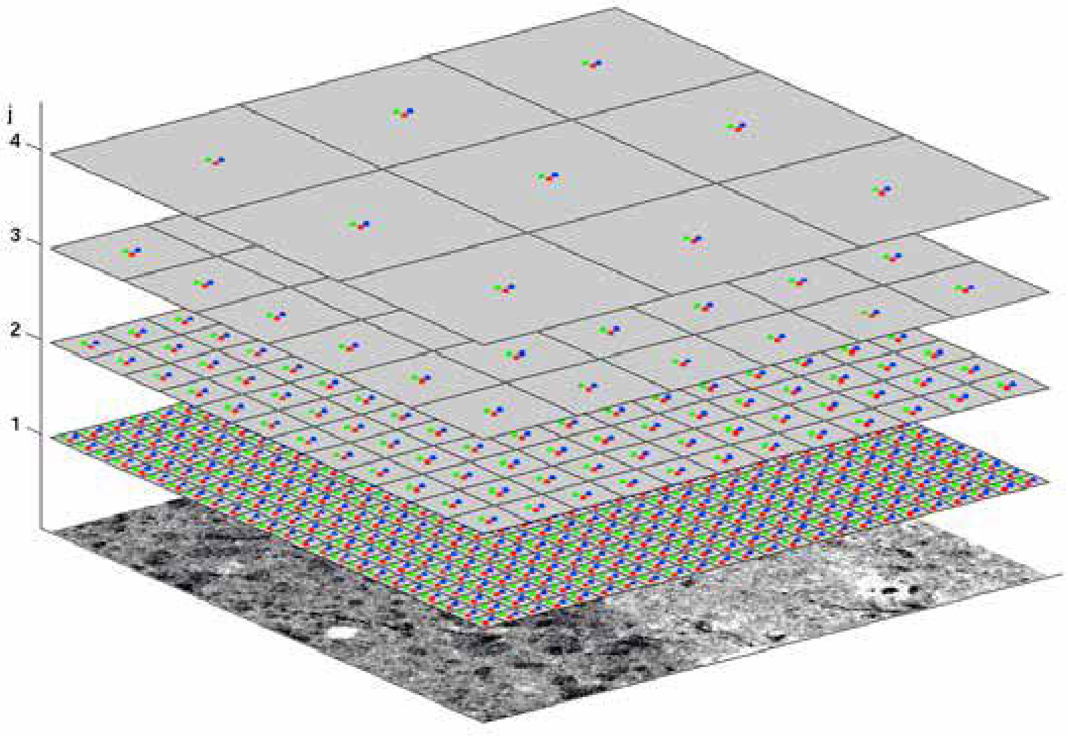
Computation of 2D wavelet coefficients



Computation of 2D wavelet coefficients



Computation of 2D wavelet coefficients



Notations for wavelets on $\mathbb{R}^d$

Dyadic cubes

$$\text{If } k = (k_1, ...k_d), \quad \lambda = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right)$$

Wavelets

$$\psi_\lambda(x) = \psi^i(2^j x - k) \qquad i = 1, \cdots, 2^d - 1$$

Wavelet coefficients

$$c_\lambda = 2^{dj} \int f(x)\psi^i(2^j x - k)dx$$

Dyadic cubes at scale j

$$\Lambda_j = \{\lambda : \quad |\lambda| = 2^{-j}\}$$

Wavelet expansion of f

$$f(x) = \sum_j \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda(x)$$

The wavelet scaling function on $\mathbb{R}^d$

$$\forall p > 0, \qquad 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p) j}$$

With two variants : The first one is

$$\zeta_f(p) = \liminf_{j \to +\infty} \frac{\log \left(2^{-dj} \displaystyle\sum_{\lambda \in \Lambda_j} |c_\lambda|^p \right)}{\log(2^{-j})}$$

This liminf definition coincides with Kolmogorov's scaling function

$$\forall p > 0, \qquad \zeta_f(p) = \sup \left\{ s : \quad f \in B_p^{s/p, \infty} \right\}$$

Scale invariance

The second variant is :

$$\zeta_f(p) = \lim_{j \to +\infty} \frac{\log\left(2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p\right)}{\log(2^{-j})}$$

It means that quantities defined at each scale j satisfy some scale invariance property in the limit of small scales

This condition is usually met in signal and image processing

Additional range of parameters available for classification : $p \in (0, 1)$

Indeed, L^p or Sobolev spaces do not make sense for $p < 1$. One has to use wavelet based substitutes

The role of the wavelet scaling function

$$\begin{aligned}\zeta_f(p) &= \sup \{s : f \in L^{p,s/p}\} \qquad (p \geq 1) \\ &= \sup \{s : f \in B_p^{s/p,\infty}\} \quad (p > 0)\end{aligned}$$

- If $\zeta_f(1) > 1$, then $f \in BV$
- If f is a measure, then $\zeta_f(1) \geq 0$
- If $\zeta_f(1) > 0$, f then belongs to L^1
- If $\zeta_f(2) > 0$, then $f \in L^2$

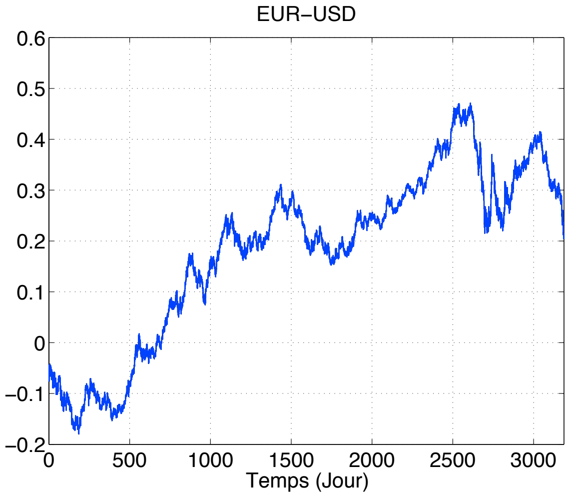
Besov spaces have an alternative definition which is not based on a wavelet expansion

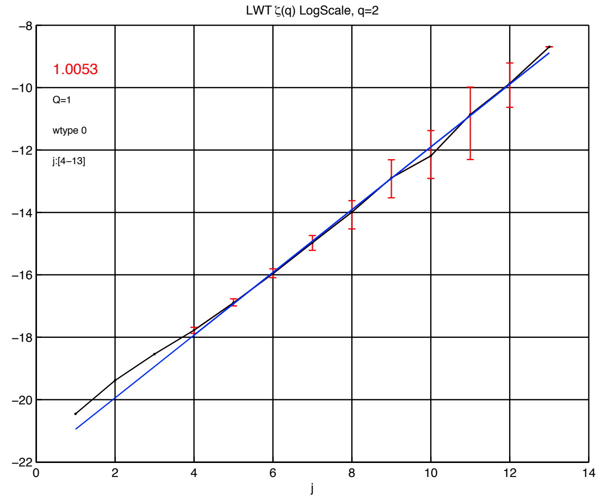
$$\Longrightarrow$$

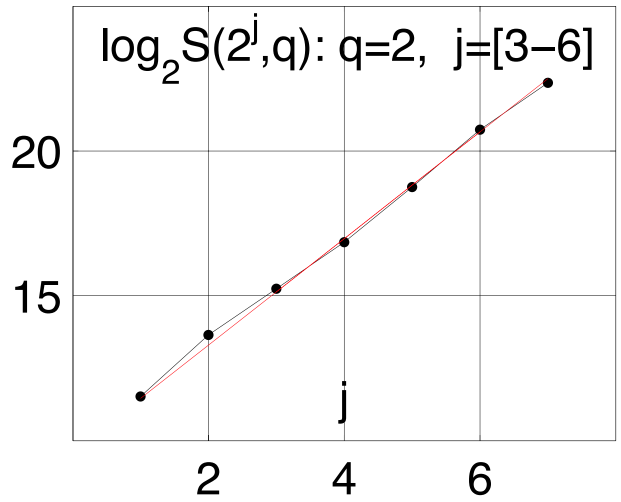
The wavelet scaling function is independent of the (smooth enough) wavelet basis chosen

Log-log plot regression

Power-law behavior : Key requirement for the numerical derivation of the scaling function (here at $p = 1$)





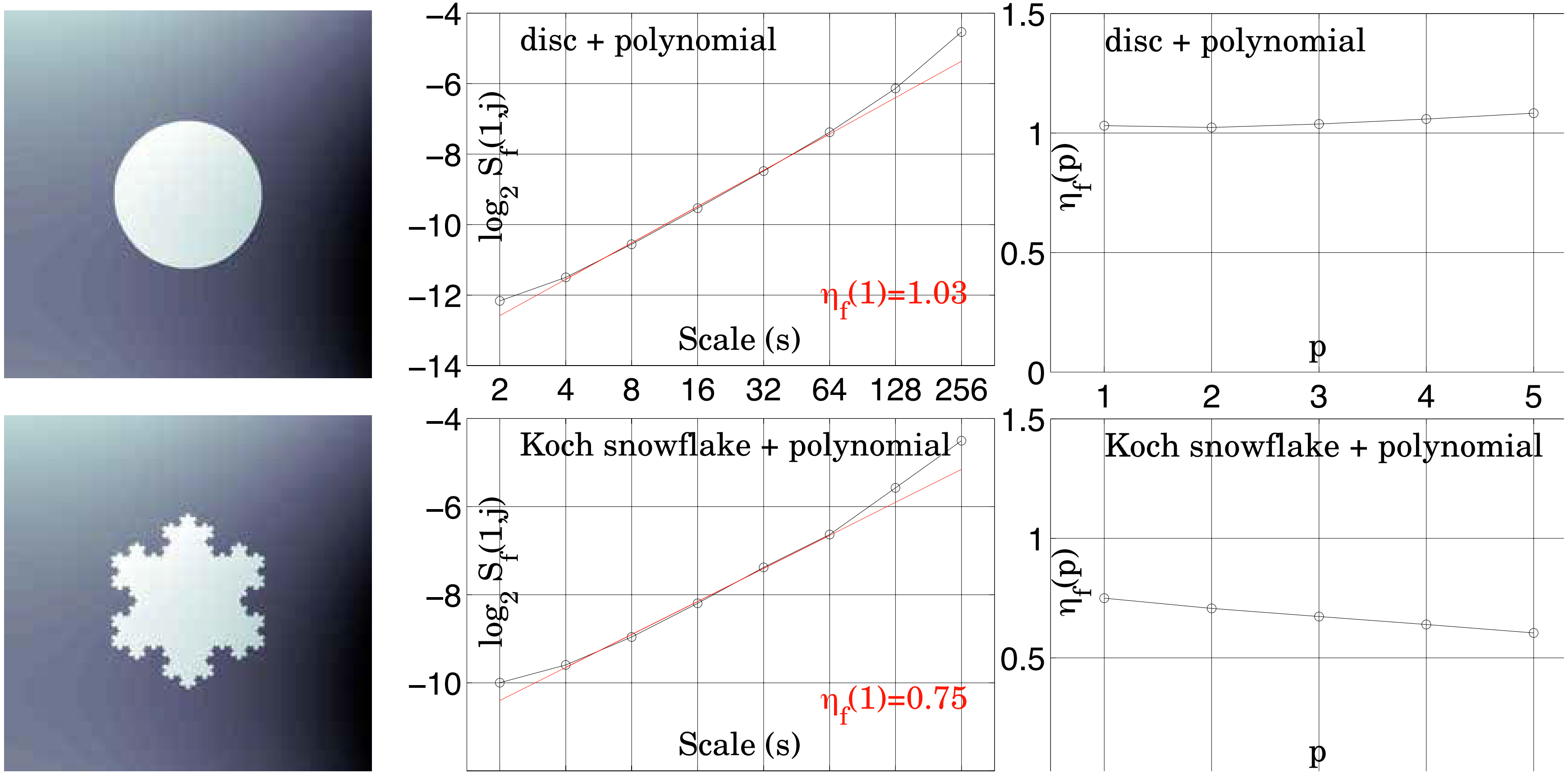


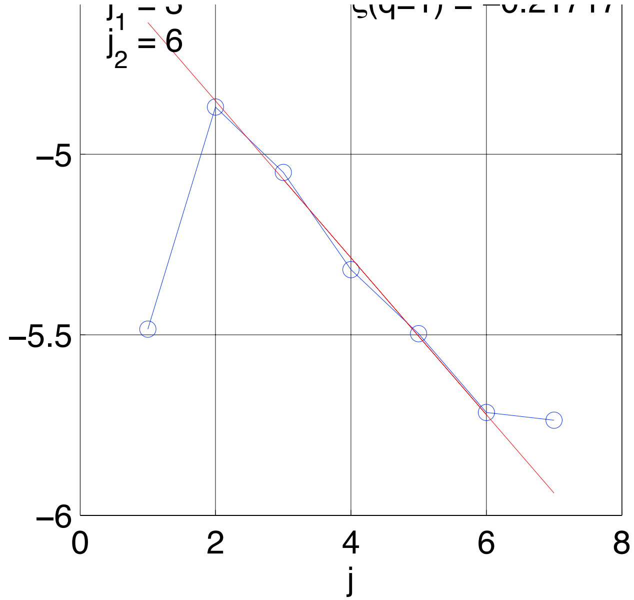
Data courtesy Vivienne Investissement

Wavelet scaling functions of synthetic images

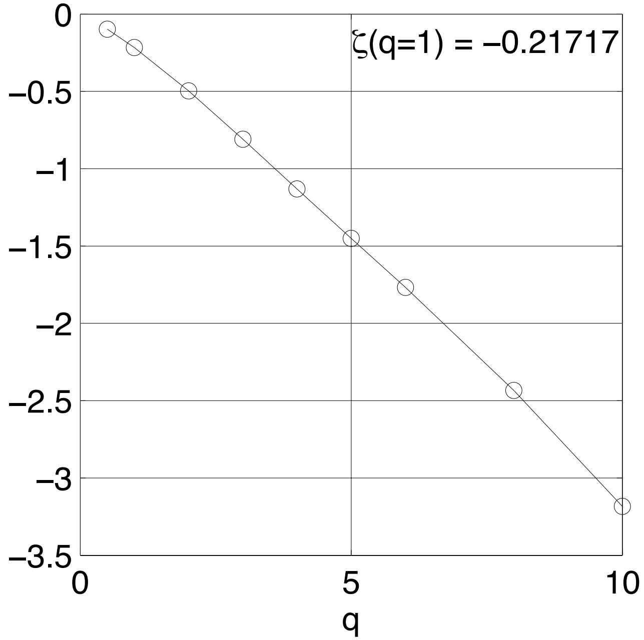
Scaling functions $\begin{cases} \zeta_f(p) = 1 & \\ \zeta_f(p) = 2 - \frac{\log 4}{\log 3} \sim 0.74 & \end{cases}$ Disk / Van Koch snowflake

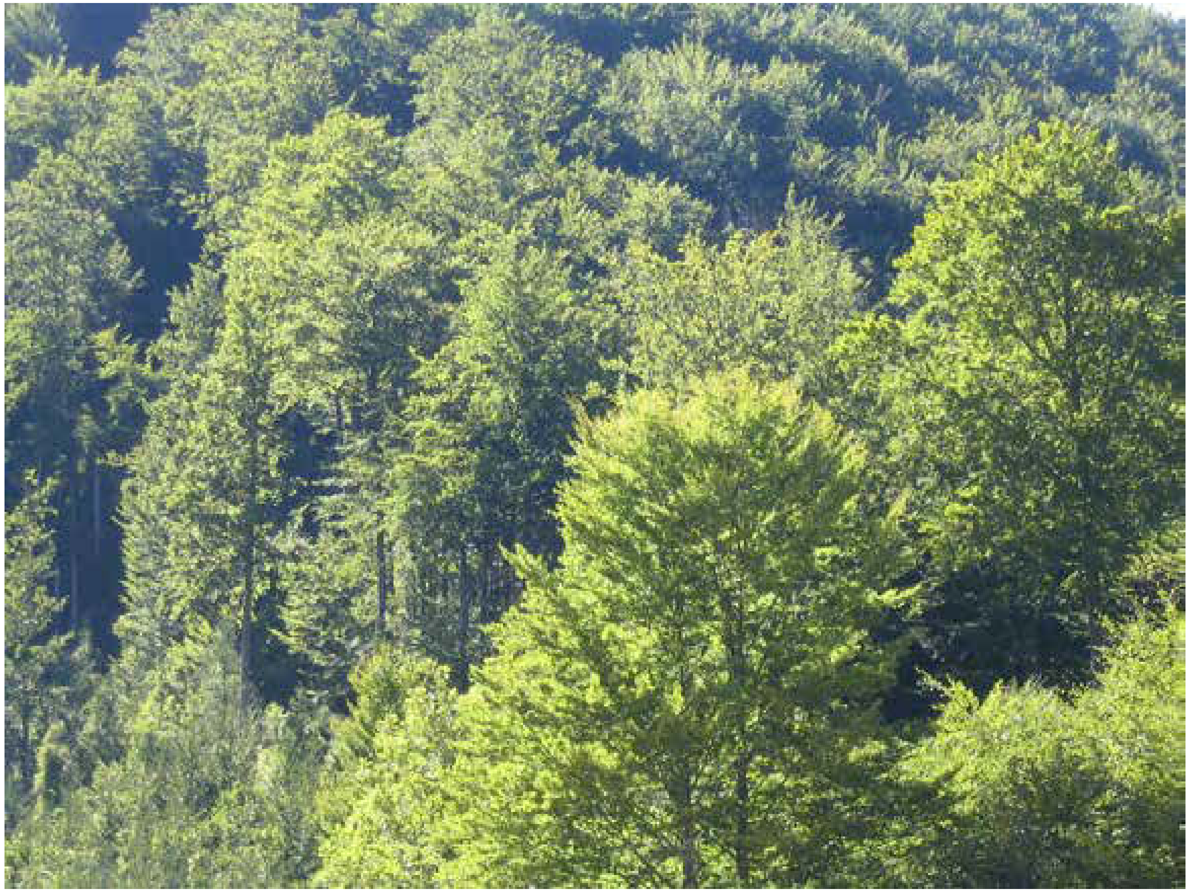
- Disk: $\zeta_f(p) = 1$
- Van Koch snowflake: $\zeta_f(p) = 2 - \frac{\log 4}{\log 3} \sim 0.74$

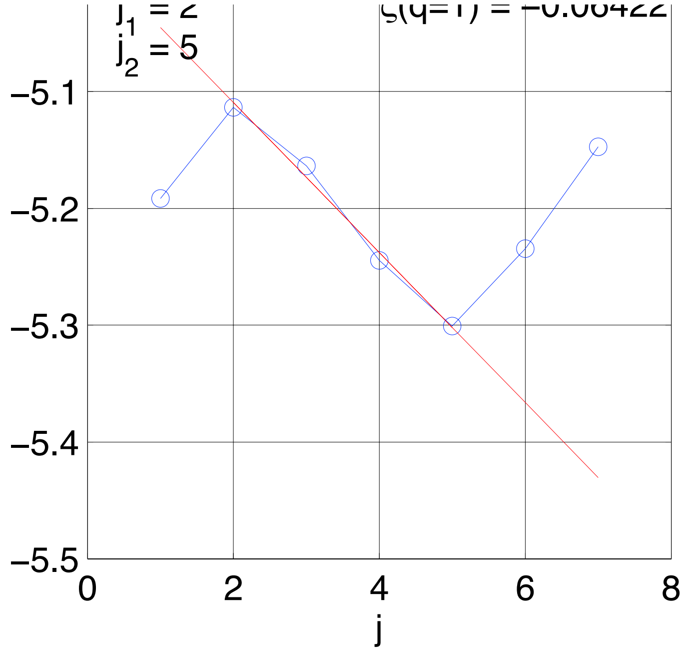




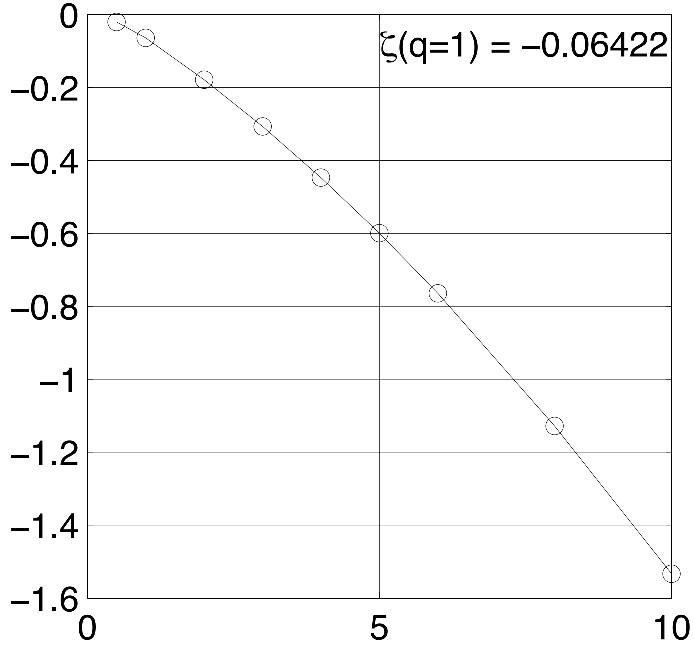
DWT fonction d'echelle $\zeta(q)$

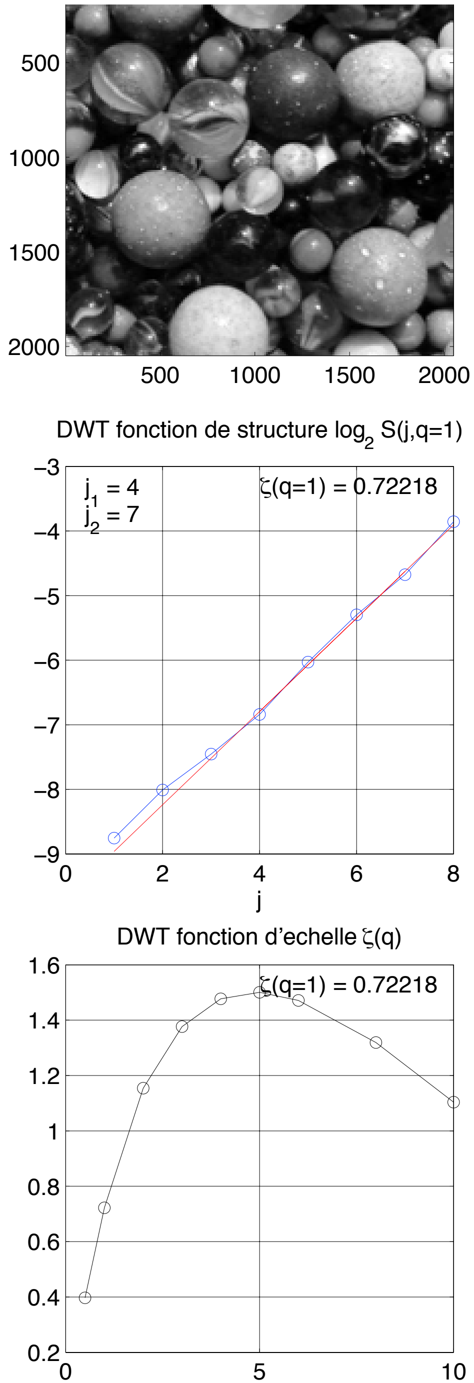


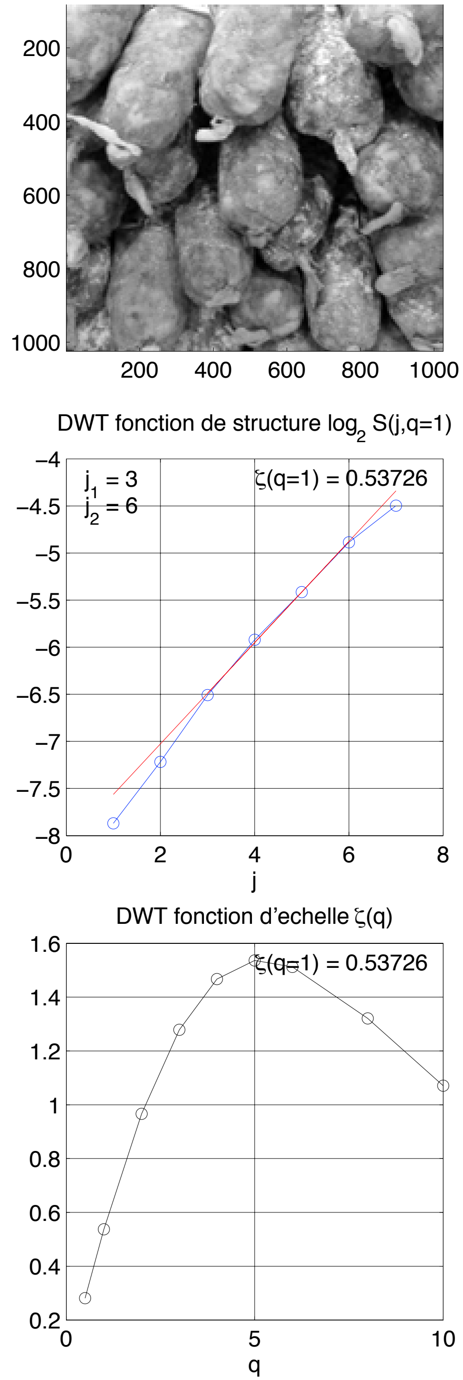




DWT fonction d'echelle ζ(q)




500
1000
1500
2000
500 1000 1500 2000
DWT fonction de structure log2 S(j,q=1)
j1 = 4
j2 = 7
ζ(q=1) = 0.72218
−3
−4
−5
−6
−7
−8
−9
0 2 4 6 8
j
DWT fonction d'echelle ζ(q)
ζ(q=1) = 0.72218
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0 5 10



200
400
600
800
1000
DWT fonction de structure log2 S(j,q=1)
j1 = 3
j2 = 6
ζ(q=1) = 0.53726
−4
−4.5
−5
−5.5
−6
−6.5
−7
−7.5
−8
0
2
4
6
8
j
DWT fonction d'echelle ζ(q)
ζ(q=1) = 0.53726
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
5
10
q

Uniform Hölder regularity

Hölder spaces : Let $\alpha \in (0, 1)$; $f \in C^\alpha(\mathbb{R}^d)$ if

$$\exists C, \forall x, y : \qquad |f(x) - f(y)| \leq C\, |x - y|^\alpha$$

If $\alpha > 1$, $f \in C^\alpha(\mathbb{R})$ if and only if $f^{([\alpha])} \in C^{\alpha - [\alpha]}$

If $\alpha < 0$, $f \in C^\alpha(\mathbb{R})$ if and only if $f = g^{(-[\alpha])}$ with $g \in C^{\alpha - [\alpha]}$

The uniform Hölder exponent of f is

$$H_f^{min} = \sup\{\alpha : \; f \in C^\alpha(\mathbb{R}^d)\}$$

Numerical computation

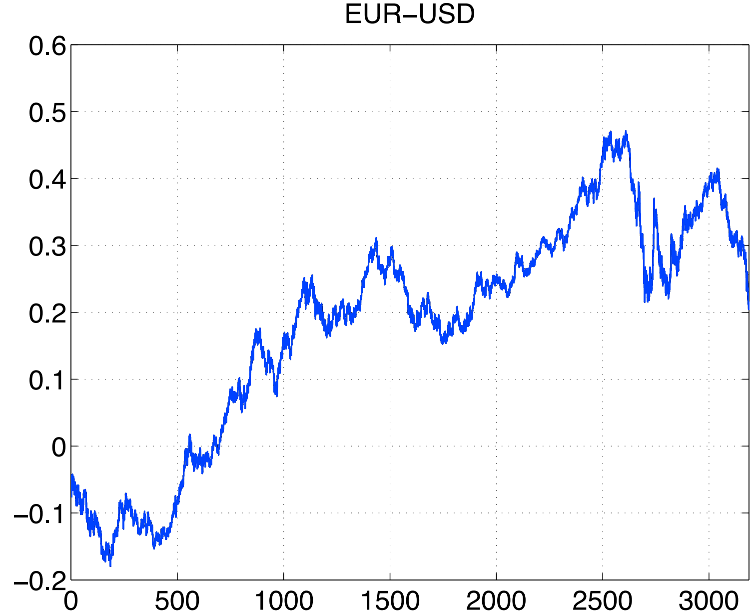
$$\text{Let} \qquad \omega_j = \sup_{\lambda \in \Lambda_j} |c_\lambda| \qquad \text{then} \qquad H_f^{min} = \liminf_{j \to +\infty} \frac{\log(\omega_j)}{\log(2^{-j})}$$

$$H_f^{min} > 0 \quad \Longrightarrow \quad f \text{ is continuous}$$

$$H_f^{min} < 0 \quad \Longrightarrow \quad f \text{ is not locally bounded}$$

Validity of jump models in finance

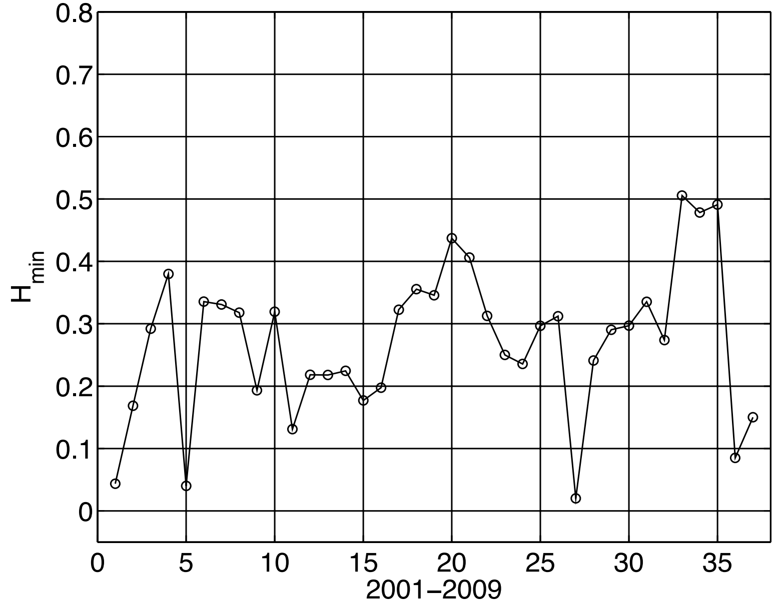
Euro vs. USD
2001-2009



H_f^{min}

local estimation

(3 months window)

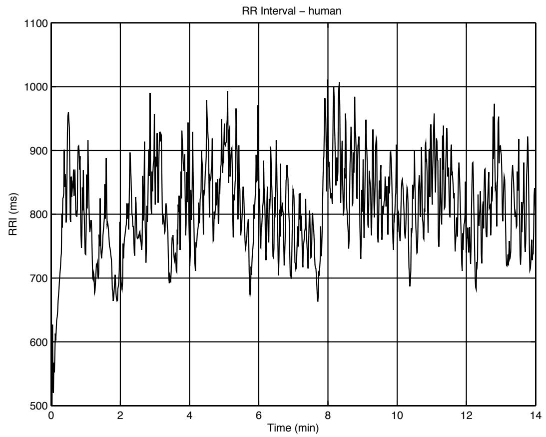


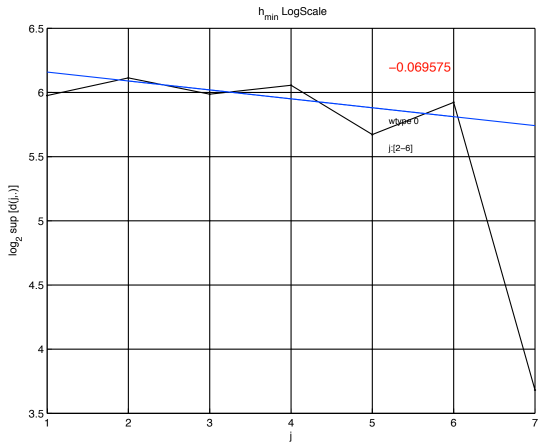
Data supplied by Vivienne Investissement

Classification based on the uniform Hölder exponent

Heartbeat intervals

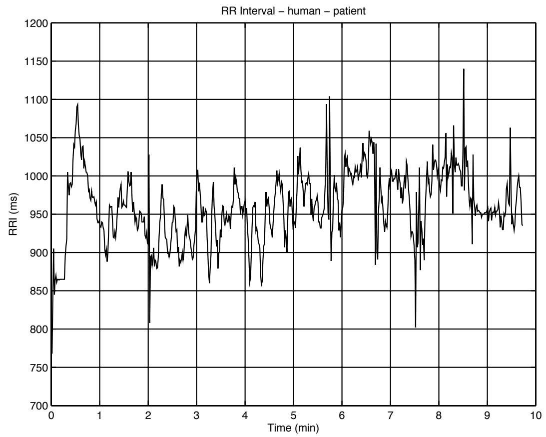
Healthy

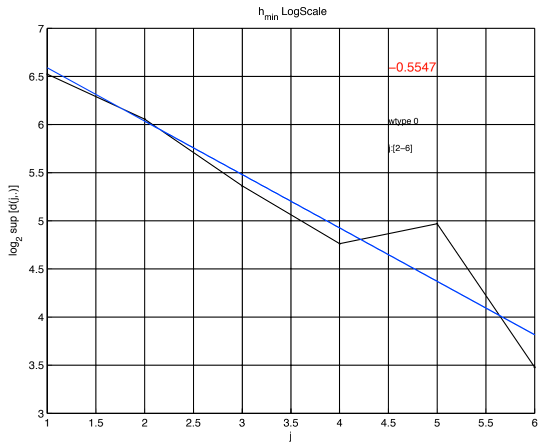




$$H_f^{min} = -0.06$$

Heart failure





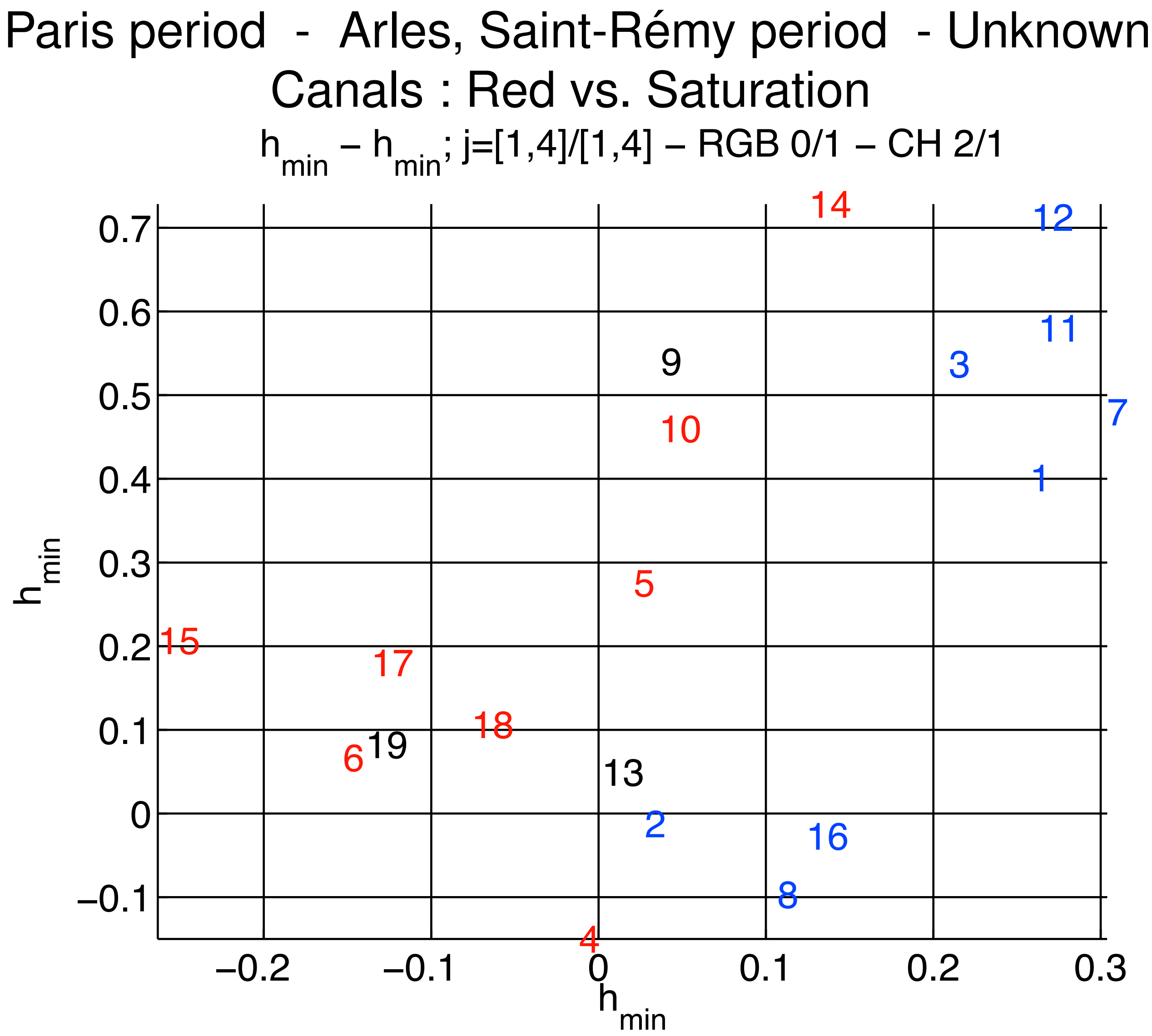
$$H_f^{min} = -0.55$$

Analysis of paintings : Van Gogh challenge

(in collaboration with D. Rockmore)

Arles and Saint-Rémy period   Van Gogh   Paris period

Challenge : Date

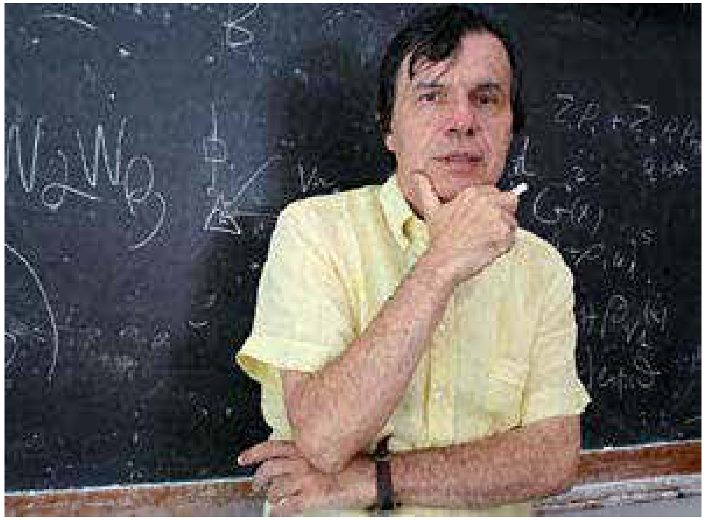


Drawbacks

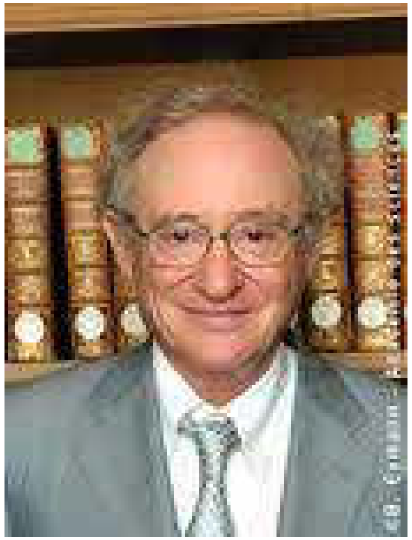
Classification only based on the wavelet scaling function proved insufficient in several occurrences (turbulence,...)

New ideas initiated by U. Frisch and G. Parisi are required

Beyond scale invariance : Multifractal Analysis



Giorgio Parisi



Uriel Frisch

Pointwise regularity

Let f be a locally bounded function $\mathbb{R}^d \to \mathbb{R}$ and $x_0 \in \mathbb{R}^d$; $f \in C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial P of degree less than α such that

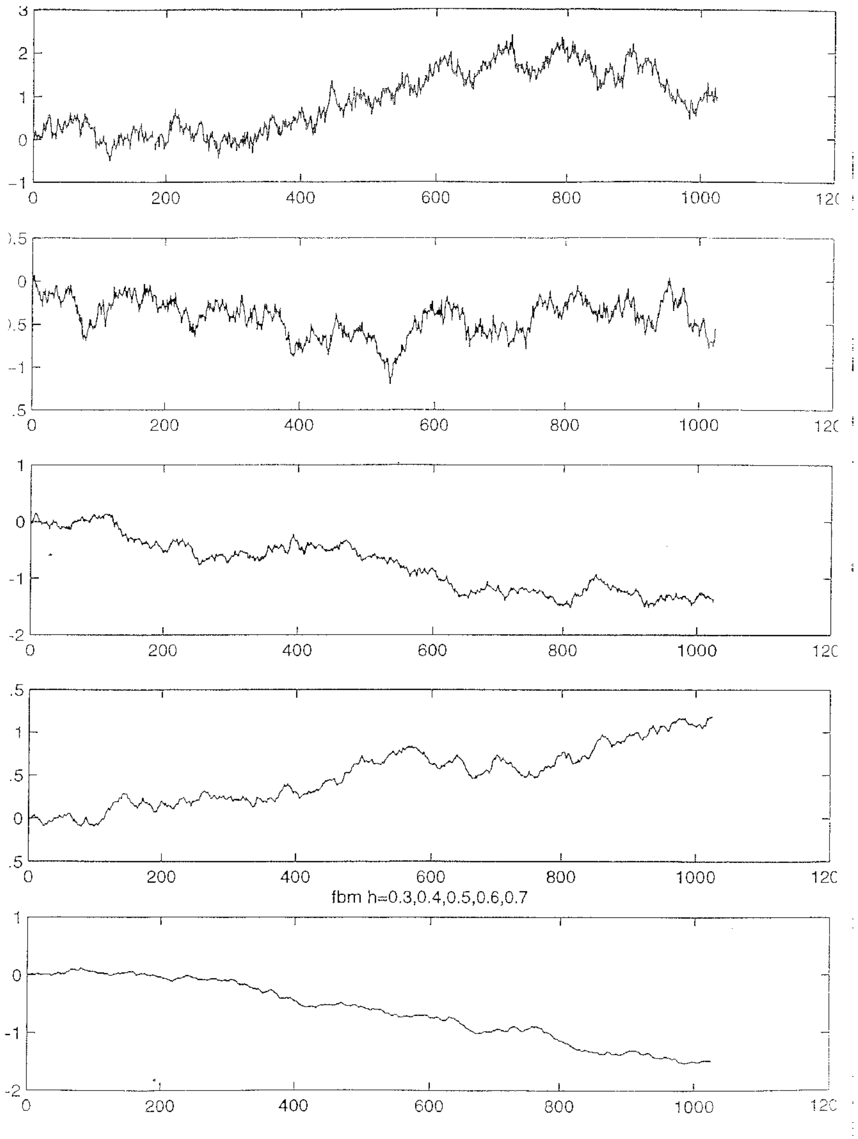
$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha : \quad f \in C^\alpha(x_0)\}$$

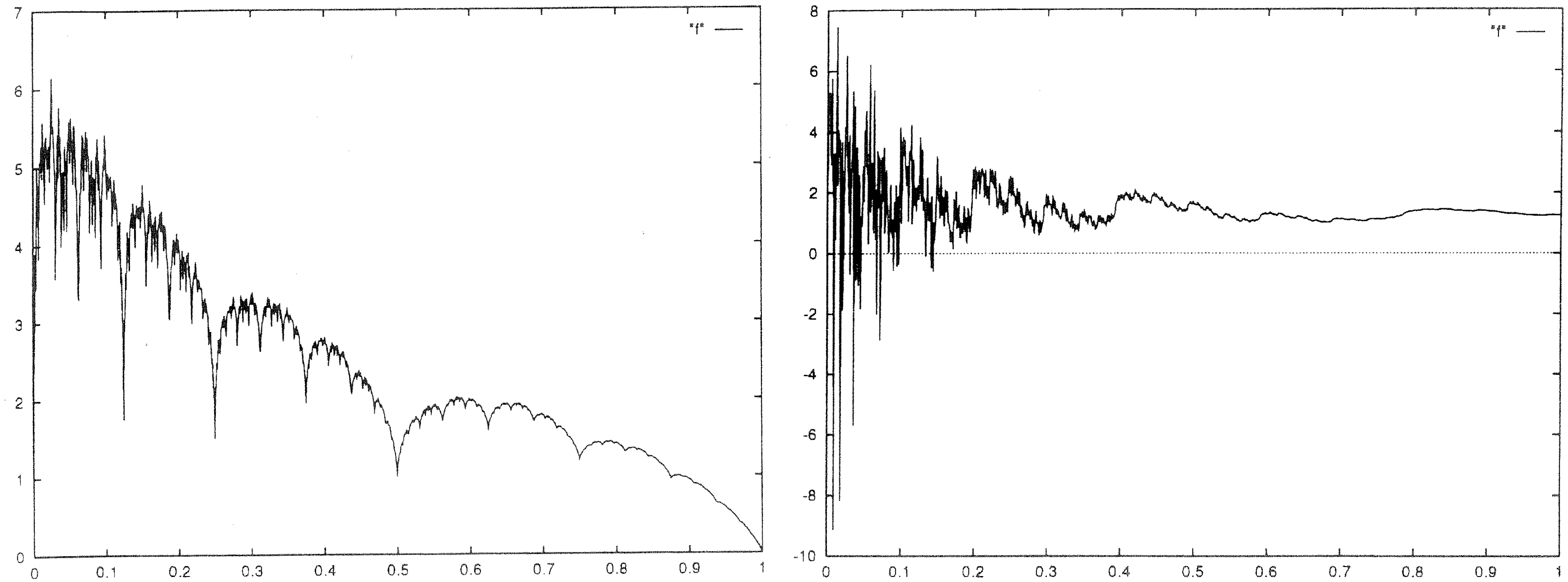
Heuristics : $\quad |f(x) - f(x_0)| \sim |x - x_0|^{h_f(x_0)}$

Constant Hölder exponents



FBMs with Hölder exponents 0.3, 0.4, 0.5, 0.6 and 0.7

Varying Hölder exponent

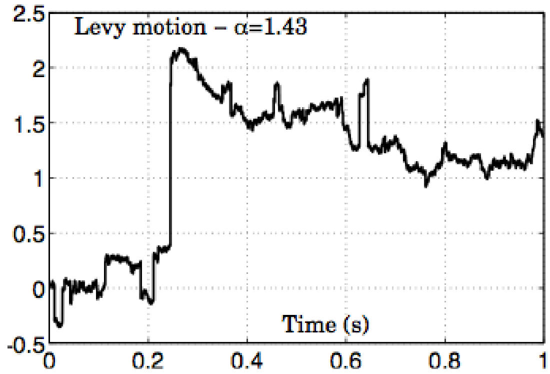


$h_f(x) = x$
(Constructions of K. Daoudy, J. Lévy-Véhel and Y. Meyer)

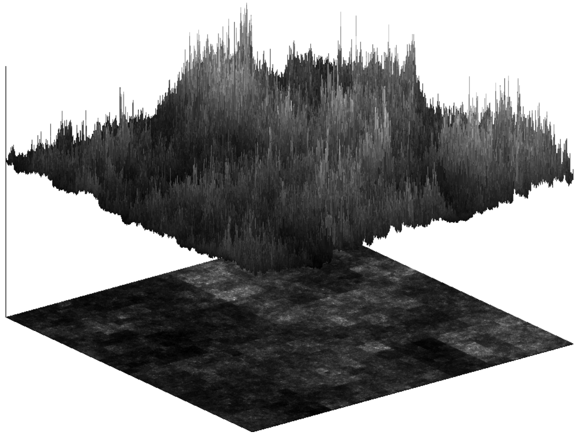
Difficulty to use directly the pointwise regularity exponent for classification

For classical models, such exponents are extremely erratic

Lévy processes



multiplicative cascades



The function *h* is random and everywhere discontinuous

Goal : Recover some information on $h(x)$ from (time or space) averaged quantities which is :

- numerically computable on a sample path of the process, or on real-life data,
- deterministic (independent of the sample path)

Hausdorff dimension

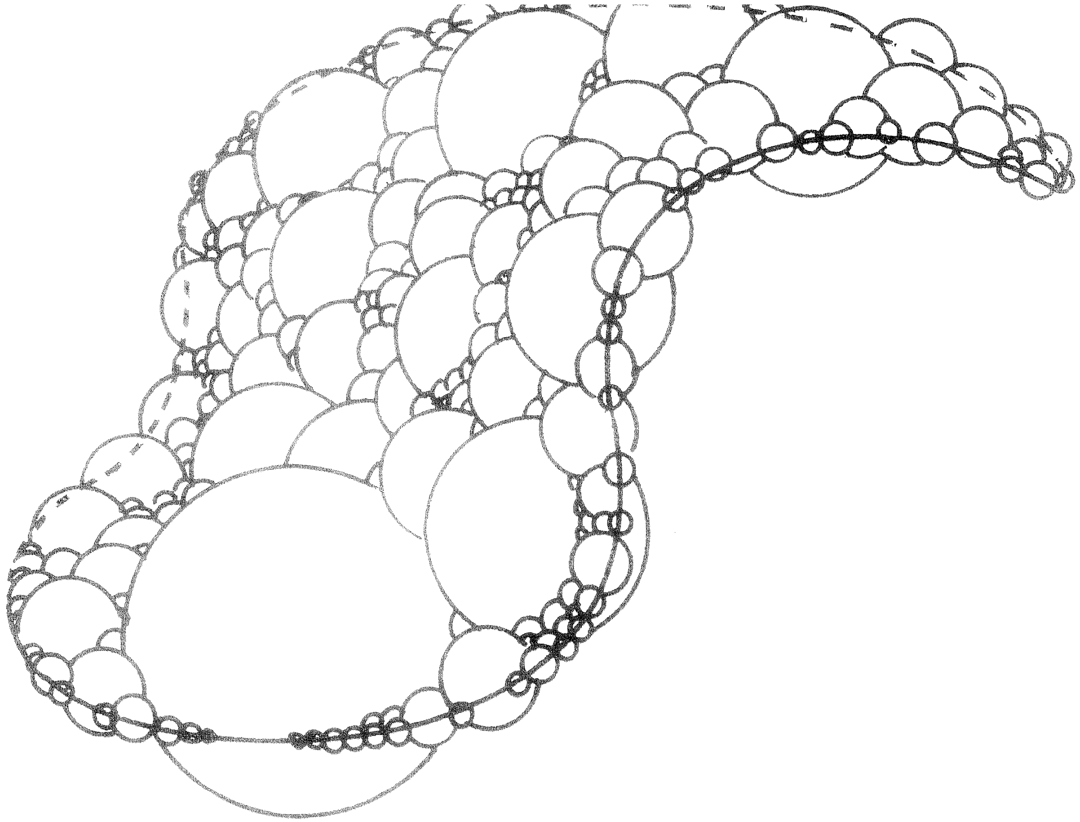
ε-coverings

Definition : Let $A \subset \mathbb{R}^d$; an ε-covering of A is a family $\mathcal{C}_\varepsilon = \{B_i\}_{i \in \mathbb{N}}$ of sets B_i such that

- $\forall i$, $diam\,(B_i) \leq \varepsilon$
- $A \subset \bigcup B_i$

Example 1 : If A is a smooth curve, "economical" ε-coverings satisfy

$$\sum_i (diam\,(B_i)) \sim L(A)$$



Example 2 : If A is a smooth surface, “economical” ε-coverings satisfy

$$\sum_i (diam\,(B_i))^2 \sim S(A)$$

Hausdorff dimension

The Hausdorff dimension of A is the exponent $\dim(A)$ satisfying

$$\text{for } \varepsilon \text{ small enough,} \qquad \inf_{\mathcal{C}_\varepsilon} \sum_i (diam\,(B_i))^{dim(A)} \sim 1.$$

Convention : $\dim(\emptyset) = -\infty$.

If the (B_i) have the same width ε, their number $N(\varepsilon)$ satisfies

$$N(\varepsilon) \sim \varepsilon^{-\dim(A)}$$

$\Longrightarrow$ The Hausdorff dimension is smaller than the box dimension

Advantage : Good mathematical properties

Drawback : Not computable for real-life data

Spectrum of singularities

The isohölder sets of f are the sets

$$E_H = \{x_0 : \quad h_f(x_0) = H\}$$

Let f be a locally bounded function

The spectrum of singularities of f is

$$D_f(H) = \dim\,(E_H)$$

(by convention, $\dim\,(\emptyset) = -\infty$)

Classification based on a (potentially) infinite number of fractal sets

Can one estimate the spectrum of singularities of signals and images ?

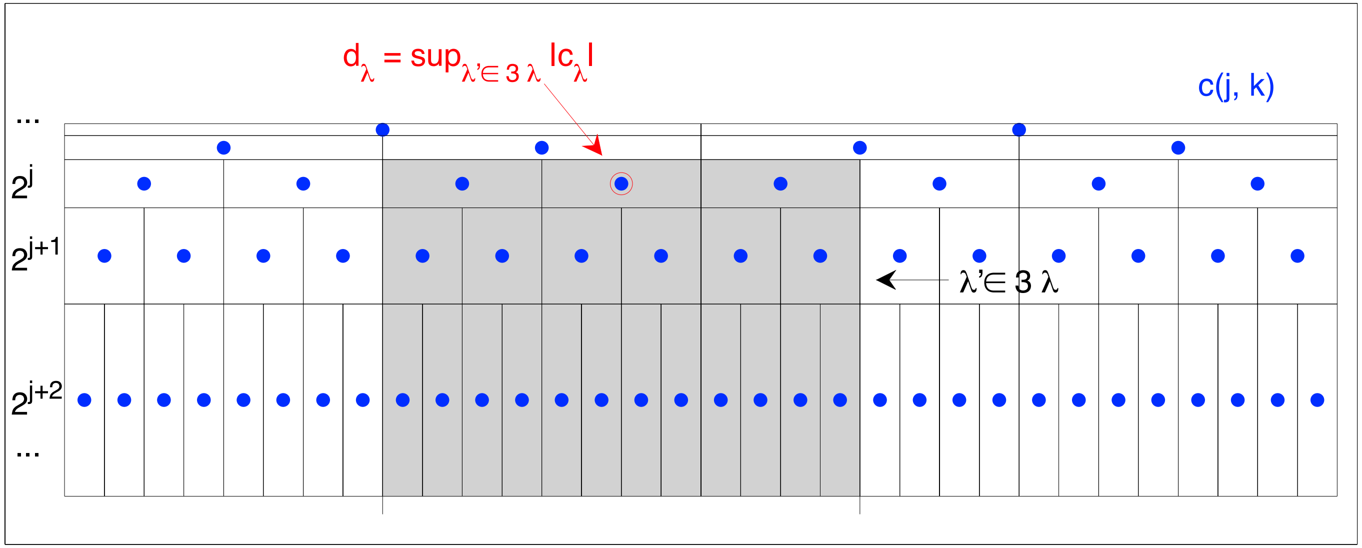
Can it yield new classification parameters ?

Wavelet leaders

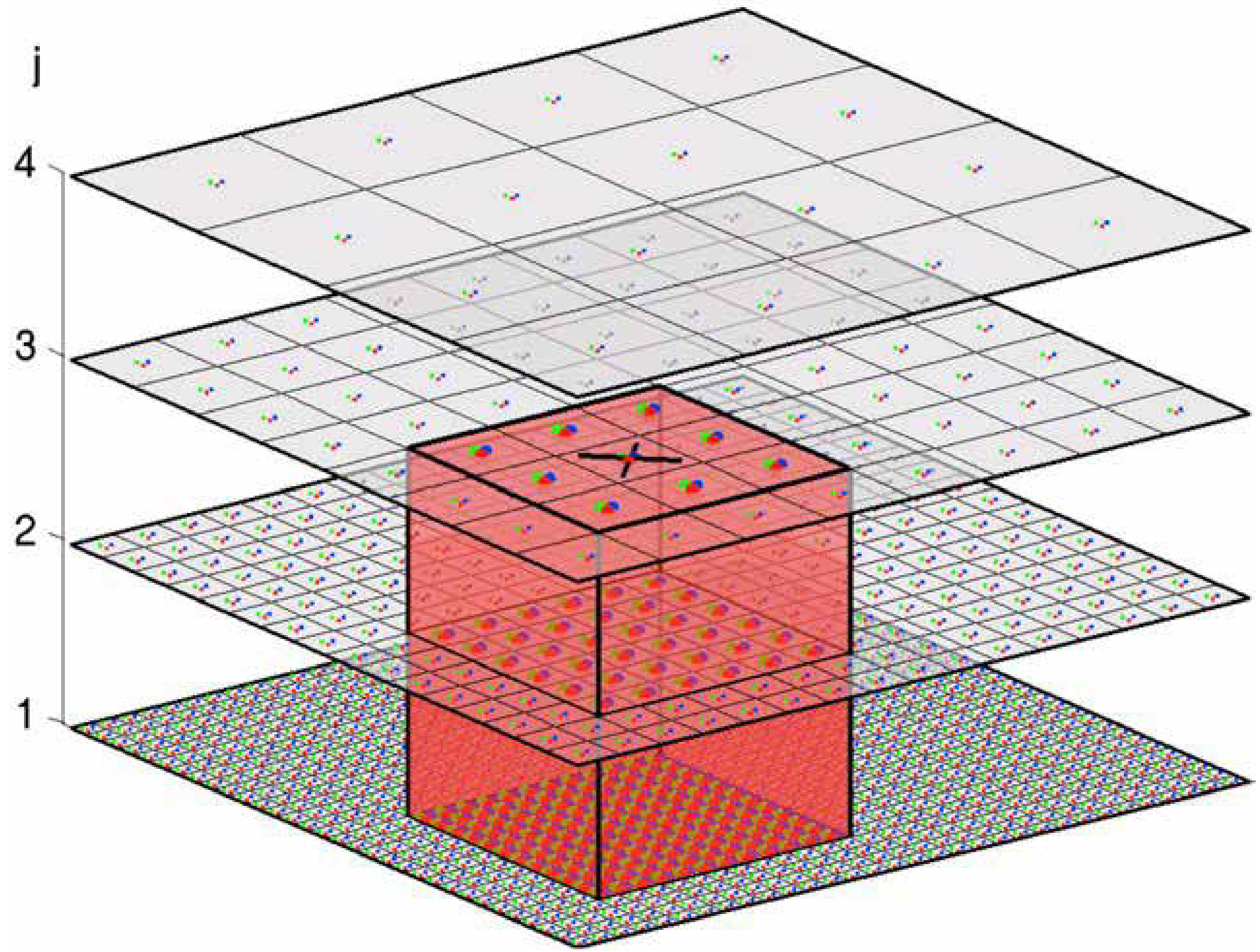
Let λ be a dyadic cube ; 3λ is the cube of same center and three times wider

Let f be a locally bounded function ; the wavelet leaders of f are the quantities

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$



Computation of 2D wavelet leaders



Wavelet characterization of pointwise smoothness

Let $\lambda_j(x_0)$ denote the dyadic cube of width 2^{-j} that contains x_0

Theorem If $H_f^{min} > 0$, then

$$\forall x_0 \in \mathbb{R}^d \qquad h_f(x_0) = \liminf_{j \to +\infty} \frac{\log(d_{\lambda_j(x_0)})}{\log(2^{-j})}.$$

Sketch of proof :

A bounded function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $f \in C^\alpha(x_0)$ ($\alpha > 0$) if there exist $C > 0$ and a polynomial P of degree less than α such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

$$\Longleftrightarrow$$

$$\sup_{x \in B(x_0, r)} |f(x) - P(x - x_0)| \leq C \cdot r^\alpha$$

Wavelet characterization of pointwise smoothness

$$\sup_{x \in B(x_0, r)} |f(x) - P(x - x_0)| \leq C \cdot r^{\alpha} \iff \| f - P \|_{L^{\infty}(B(x_0, r))} \leq C \cdot r^{\alpha}$$

One uses the “wavelet characterization” of L^{∞} :

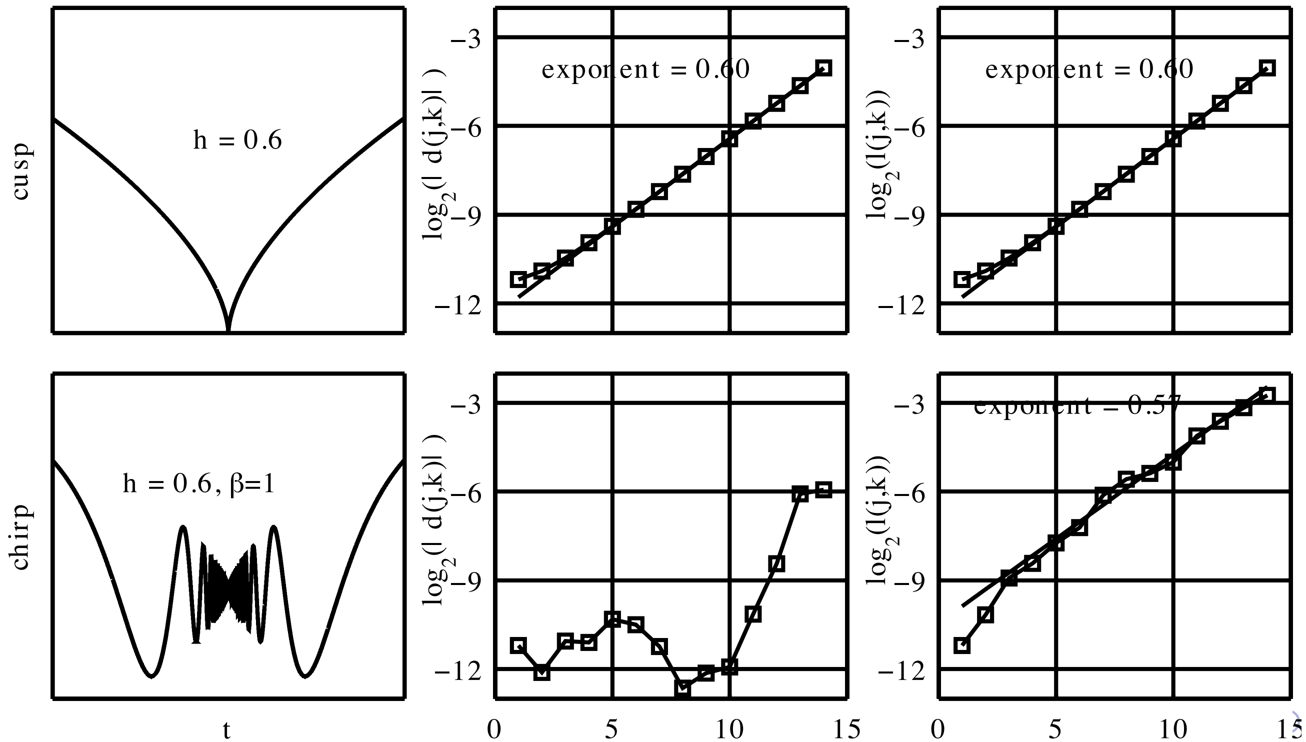
$$\text{if} \quad f \in L^{\infty} \quad \text{then} \quad \sup_{\lambda} |c_{\lambda}| \leq C$$

$|x - x_0|^h$

$h = 0.6$

$|x - x_0|^h \sin\left(\frac{1}{|x - x_0|^{\beta}}\right)$

$h = 0.6 \ \beta = 1$



Associated scaling function and function spaces

Wavelet scaling function

$$2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p)j}$$

Leader scaling function

$$2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p \sim 2^{-\eta_f(p)j}$$

Two possible definitions depending whether the quotient of logarithms only has a liminf or a real limit.

Oscillation spaces : Let $p > 0$; $f \in \mathcal{O}_p^s(\mathbb{R}^d)$ if

$$\exists C, \forall j : \qquad 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p \leq 2^{-spj}$$

$$\forall p > 0, \qquad \eta_f(p) = p \cdot \sup \left\{ s : \ \ p \in \mathcal{O}_p^s \right\}$$

$\mathcal{O}_p^s$ and $B_p^{s,\infty}$ coincide if $s > d/p$

If $s < d/p$, one can derive information on the inter-scales correlations of wavelet coefficients

The nature of pointwise singularities

Comparing Besov and Oscillation spaces which contain f yields information on the type of pointwise singularities present in the function

Cusp singularities : $f(x) = C|x - x_0|^\alpha + o(|x - x_0|^\alpha)$
When the Besov and oscillation spaces that contain f coincide

Chirp singularities : $f(x) = C|x - x_0|^\alpha \sin\left(\frac{1}{|x - x_0|^\beta}\right) + o(|x - x_0|^\alpha)$
When the Besov and oscillation spaces that contain f differ

Heuristic justification : For cusp-like singularities, the supremum in the leaders in reached at the root of the dyadic tree

$$2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p$$

For cusps : $d_\lambda := \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}| \sim c_\lambda$

Estimation of the p-variation

Let $f : [0,1]^d \to \mathbb{R}$ be a continuous function. If $A \subset [0,1]^d$, the first order oscillation of f on A is

$$Os_f(A) = \sup_A f - \inf_A f$$

Definition : Let $p \geq 1$; f belongs to V_p^s if

$$\exists C \quad \forall j \qquad 2^{-dj} \sum_{\lambda \in \Lambda_j} (Os_f(3\lambda))^p \leq C 2^{-spj}$$

Definition : f has bounded p-variation if

$$\exists C \;\; \forall j \geq 0 \qquad \sum_{\lambda \in \Lambda_j} (Os_f(3\lambda))^p \leq C$$

Therefore If $f \in V_p^{d/p}$, then f has bounded p-variation

Estimation of the p-variation

Theorem : Let $p \geq 1$; then

$$\forall \varepsilon > 0 \qquad C^{\varepsilon} \cap V_p^s \hookrightarrow \mathcal{O}_p^s \hookrightarrow V_p^{s+\varepsilon}$$

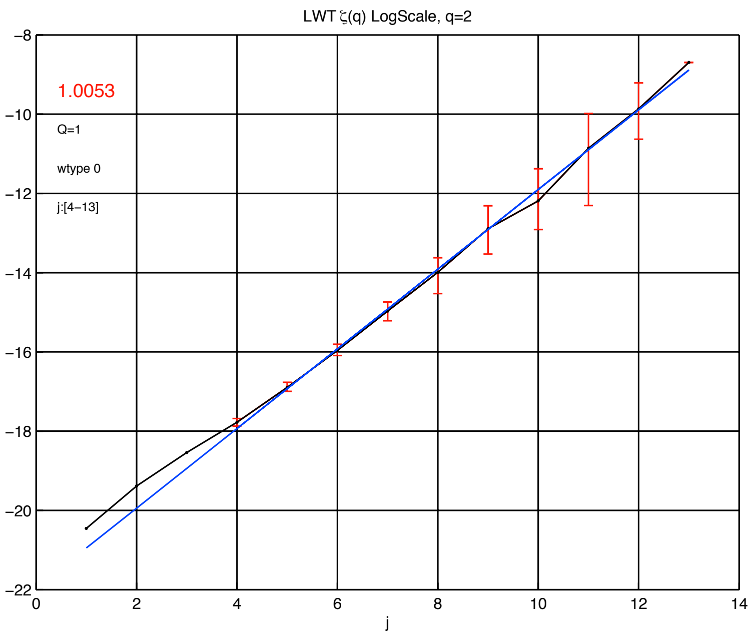
Corollary : If $H_f^{min} > 0$, one can determine whether f has bounded p-variation or not by inspecting the value of its leader scaling function at p, indeed :

- If $\zeta_f(p) > d$, then f has bounded p-variation
- if $\zeta_f(p) < d$, then the p-variation of f is unbounded

Quadratic variation

Wavelet criterium

If $H_f^{min} > 0$ and $\zeta_f(2) > 1$ then f has bounded quadratic variation



Euro vs. USD
2001-2009

$\zeta(2) = 1.0053$

Multifractal formalism :

The Frisch-Parisi derivation revisited

Leader structure function : $T_{p,j} = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p \sim 2^{-\eta_f(p) j}$

Contribution to $T_{p,j}$ of the points x_0 such that $h(x_0) = H$:

$$\sim 2^{-dj} \cdot \left(2^{-j}\right)^{D_f(H)} \cdot (2^{-Hj})^p = (2^{-j})^{d+Hp-D_f(H)}$$

$$\eta_f(p) = \inf_H \left(d + Hp - D_f(H)\right)$$

η_f is expected to be the Legendre transform of D_f

Conjecture : $D_f(H) = \inf_{p \in \mathbb{R}} \left(d + Hp - \eta_f(p)\right)$

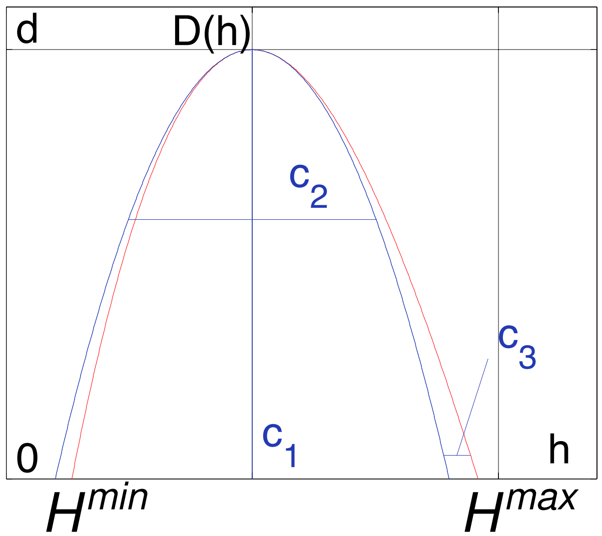
We define the leader spectrum as :

$$L_f(H) = \inf_{p \in \mathbb{R}} \left(d + Hp - \eta_f(p)\right)$$

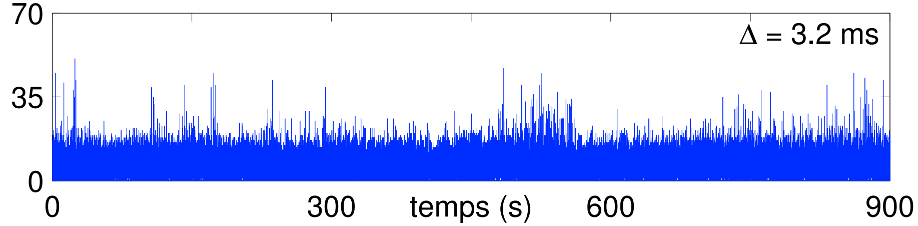
Theorem : If $f \in C^\varepsilon(\mathbb{R}^d)$ for an $\varepsilon > 0$ then $\forall H \in \mathbb{R}, \quad D_f(H) \leq L_f(H)$

Classification parameters deduced from the multifractal spectrum

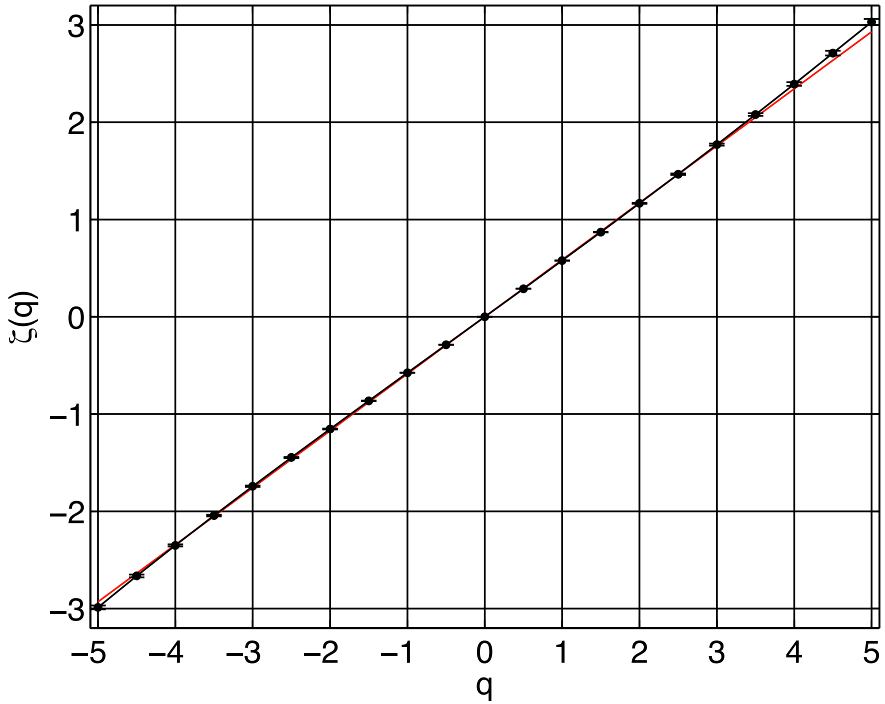
- H^{min} : Minimal regularity
- H^{max} : Maximal regularity
- c_1 : Position of the max
- c_2 : Width
- c_3 : Asymetry



Monohölder vs. Multifractality



Internet traffic

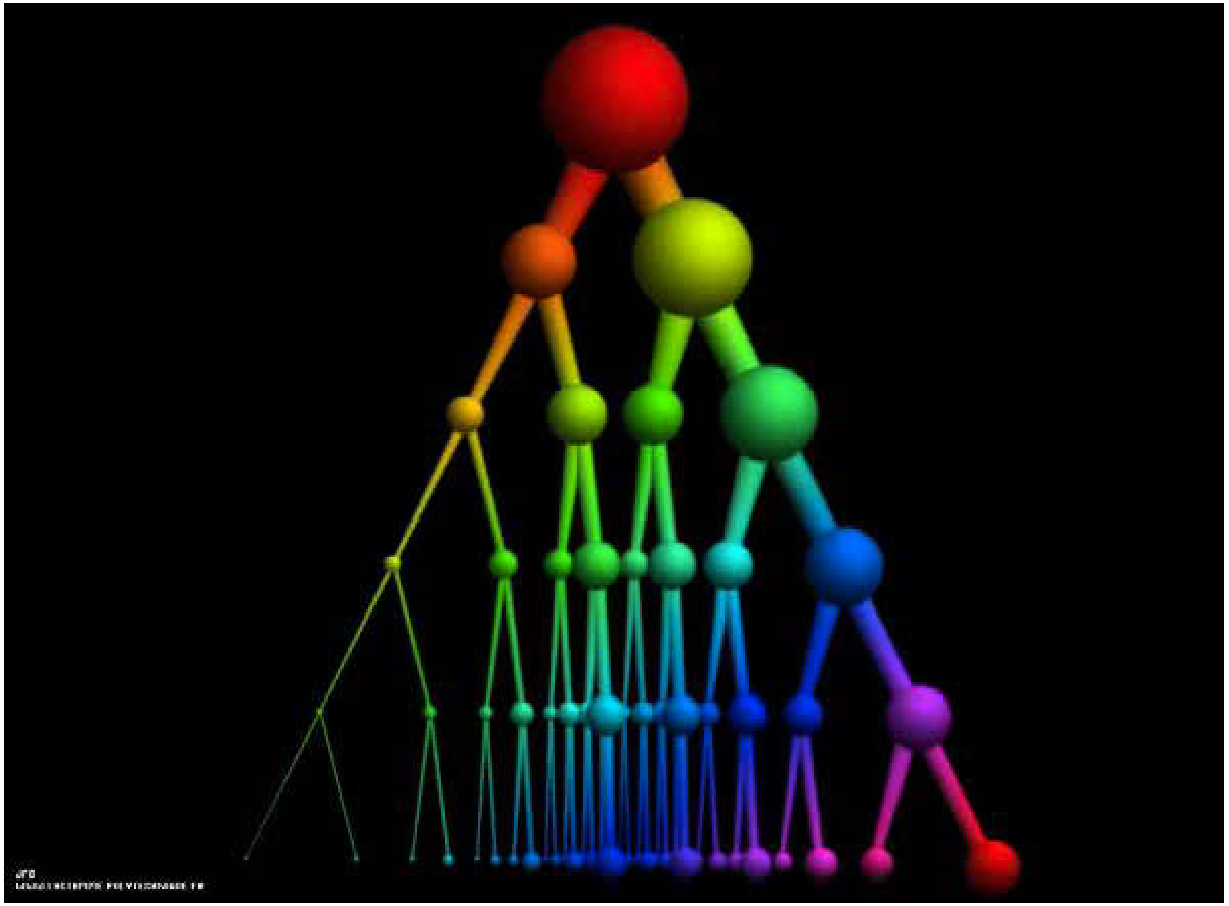


Wavelet leader scaling function

Modeling using Fractional Brownian Motion is reasonable

Data from http ://mawi.wide.ad.jp/mawi/

Cascade models : Binomial cascade

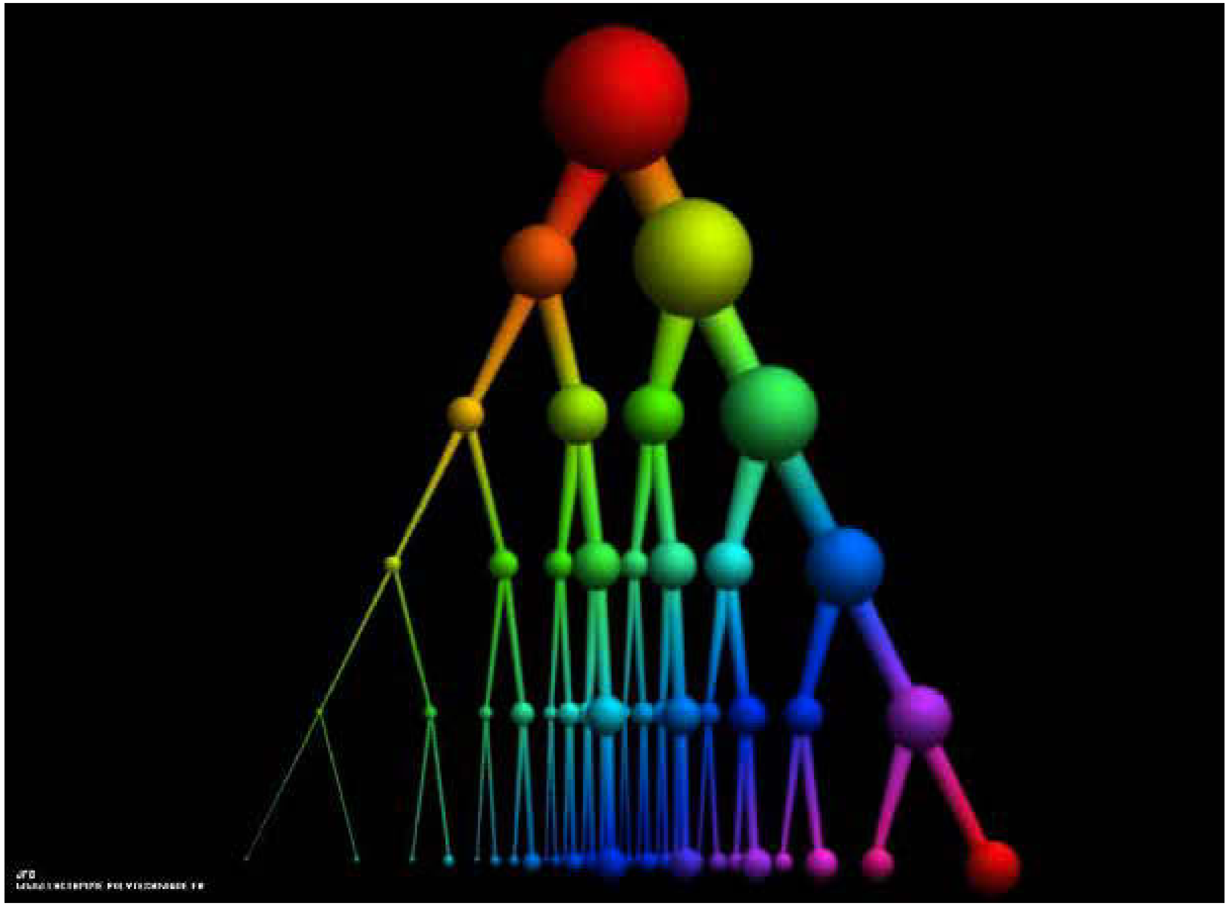


We obtain a measure μ on the interval $[0, 1]$:

$$\mu(I) = \text{ amount of sand that fell on } I$$

Courtesy of Jean-François Colonna, LACTAMME

Cascade models : Binomial cascade

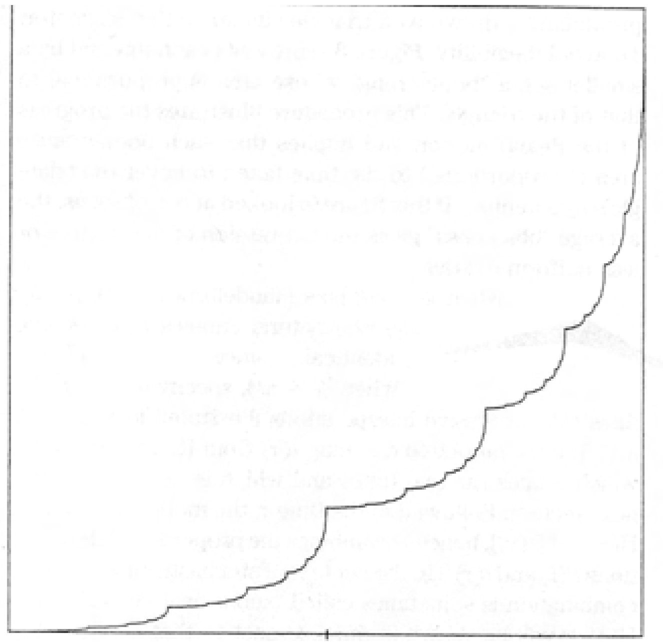


Intervals of length $\left(\frac{1}{2}\right)^n$:

far left : $\mu(I) = \left(\frac{1}{4}\right)^n = |I|^2$ far right : $\mu(I) = \left(\frac{3}{4}\right)^n = |I|^{\log(4/3)/\log 2}$

average : $\mu(I) = \left(\frac{1}{4}\right)^{n/2} \left(\frac{3}{4}\right)^{n/2} = |I|^{\log(4/\sqrt{3})/\log 2}$

Cascade models : Binomial cascade



Repartition function of the measure μ :
$f(x) = \mu([0, x]) =$ amount of sand in $[0, x]$

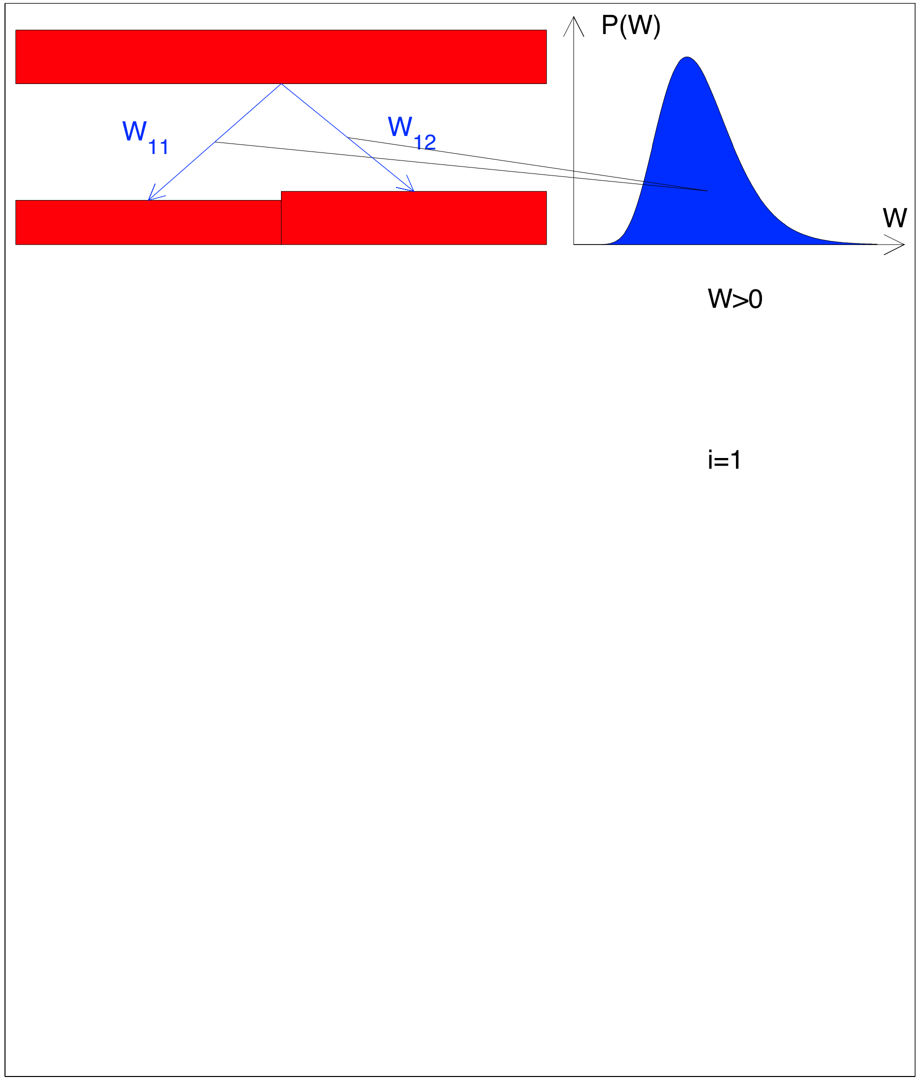
$$f(x+\delta) - f(x) = \mu([x, x+\delta]) \sim \delta^{h(x)}$$

$$h(x) \in \left[\frac{\log(4/3)}{\log 2}, 2\right]$$

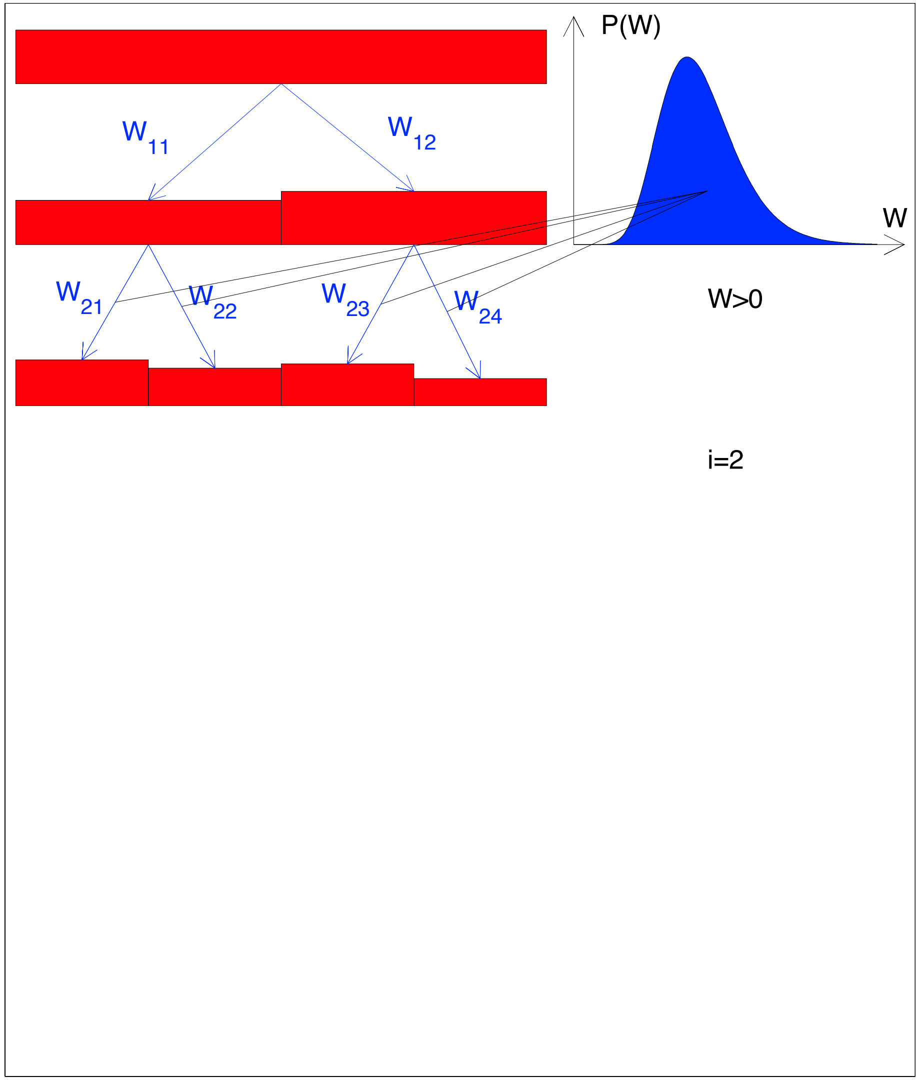
Spectrum of singularities of f :

$$d_f(H) = \dim(\{x : h(x) = H\})$$

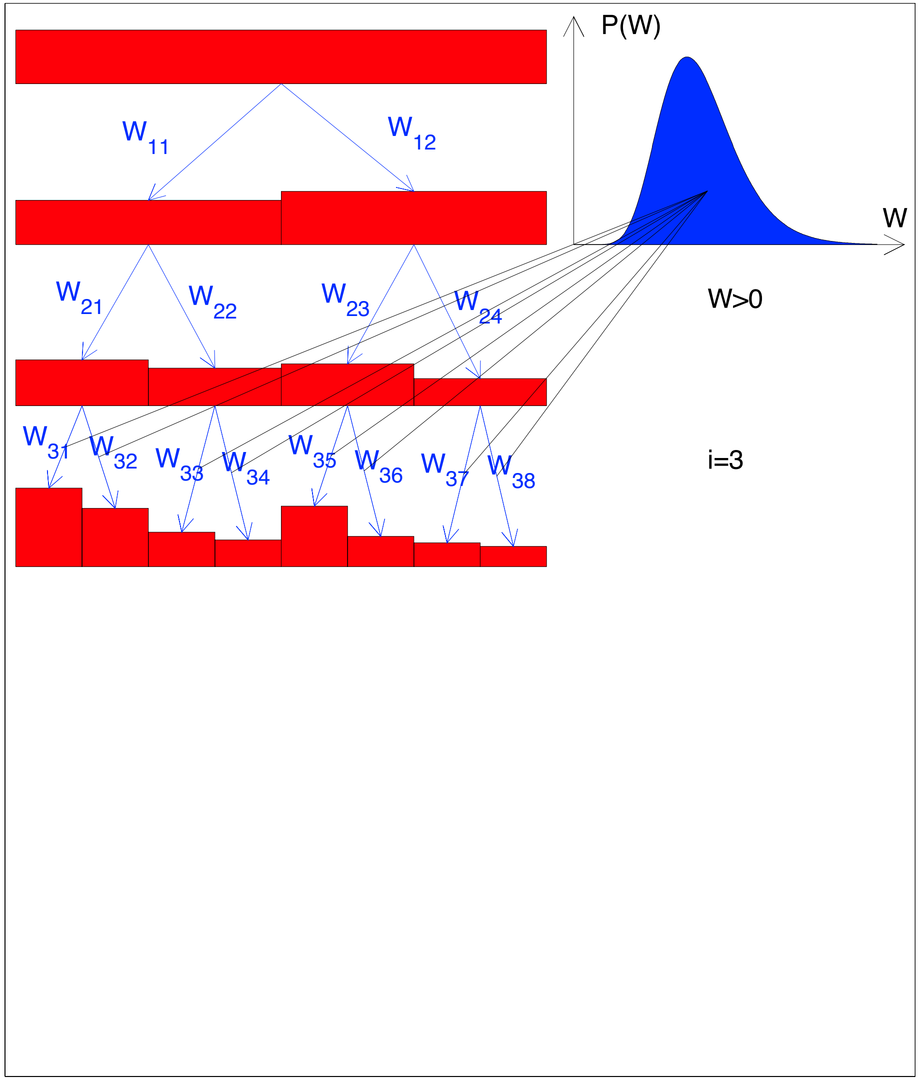
Multiplicative model : Mandelbrot Cascades



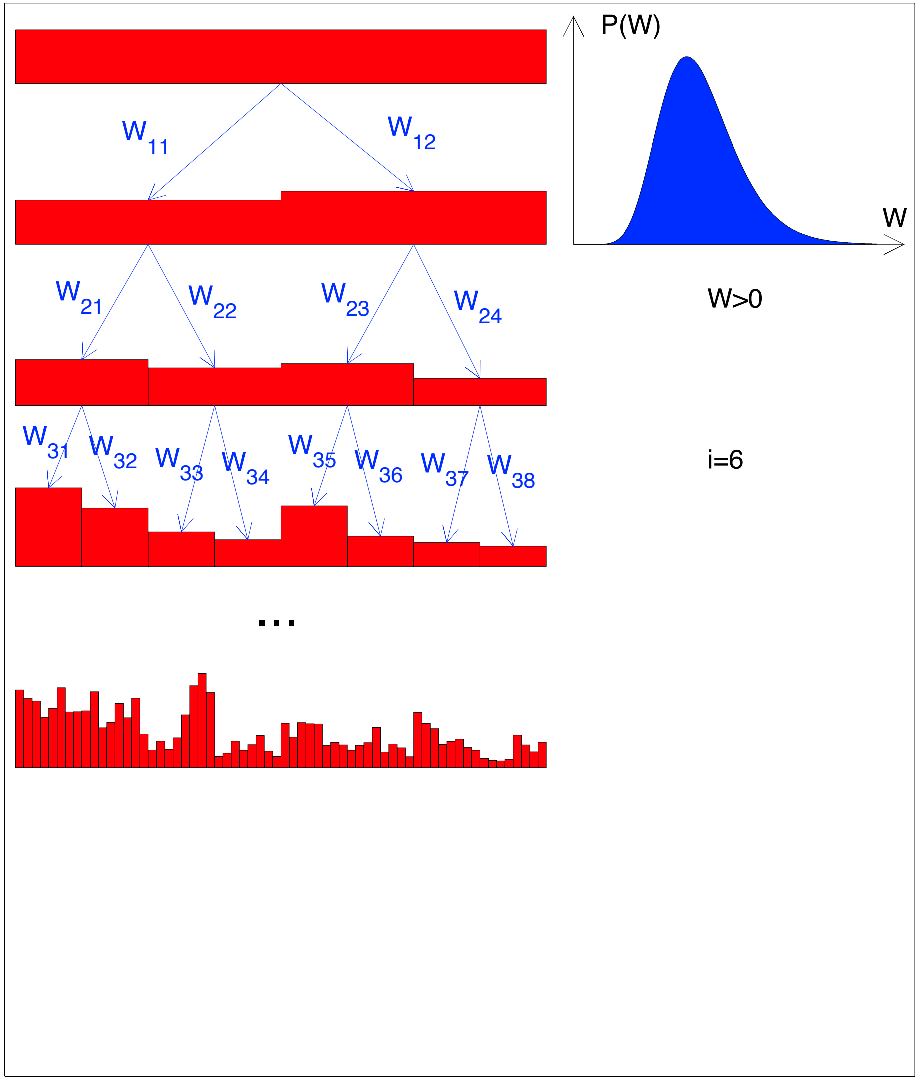
Multiplicative model : Mandelbrot Cascades



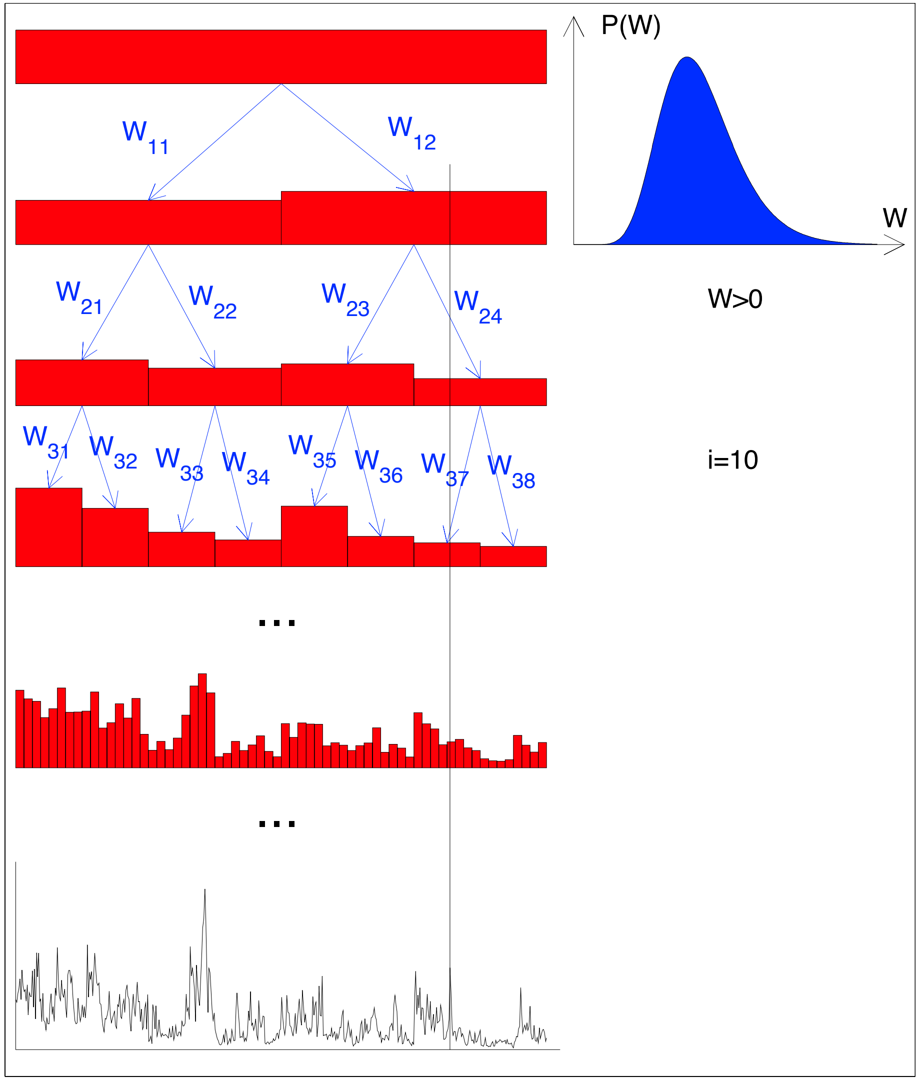
Multiplicative model : Mandelbrot Cascades



Multiplicative model : Mandelbrot Cascades



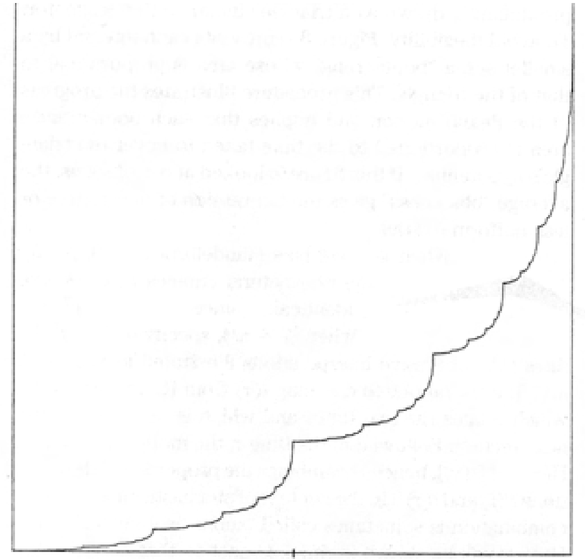
Multiplicative model : Mandelbrot Cascades



Multifractal models for signals

The repartition function of a probability measure is not appropriate in signal modeling. One can make it oscillate randomly by taking $f(t)$ for time change

$$X(t) = B_H(f(t))$$



Fractional Brownian motion in multifractal time

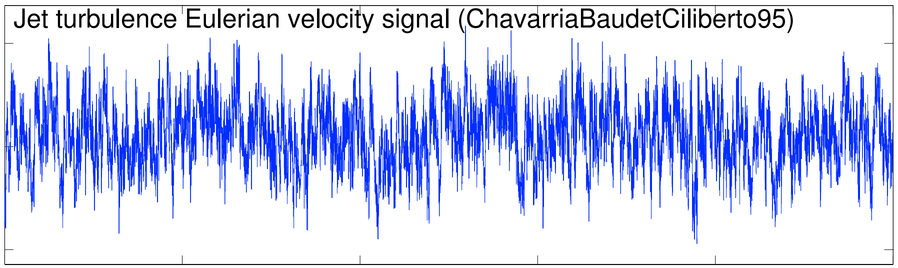
proposed by B. Mandelbrot for financial modeling

Alternatives :

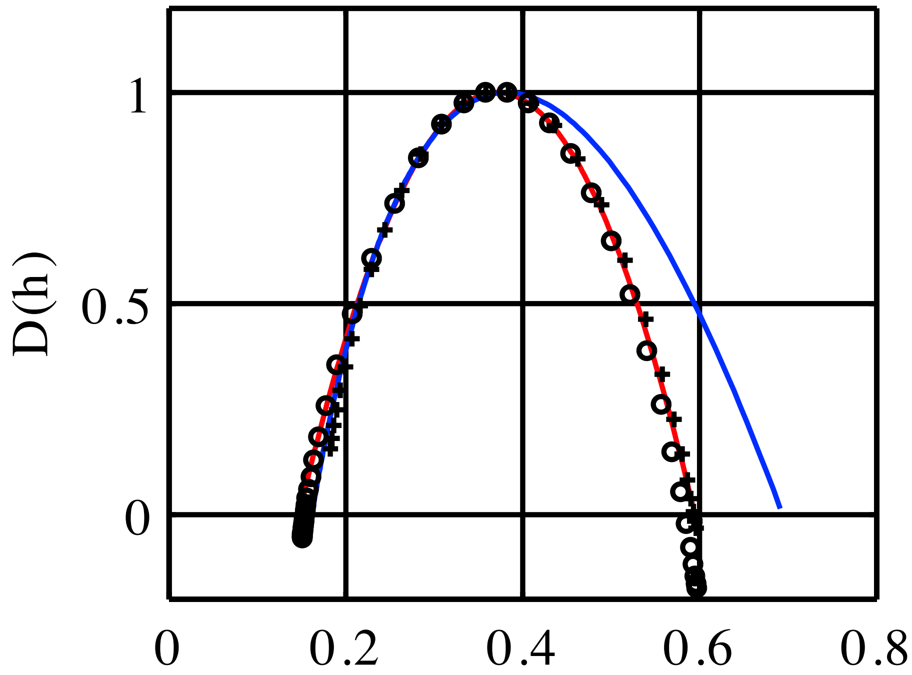
- Orthonormal wavelet series whose coefficients are derived from the measure : A. Arneodo, J. Barral and S. Seuret
- ...

Model refutation

(joint work with Bruno Lashermes)



Log-normal vs. She-Leveque model



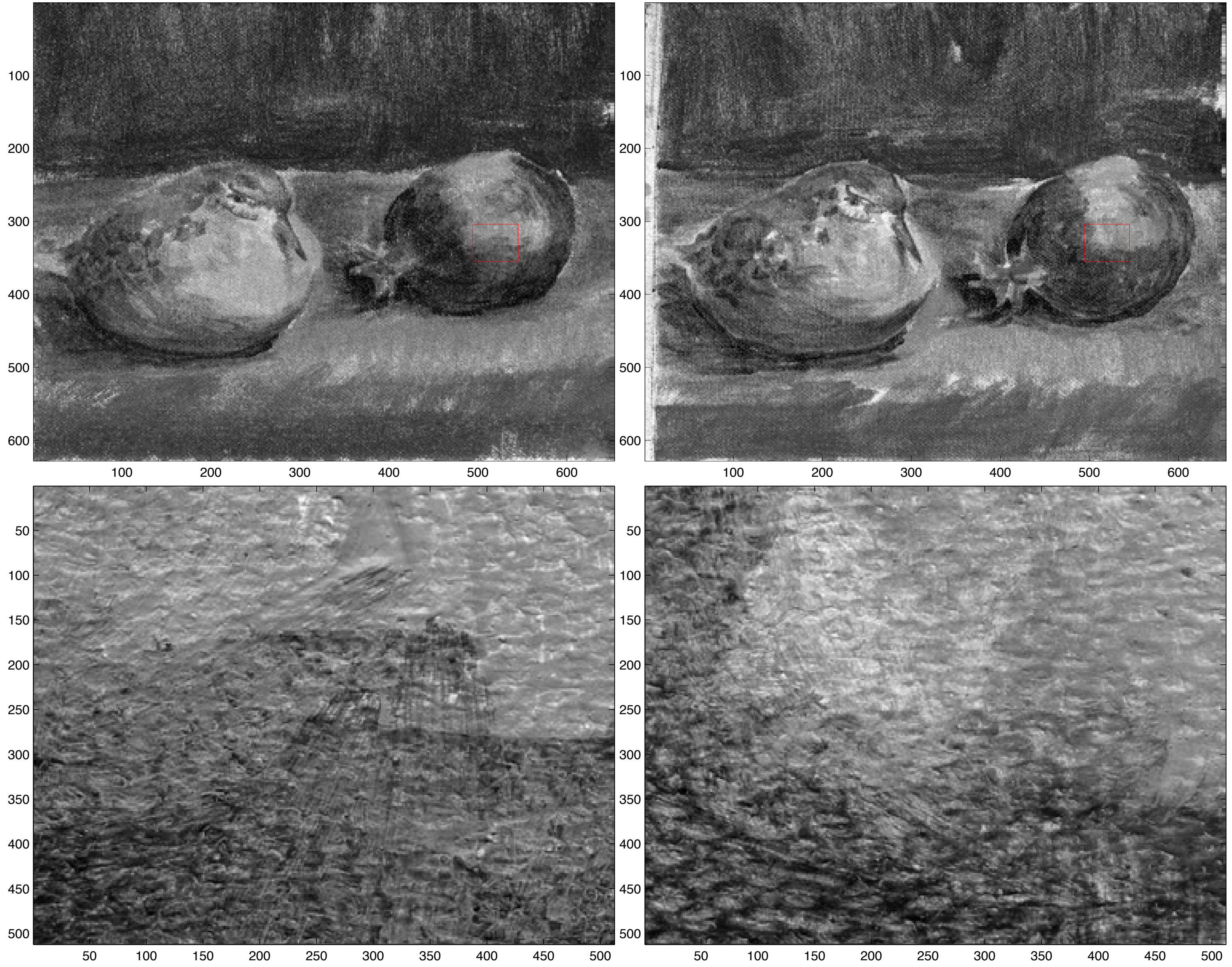
Princeton Experiment : Stylometry issues

- Experiment design :
  - same Painter (Charlotte Casper) does Original and Copies
  - a series of 7 small paintings,
  - different set of materials (various brushes, grounds, paints)
  - Original and Copies with same set of materials
  - high resolution digitalisation, under uniform conditions.
- Description :

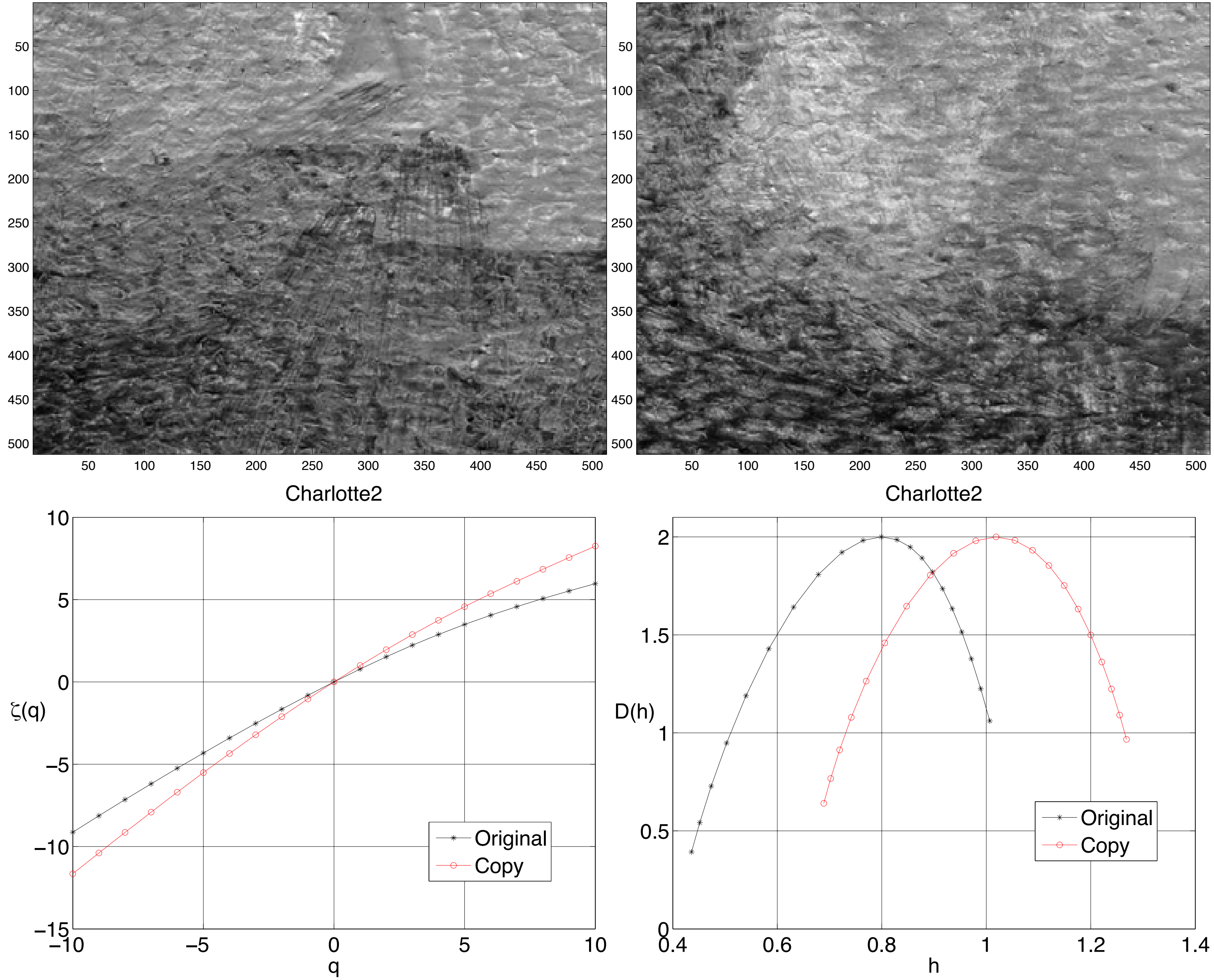
Pair	Ground	Paint	Brushes
1	CP Canvas	Oils	S& H
2	CP Canvas	Acrylics	S& H
3	Smooth CP Board	Oils	S& H
4	Bare linen canvas	Oils	S
5	Chalk and Glue	Oils	S
6	CP Canvas	Acrylics	S
7	Smooth CP Board	Oils	S

Charlotte2's Original & Copy

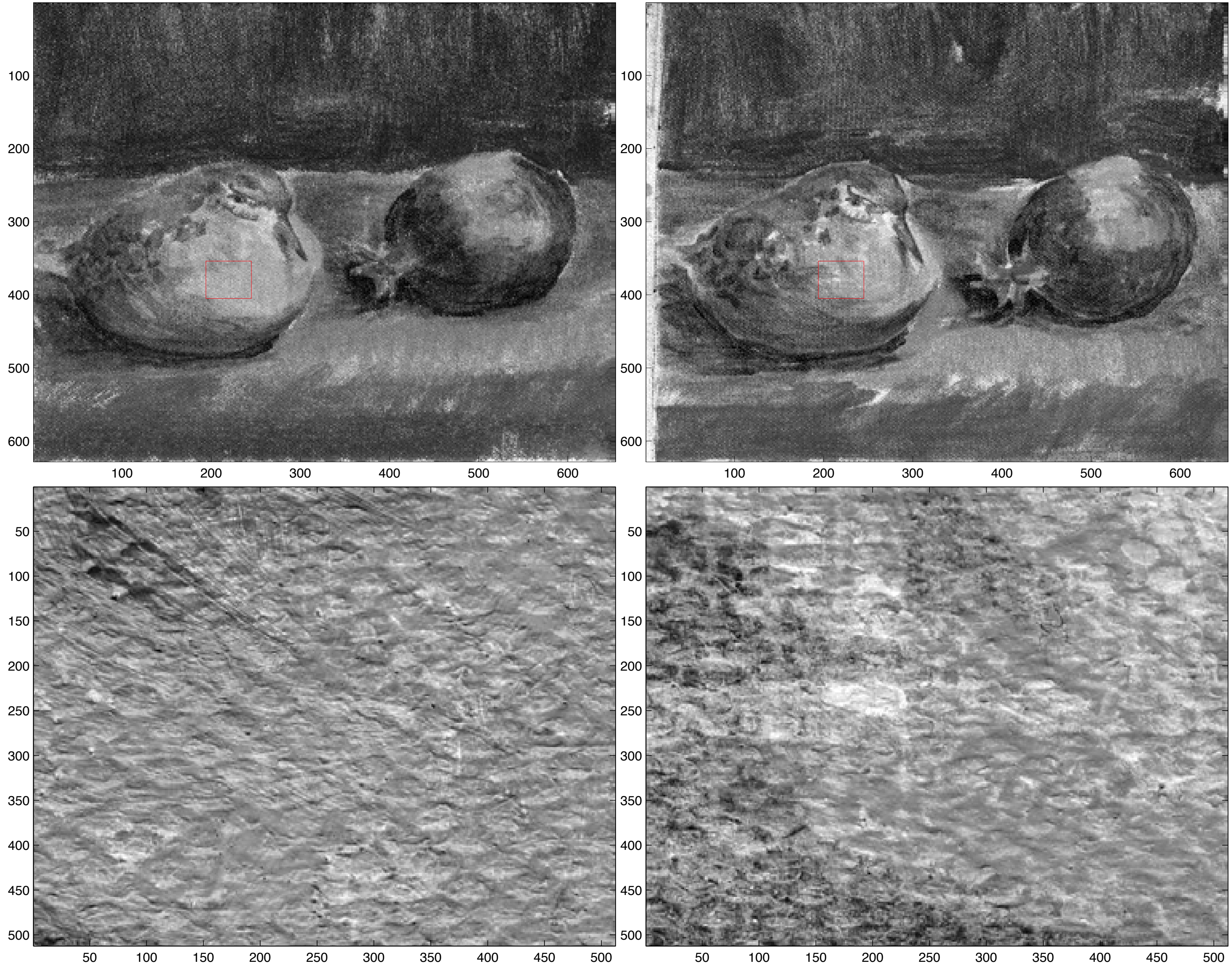
Charlotte2 MF



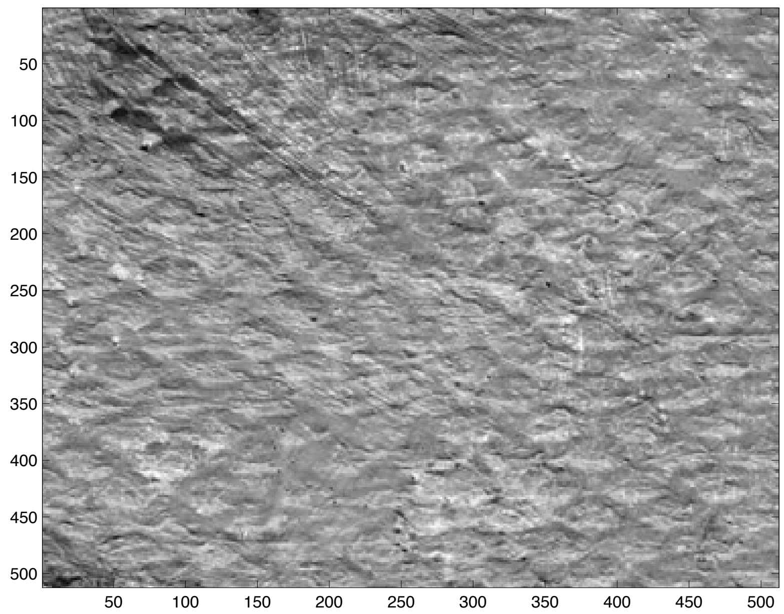
Charlotte2 MF

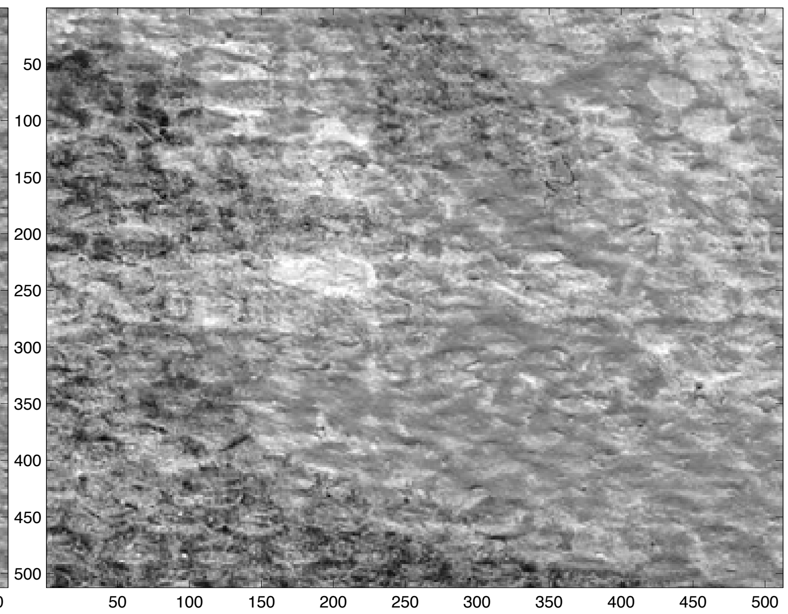


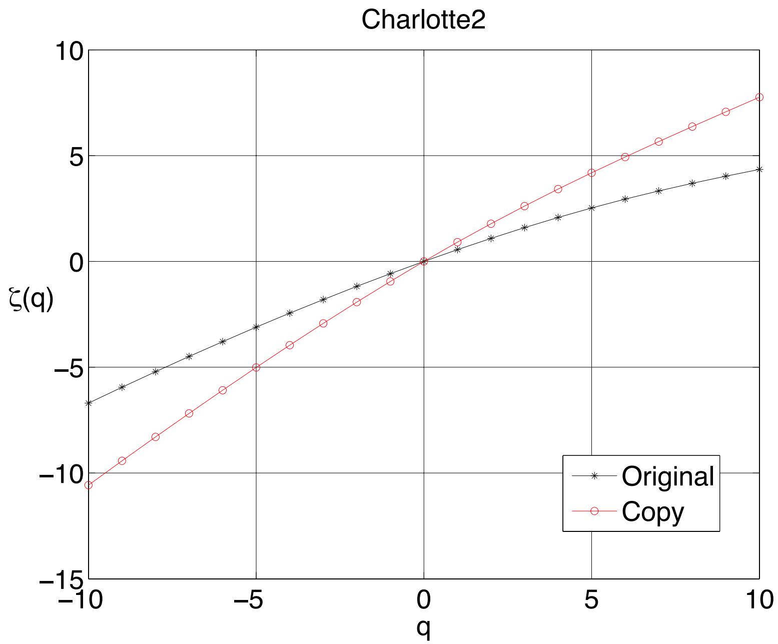
Charlotte2 MF

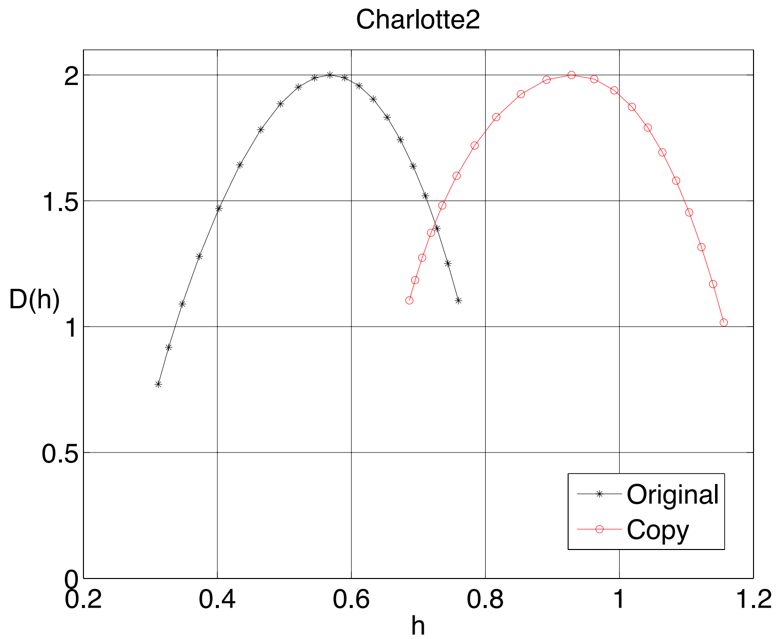


Charlotte2 MF

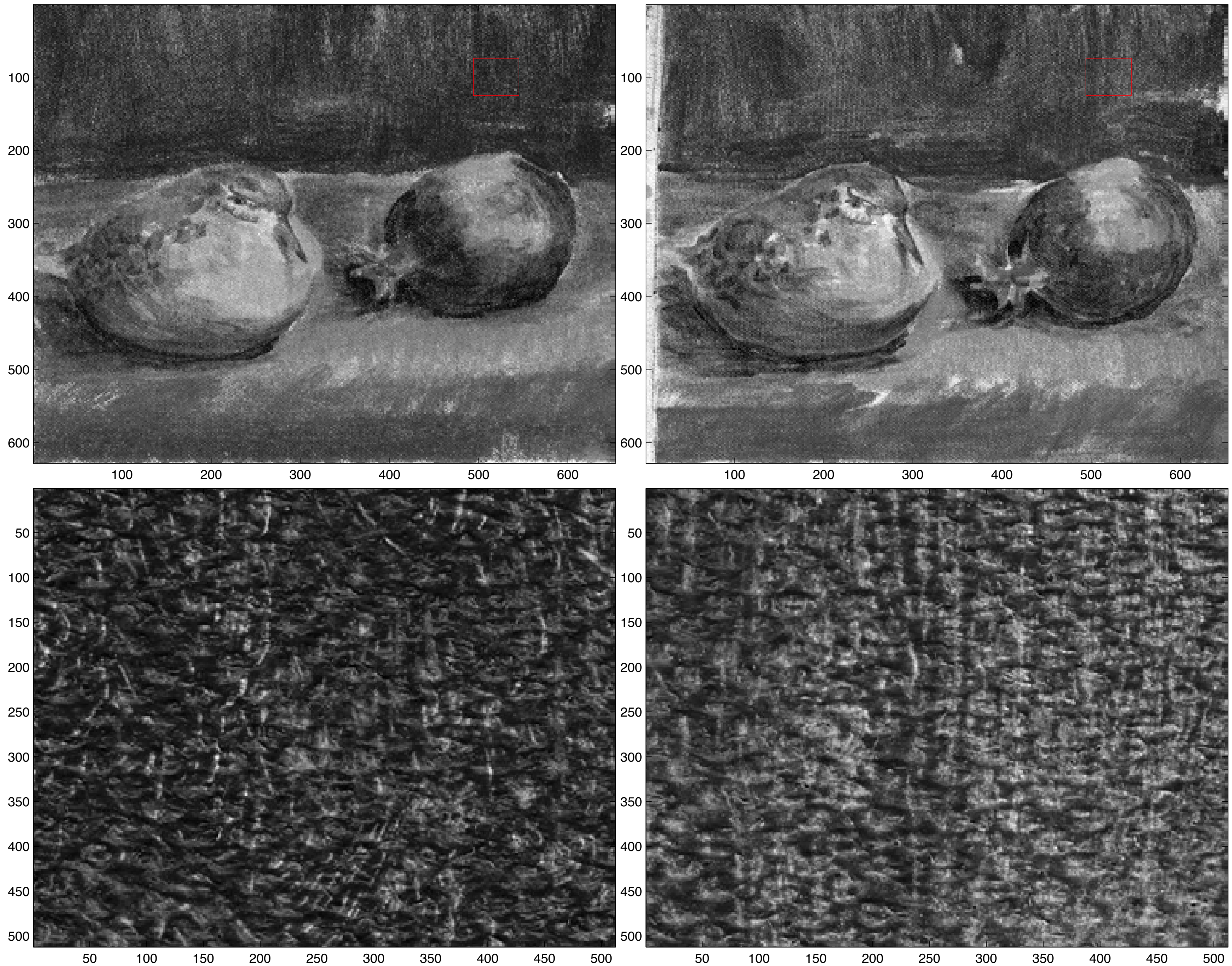




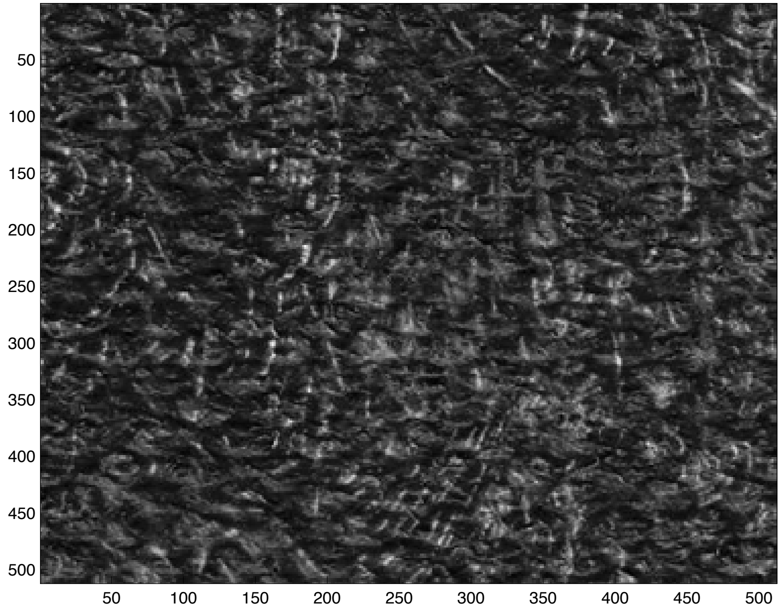


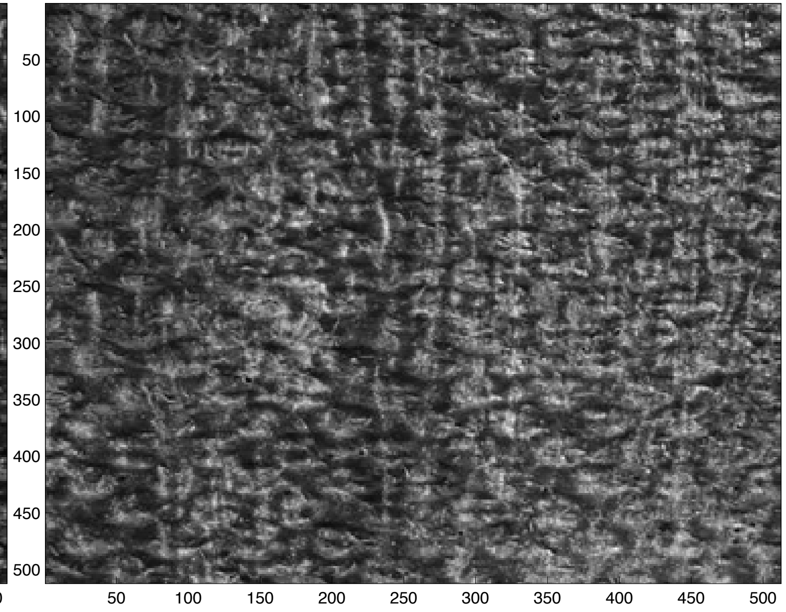


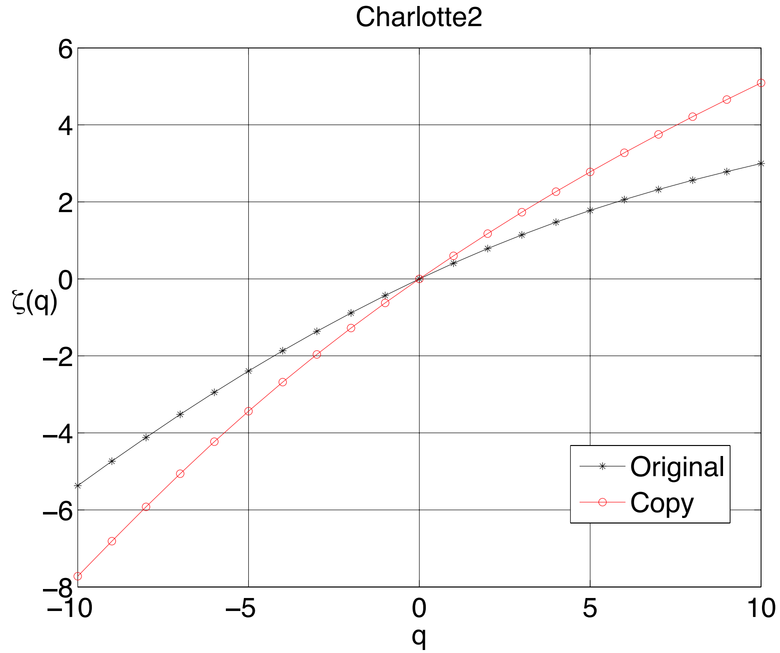
Charlotte2 MF

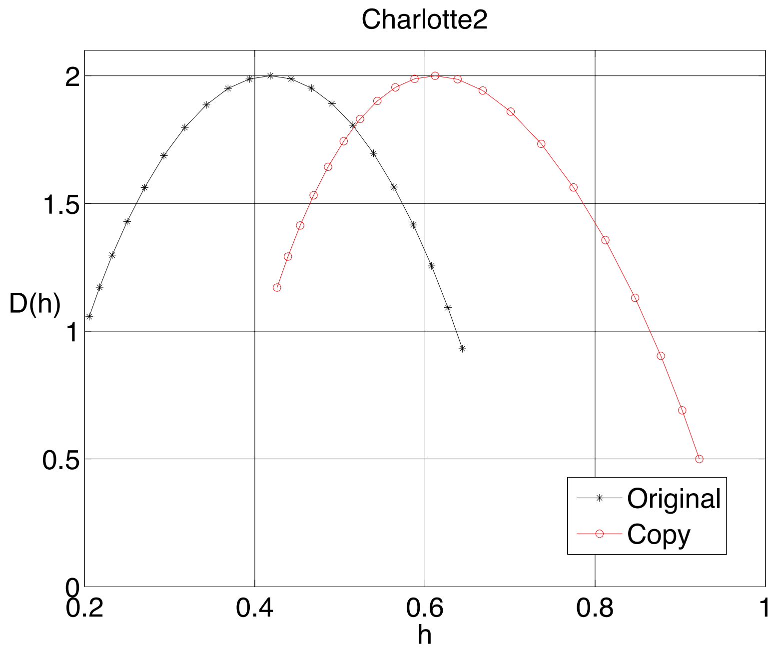


Charlotte2 MF

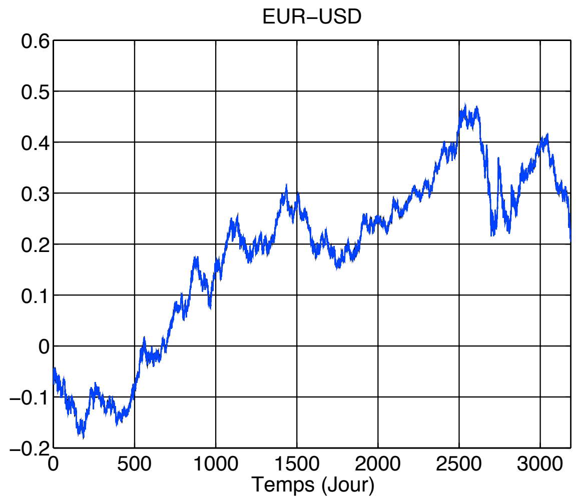




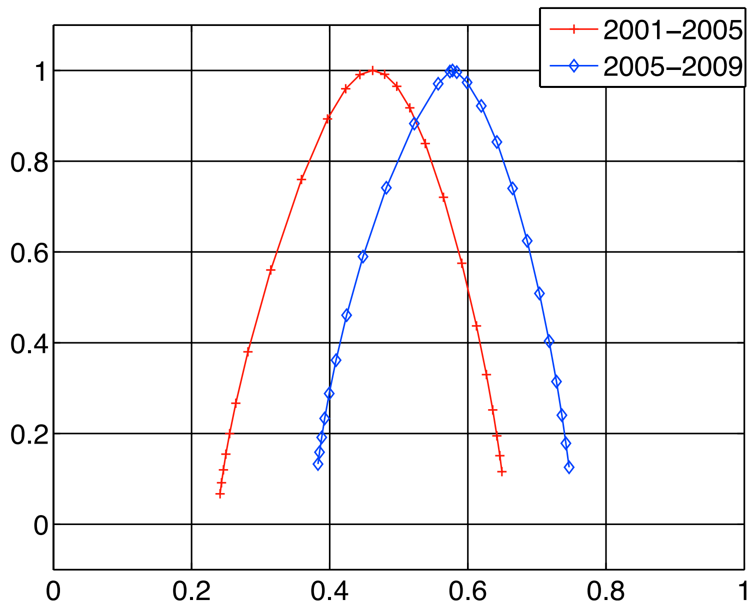




Multifractal analysis of the USD-Euro change rate



Legendre spectra



Uniform Hölder regularity

Hölder spaces : Let $\alpha \in (0,1)$; $f \in C^\alpha$ if

$$\exists C, \ \ \forall x, y : \qquad |f(x) - f(y)| \leq C \cdot |x - y|^\alpha$$

The uniform Hölder exponent of f is

$$H_f^{min} = \sup\{\alpha : \ f \in C^\alpha\}$$

Numerical computation

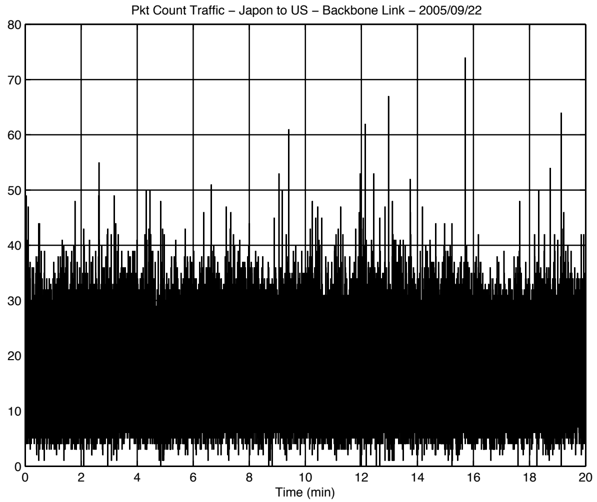
Let $\omega_j = \sup\limits_{\lambda \in \Lambda_j} |c_\lambda|$ then $H_f^{min} = \liminf\limits_{j \to +\infty} \dfrac{\log(\omega_j)}{\log(2^{-j})}$

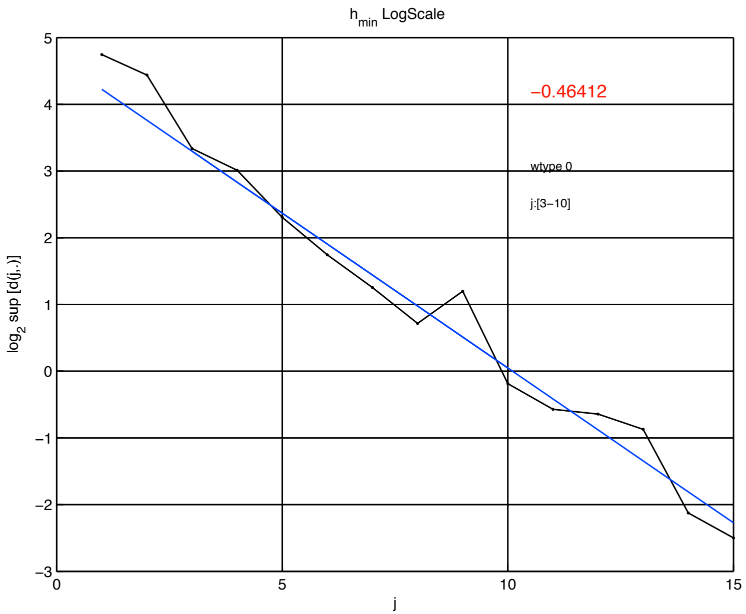
$$H_f^{min} > 0 \quad \Longrightarrow \quad f \text{ is continuous}$$

$$H_f^{min} < 0 \quad \Longrightarrow \quad f \text{ is not locally bounded}$$

Is $H^{min} > 0$ fulfilled in applications ?

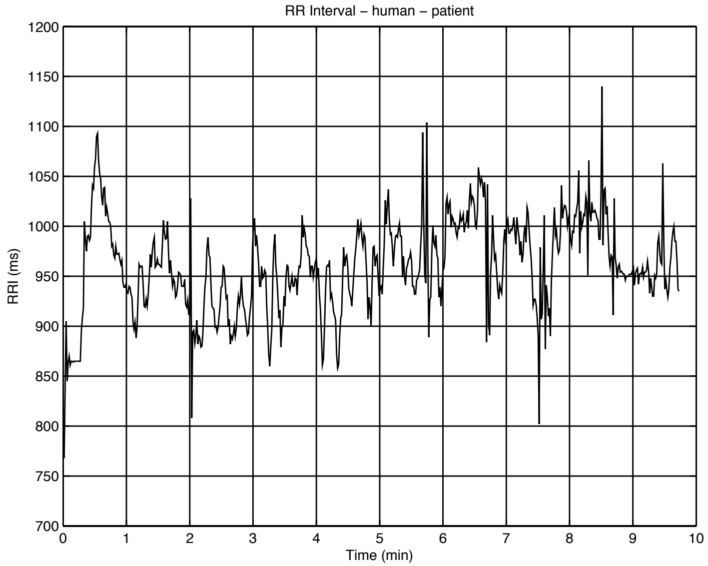
Internet Traffic

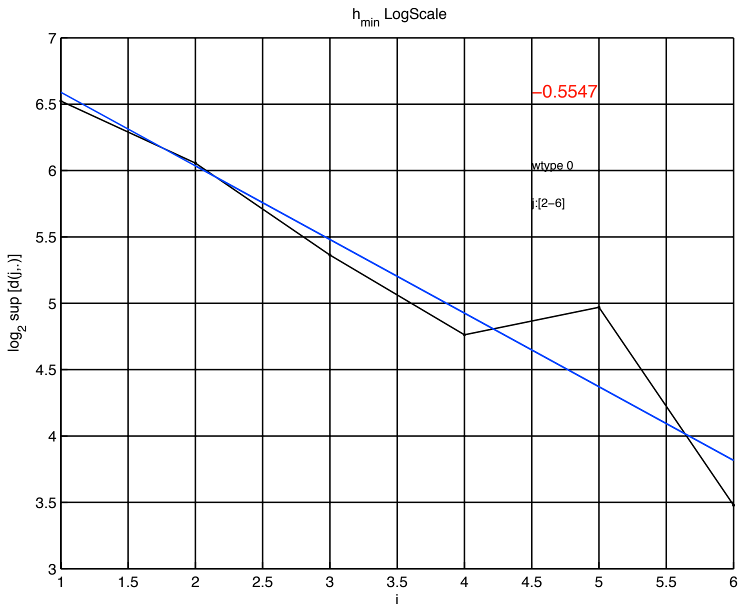




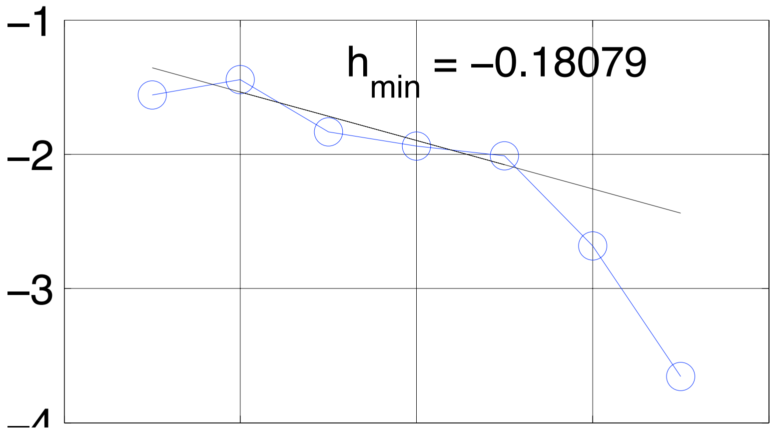
$$H_{min} = -0.46$$

Heartbeat Intervals

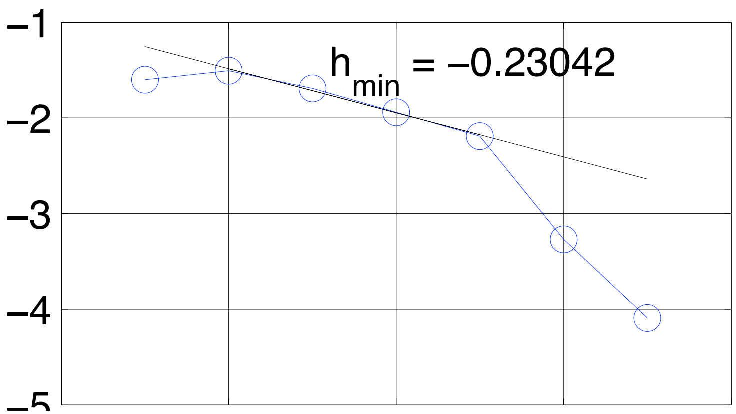




$$H_{min} = -0.55$$



−1
−2
−3
−4
h min = −0.18079



−1
−2
−3
−4
−5
h_min = −0.23042

Pointwise regularity with negative exponents ?

Pointwise Hölder regularity : $f \in C^\alpha(x_0)$ if

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

If $\alpha < 0$, this definition implies that, outside of x_0, f is locally bounded

$\Longrightarrow$ it could be used only to define isolated singularities of negative exponent

Which definition of pointwise regularity would allow for negative exponents ?

Clue : The definition of pointwise Hölder regularity can be rewritten

$$f \in C^\alpha(x_0) \iff \sup_{B(x_0,r)} |f(x) - P(x - x_0)| \leq Cr^\alpha$$

Definition (Calderón and Zygmund) : Let $f \in L^p(\mathbb{R}^d)$; $f \in T^p_\alpha(x_0)$ if there exists a polynomial P such that for r small enough,

$$\left(\frac{1}{r^d} \int_{B(x_0,r)} |f(x) - P(x - x_0)|^p dx \right)^{1/p} \leq Cr^\alpha$$

The p-exponent

Definition : Let $f \in L^p(\mathbb{R}^d)$; $f \in T^p_\alpha(x_0)$ if there exists a polynomial P such that for r small enough,

$$\left(\frac{1}{r^d} \int_{B(x_0,r)} |f(x) - P(x - x_0)|^p dx \right)^{1/p} \leq C r^\alpha$$

The p-exponent of f at x_0 is $h_p(x_0) = \sup\{\alpha : f \in T^p_\alpha(x_0)\}$

The p-spectrum of f is $d^p(H) = dim\ (\{x_0 : \quad h_p(x_0) = H\})$

Remarks :

- The case $p = +\infty$ corresponds to pointwise Hölder regularity
- The normalization is chosen so that a cusp $|x - x_0|^\alpha$ has the same p-exponent for all p : $h_p(x_0) = \alpha$ (as long as $\alpha \geq -d/p$)

How can one check that the data belong to L^p ?

The wavelet scaling function is informally defined by

$$\forall p > 0 \qquad 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p) j}$$

- If $\zeta_f(p) > 0$, then $f \in L^p$
- If $\zeta_f(p) < 0$, then $f \notin L^p$

Properties of p-exponents

Gives a mathematical framework to the notion of negative regularity exponents

The p-exponent satisfies : $h_p(x_0) \geq -\frac{d}{p}$

p-exponents may differ :

Theorem : Let f be an L^1 function, and $x_0 \in \mathbb{R}^d$. Let

$$p_0 = \sup\{p : f \in L^p_{loc}(\mathbb{R}^d) \text{ in a neighborhood of } x_0\}$$

The function $p \to h_p(x_0)$ is defined on $[1, p_0)$ and possesses the following properties :

1. It takes values in $[-d/p, \infty]$
2. It is a decreasing function of p.
3. The function $r \to h_{1/r}(x_0)$ is concave.

Furthermore, Conditions 1 to 3 are optimal.

When do p-exponents coincide ?

Notation : $h_{p,\gamma}(x_0)$ denotes the p-exponent of the fractional integral of f of order s at x_0

Definition : f is a cusp of exponent h at x_0 if $\exists p, \gamma > 0$ such that

- $h_p(x_0) = h$
- $h_{p,\gamma}(x_0) = h + \gamma$

Theorem : This notion is independent of p and γ

Types of pointwise singularities (Yves Meyer)

Typical pointwise singularities :

Cusps : $f(x) - f(x_0) = |x - x_0|^H$

After one integration :

$f^{(-1)}(x) - f^{(-1)}(x_0) \sim \frac{1}{H}|x - x_0|^{H+1}$

Oscillating singularity : $f(x) - f(x_0) = |x - x_0|^H \sin\left(\frac{1}{|x - x_0|^\beta}\right)$

After one integration :

$$f^{(-1)}(x) - f^{(-1)}(x_0) = \frac{|x - x_0|^{H+(1+\beta)}}{\beta} \cos\left(\frac{1}{|x - x_0|^\beta}\right) + \cdots$$

More generally, after a fractional integration of order s,

- If f has a cusp at x_0, then $h_{\mathcal{I}^s f}(x_0) = h_f(x_0) + s$
- If f has an oscillating singularity at x_0, then $h_{\mathcal{I}^s f}(x_0) = h_f(x_0) + (1 + \beta)s$

Further classification

Two types of oscillating singularities :

"Full singularities" : $|x - x_0|^H \sin\left(\frac{1}{|x - x_0|^\beta}\right)$ (p-exponents coincide)

"Skinny singularities " $|x - x_0|^H 1_{E_\gamma}$ where

$$E_\gamma = \bigcup \left[\frac{1}{n}, \frac{1}{n} + \frac{1}{n^\gamma}\right] \quad \text{for } \gamma > 1$$

(p-exponents differ)

Characterization

oscillating singularities : $F(x) = |x - x_0|^H g\left(\frac{1}{|x - x_0|^\beta}\right) + r(x)$

where g is indefinitely oscillating

For "full singularities" g is "large at infinity"

For "skinny singularities " g is "small at infinity"

Wavelet derivation of the p-exponent

Definition : Let $f : \mathbb{R}^d \to \mathbb{R}$ be locally in L^p ; the p-leaders of f are

$$d_\lambda^p = \left(\sum_{\lambda' \subset 3\lambda} |c_{\lambda'}|^p 2^{d(j-j')} \right)^{1/p}$$

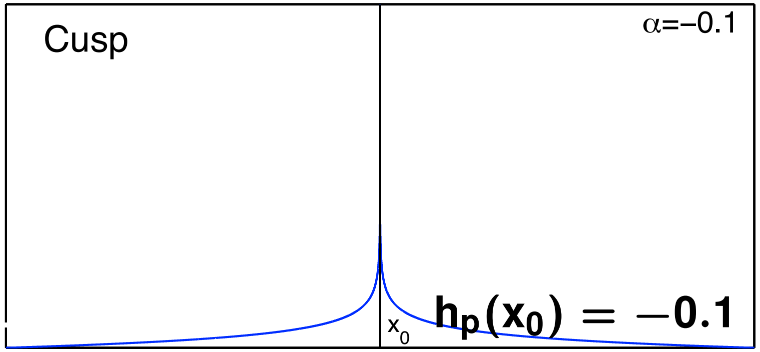
where j' is the scale associated with the subcube λ' included in 3λ (i.e. λ' has width $2^{-j'}$).

Theorem : (C. Melot)
If $\eta_f(p) > 0$, then

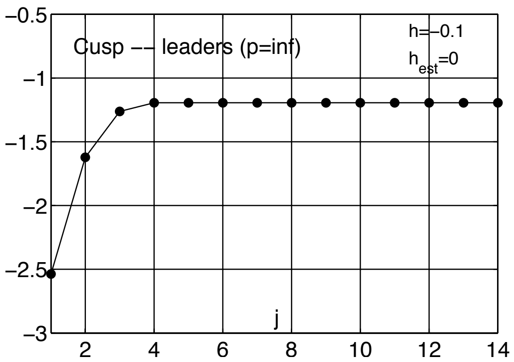
$$\forall x_0 \in \mathbb{R}^d \qquad h_f p(x_0) = \liminf_{j \to +\infty} \frac{\log(d^p_{\lambda_j(x_0)})}{\log(2^{-j})}.$$

$$h_p(x_0) = \liminf_{j \to +\infty} \left(\frac{\log\left(d^p_{\lambda_j(x_0)}\right)}{\log(2^{-j})} \right)$$

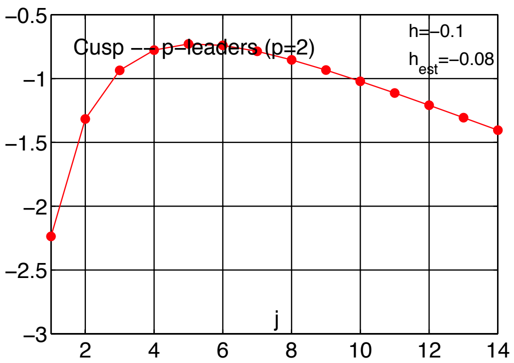
p-leaders and negative regularity

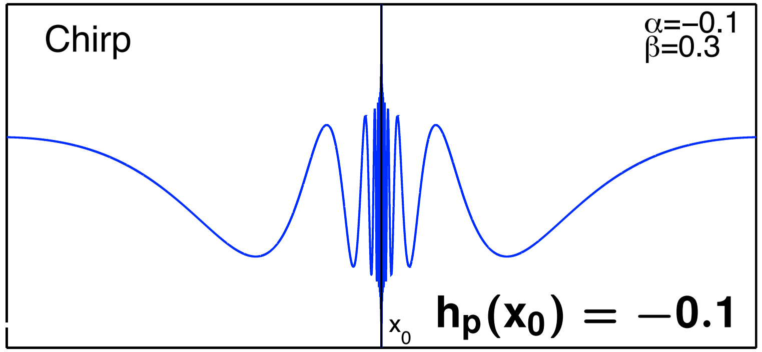


leaders : $\hat{h}(x_0) = \mathbf{0.00}$

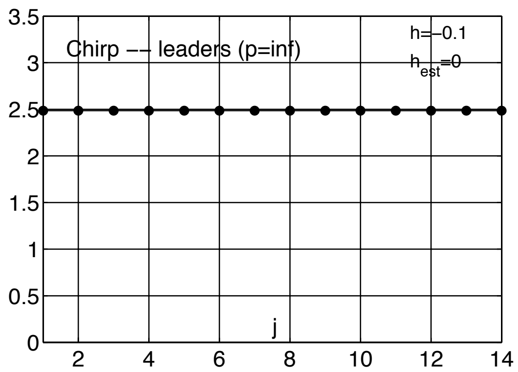


p-leaders : $\hat{h}(x_0) = \mathbf{-0.08}$

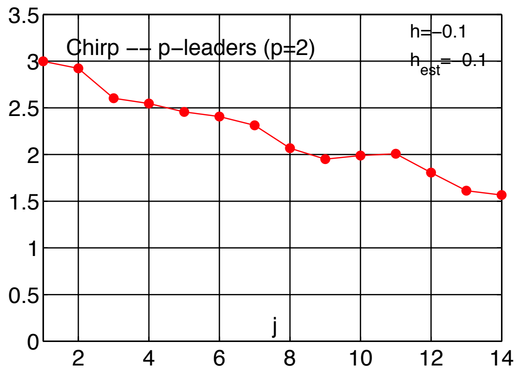




leaders : $\hat{h}(x_0) = \mathbf{0.00}$



p-leaders : $\hat{h}(x_0) = \mathbf{-0.10}$



p-Multifractal Formalism

The p-scaling function is defined informally by

$$\forall q \in \mathbb{R}, \qquad 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda^p|^q \sim 2^{-\eta_p(q) j}$$

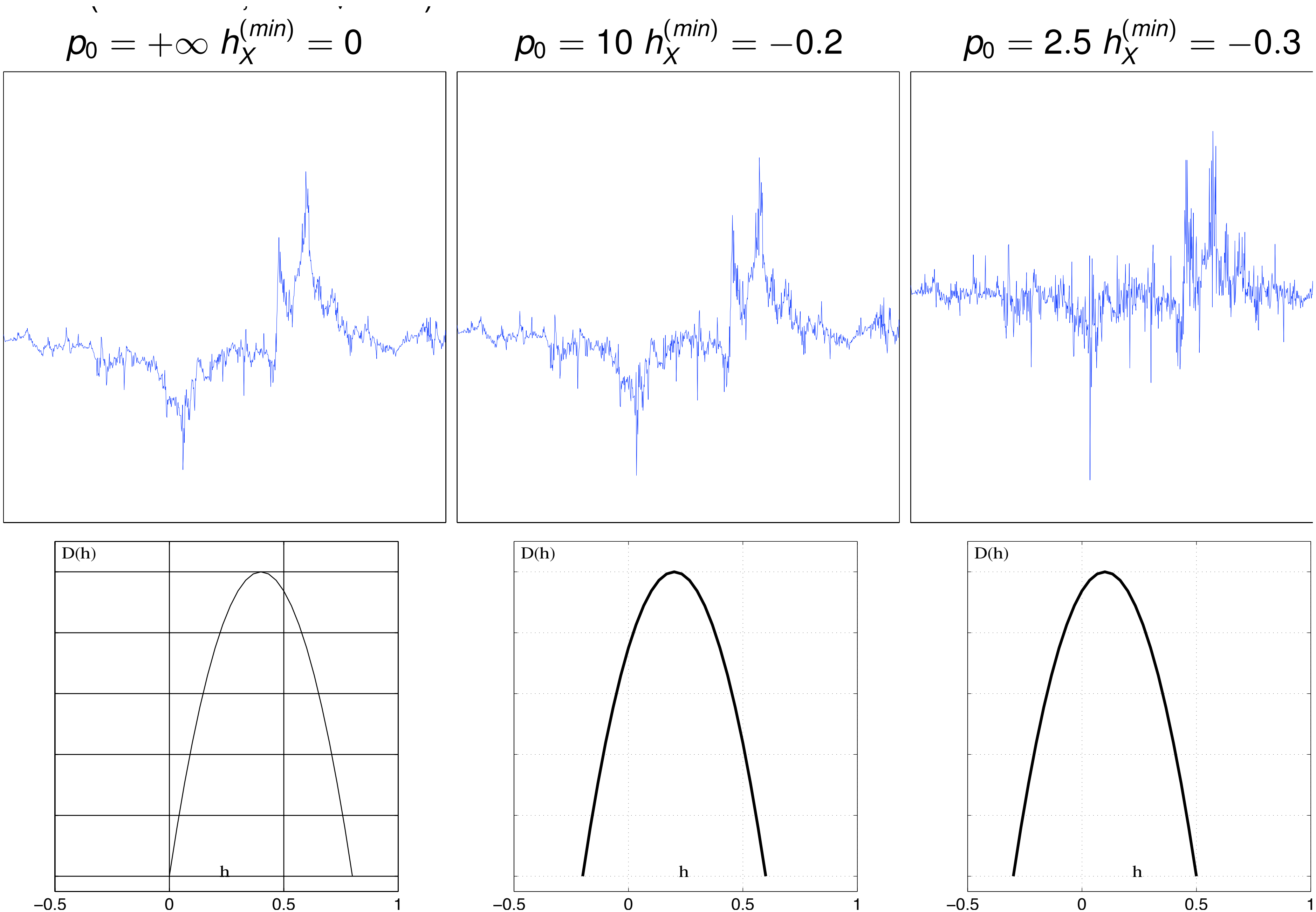
$$\eta_p(q) = \liminf_{j \to +\infty} \frac{\log \left(2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda^p|^q \right)}{\log(2^{-j})}$$

Stability properties :

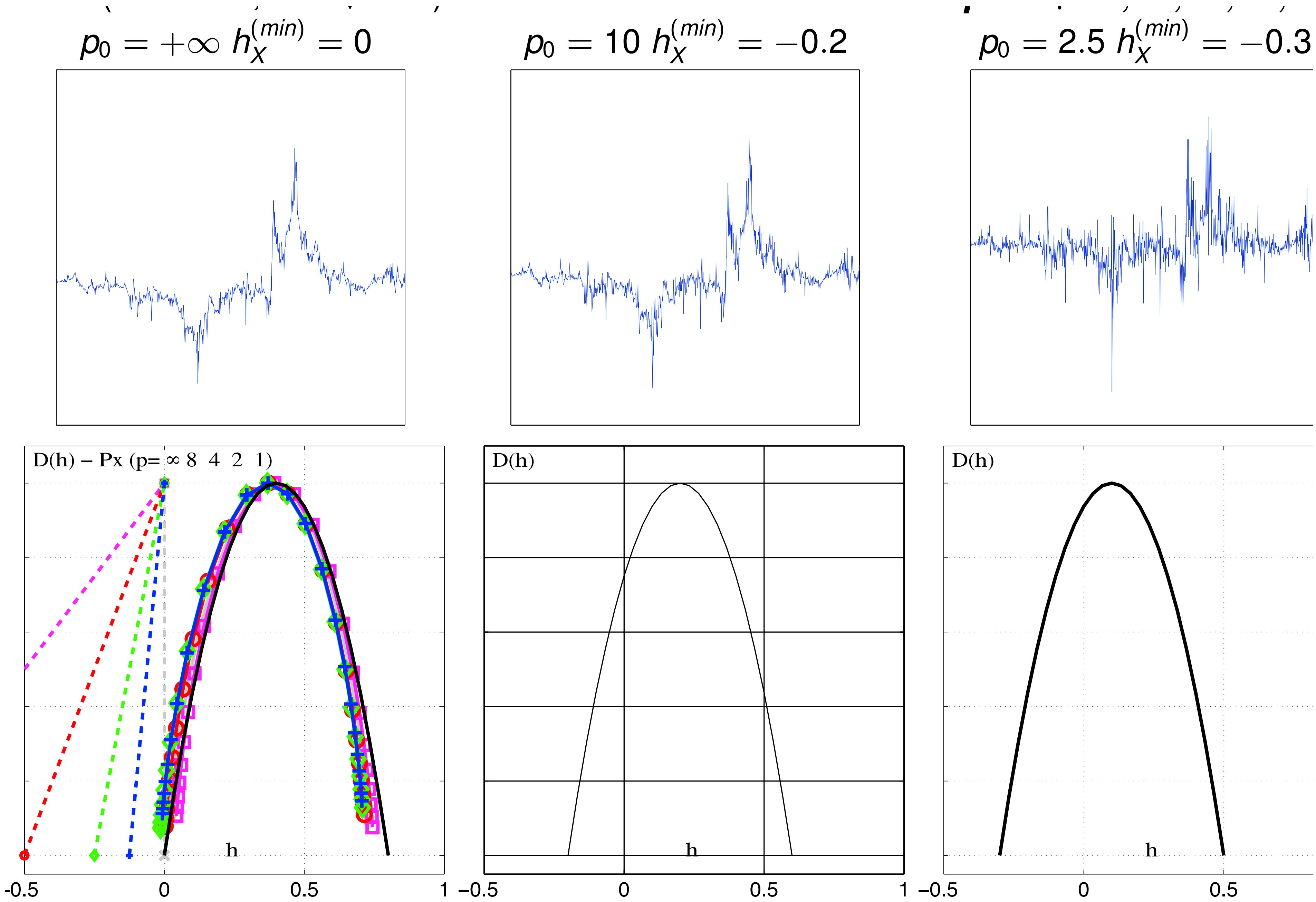
- Invariant with respect to deformations
- independent of the wavelet basis

The p-Legendre Spectrum is

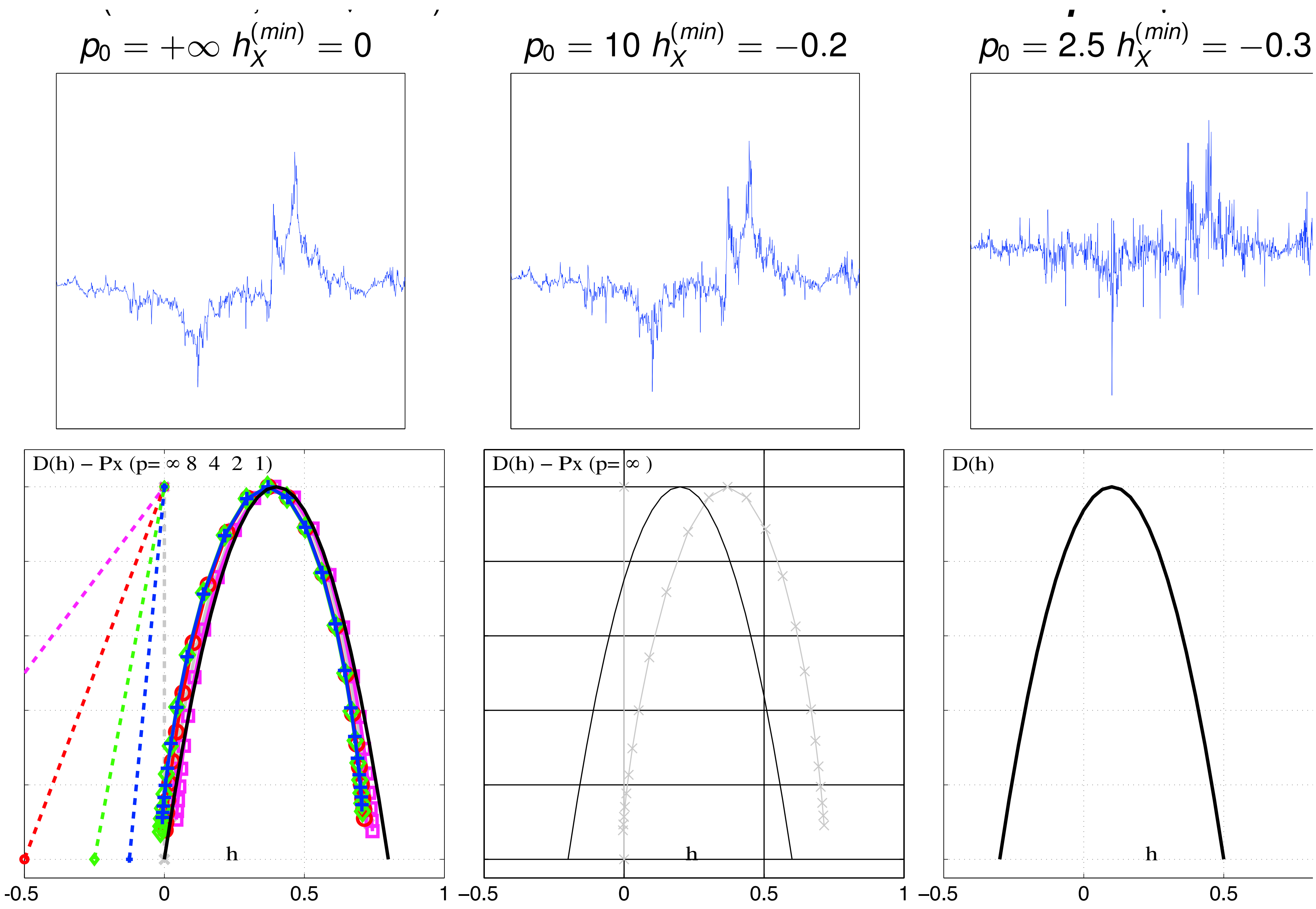
$$L_p(H) = \inf_{q \in \mathbb{R}} \left(d + Hq - \eta_p(q) \right)$$



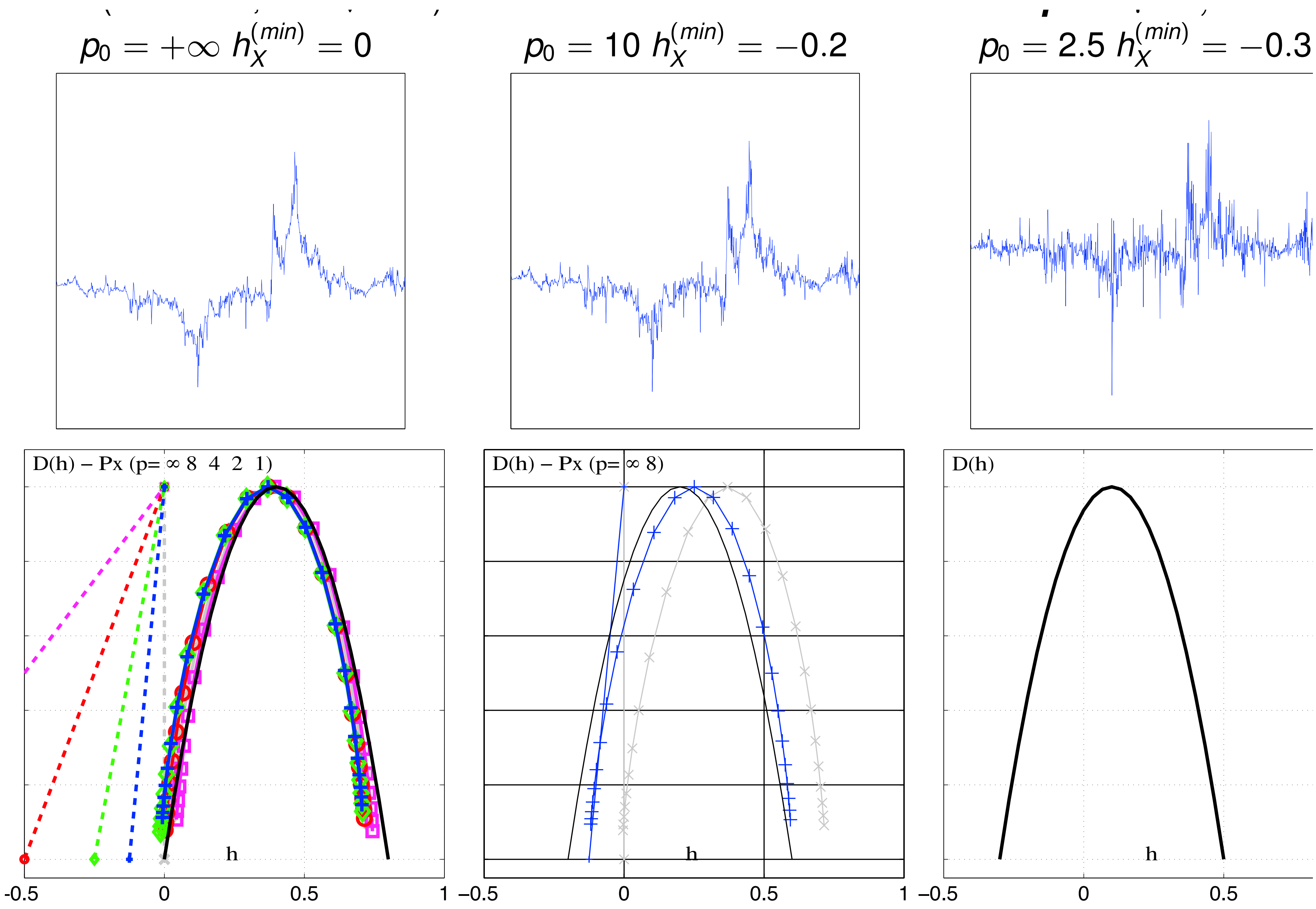
$p_0 = +\infty \; h_X^{(min)} = 0$
$p_0 = 10 \; h_X^{(min)} = -0.2$
$p_0 = 2.5 \; h_X^{(min)} = -0.3$
D(h)
h
–0.5
0
0.5
1



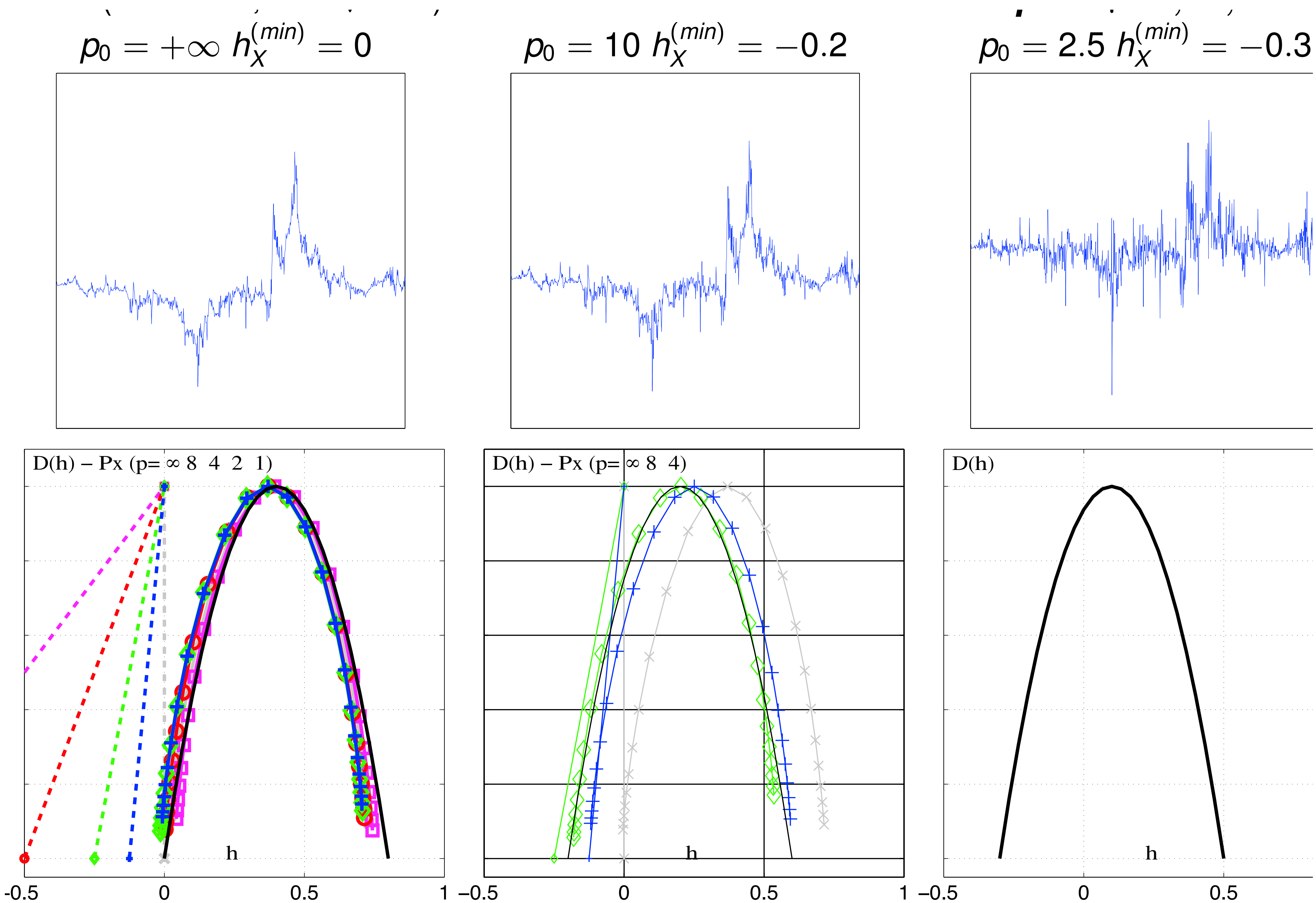
$p_0 = +\infty \; h_X^{(min)} = 0$
$p_0 = 10 \; h_X^{(min)} = -0.2$
$p_0 = 2.5 \; h_X^{(min)} = -0.3$
D(h) – Px (p= ∞ 8 4 2 1)
D(h)
D(h)
h
h
h
-0.5
0
0.5
1
−0.5
0
0.5
1
−0.5
0
0.5



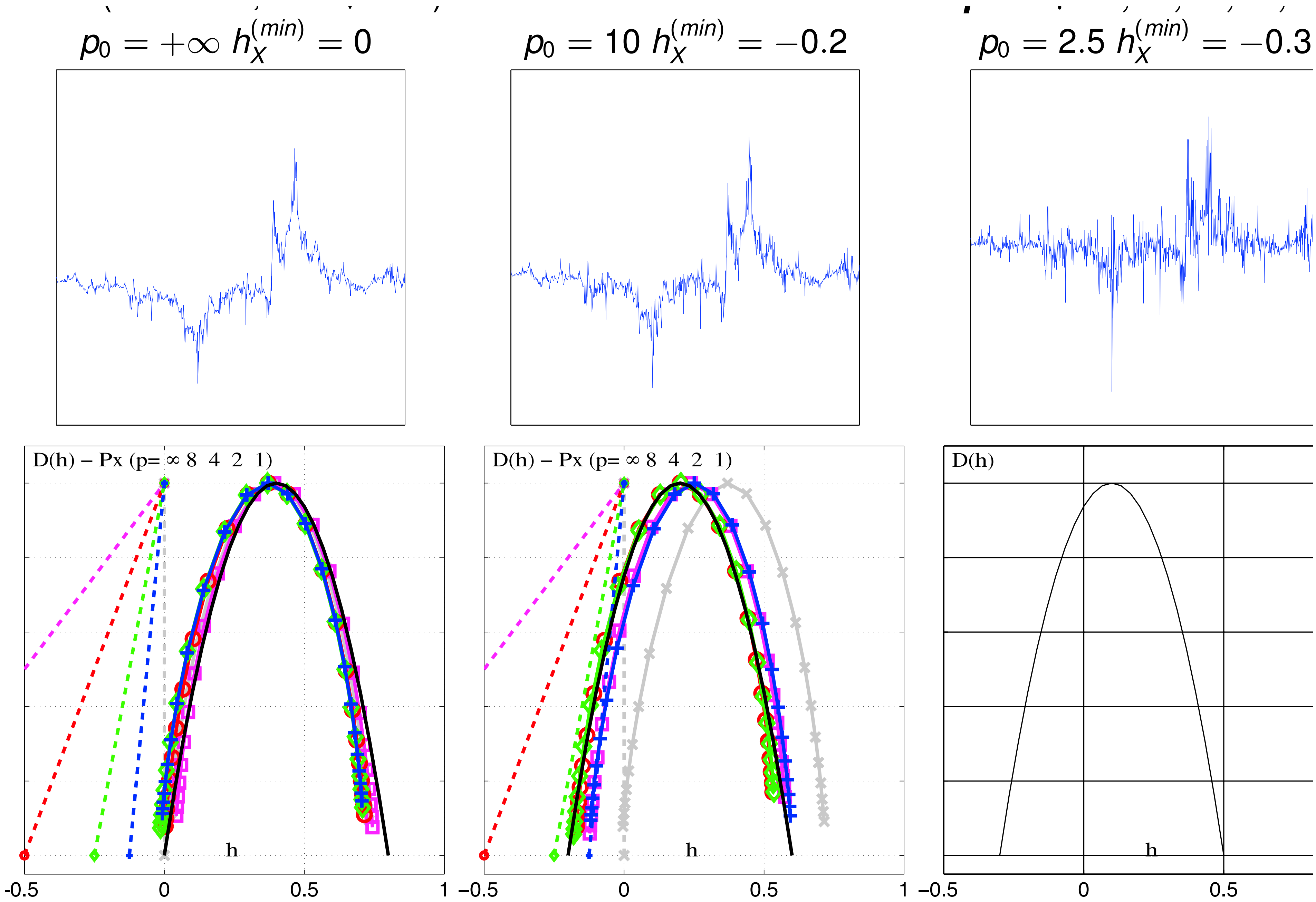
$p_0 = +\infty\ h_X^{(min)} = 0$
$p_0 = 10\ h_X^{(min)} = -0.2$
$p_0 = 2.5\ h_X^{(min)} = -0.3$
D(h) – Px (p= ∞ 8 4 2 1)
D(h) – Px (p= ∞)
D(h)
h
h
h
-0.5
0
0.5
1
−0.5
0
0.5
1
−0.5
0
0.5



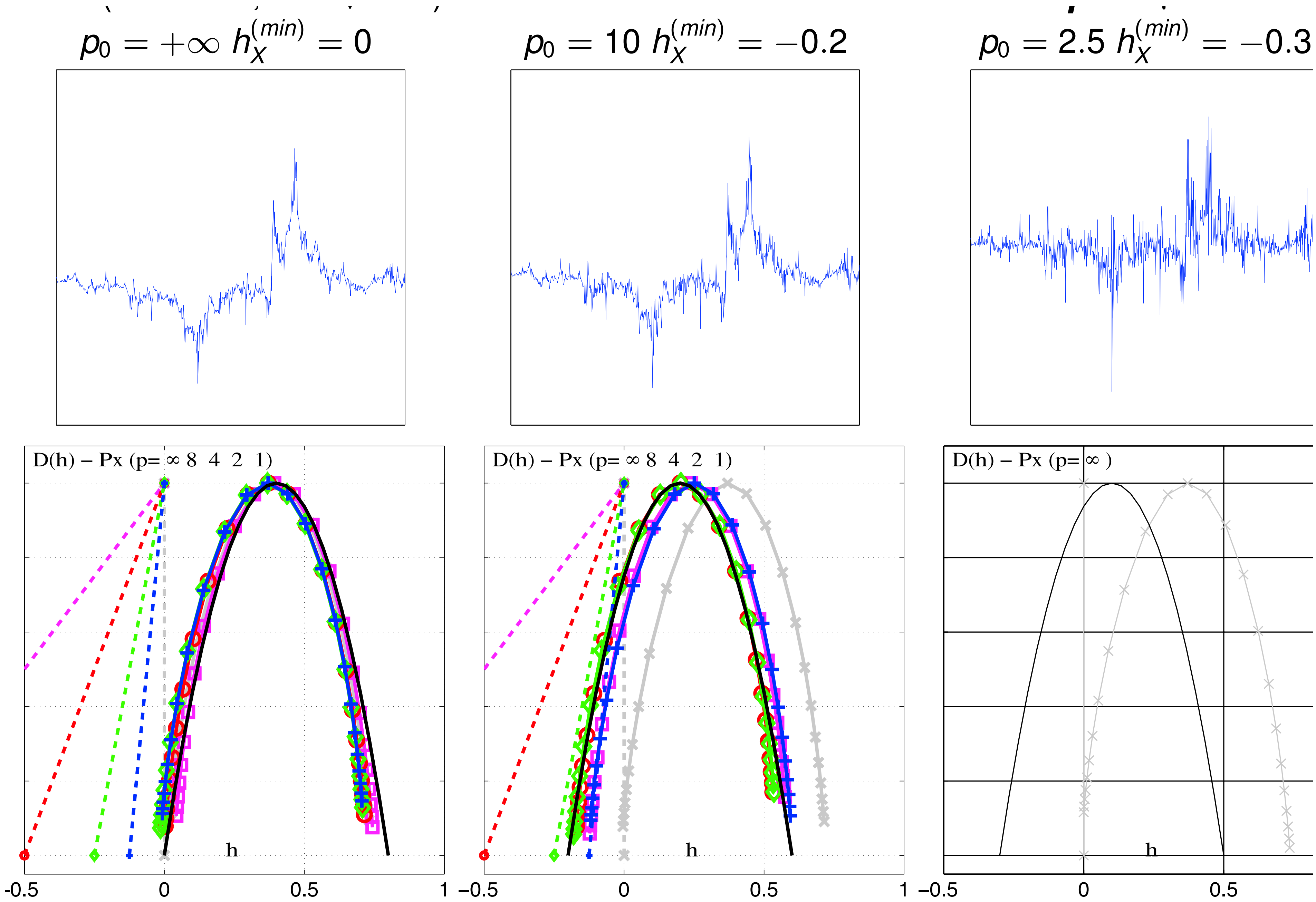
$p_0 = +\infty \; h_X^{(min)} = 0$
$p_0 = 10 \; h_X^{(min)} = -0.2$
$p_0 = 2.5 \; h_X^{(min)} = -0.3$
D(h) – Px (p= ∞ 8 4 2 1)
D(h) – Px (p= ∞ 8)
D(h)
h
h
h
-0.5
0
0.5
1
−0.5
0
0.5
1
−0.5
0
0.5



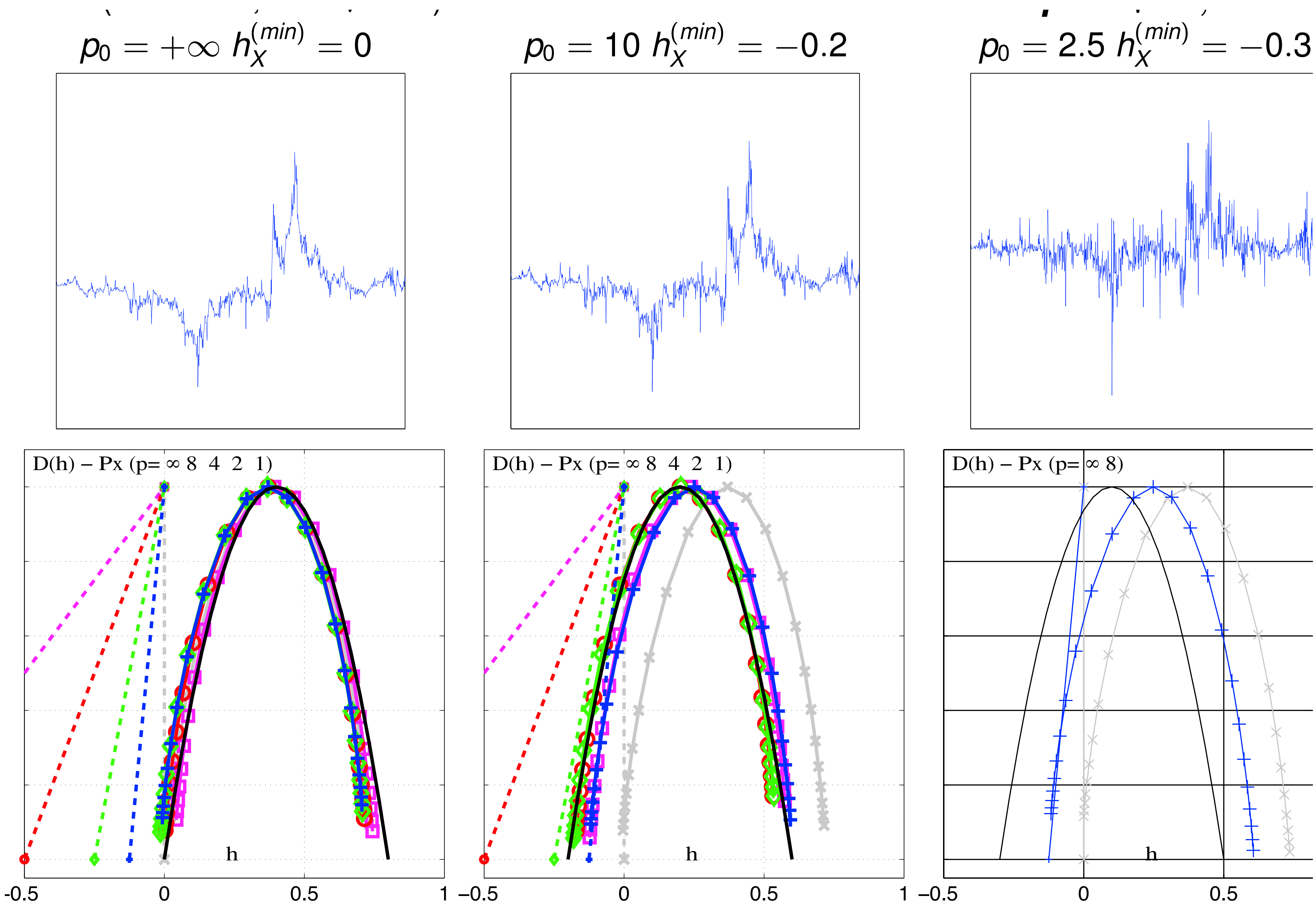
$p_0 = +\infty \; h_X^{(min)} = 0$
$p_0 = 10 \; h_X^{(min)} = -0.2$
$p_0 = 2.5 \; h_X^{(min)} = -0.3$
D(h) − Px (p= ∞ 8 4 2 1)
D(h) − Px (p= ∞ 8 4)
D(h)
h
h
h
-0.5
0
0.5
1
−0.5
0
0.5
1
−0.5
0
0.5



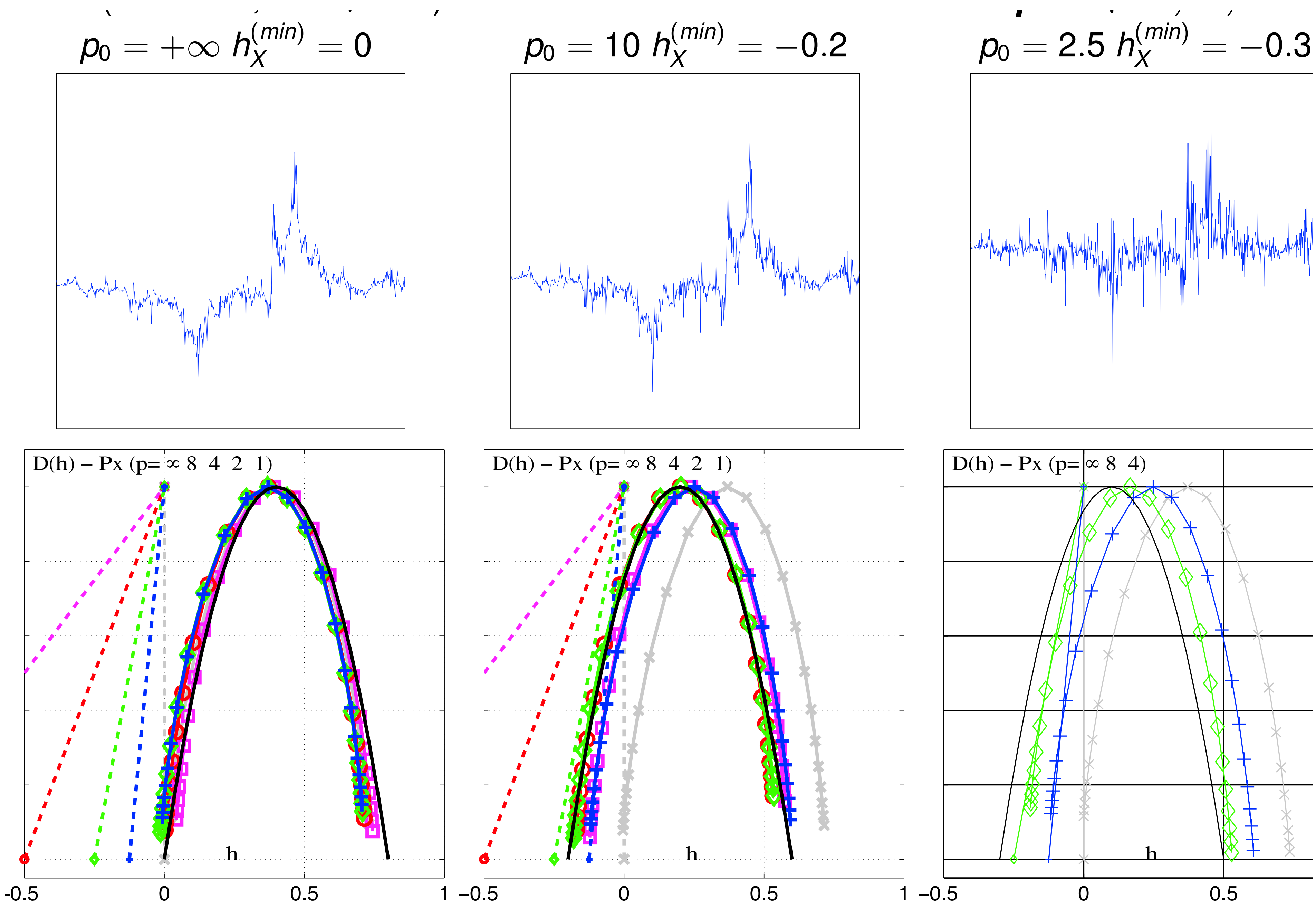
p0 = +∞ hX(min) = 0
p0 = 10 hX(min) = −0.2
p0 = 2.5 hX(min) = −0.3
D(h) − Px (p= ∞ 8 4 2 1)
D(h) − Px (p= ∞ 8 4 2 1)
D(h)
h
h
h
-0.5
0
0.5
1
−0.5
0
0.5
1
−0.5
0
0.5



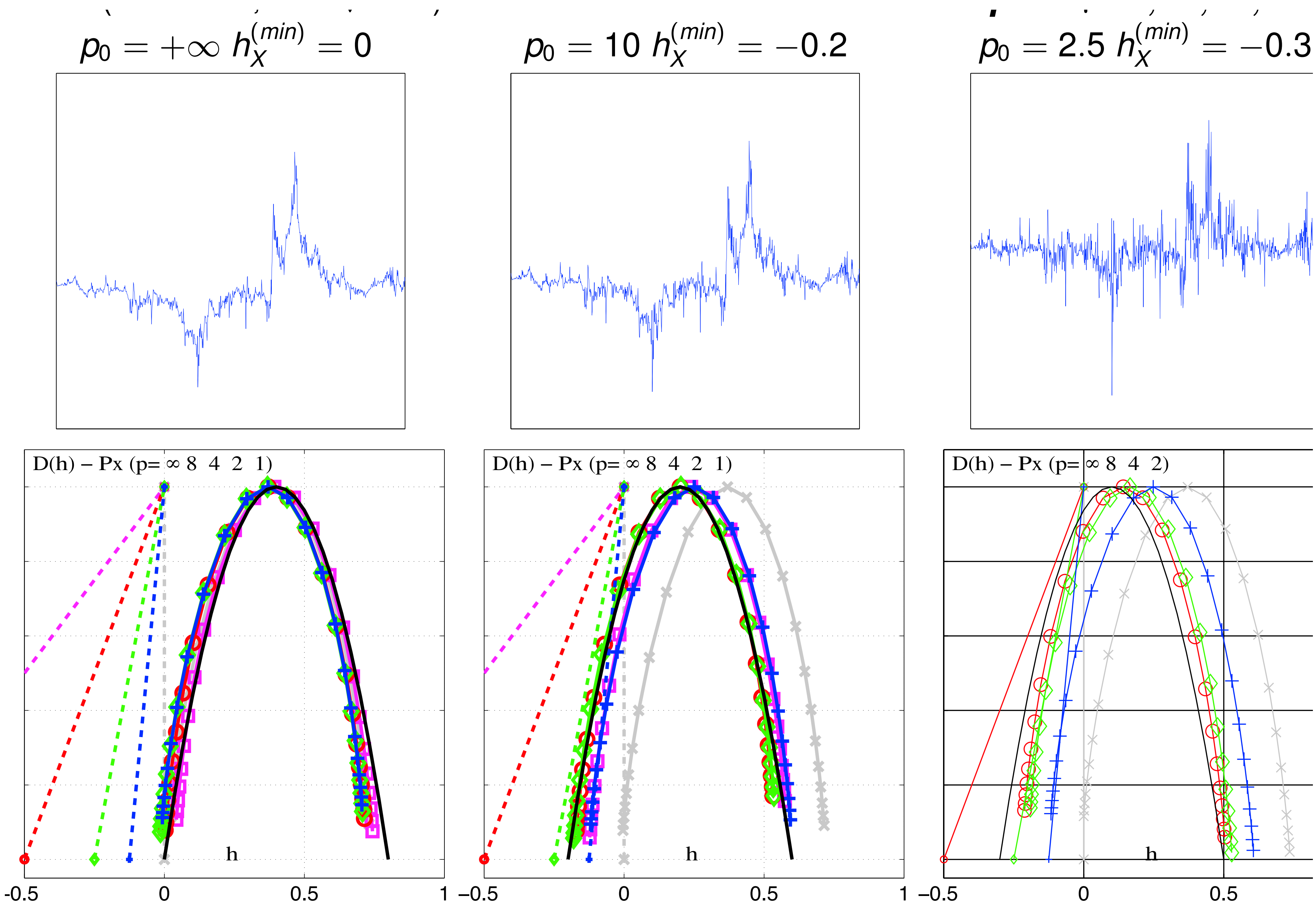
$p_0 = +\infty \; h_X^{(min)} = 0$
$p_0 = 10 \; h_X^{(min)} = -0.2$
$p_0 = 2.5 \; h_X^{(min)} = -0.3$
D(h) – Px (p= ∞ 8 4 2 1)
h
-0.5
0
0.5
1
D(h) – Px (p= ∞ 8 4 2 1)
h
−0.5
0
0.5
1
D(h) – Px (p= ∞)
h
−0.5
0
0.5



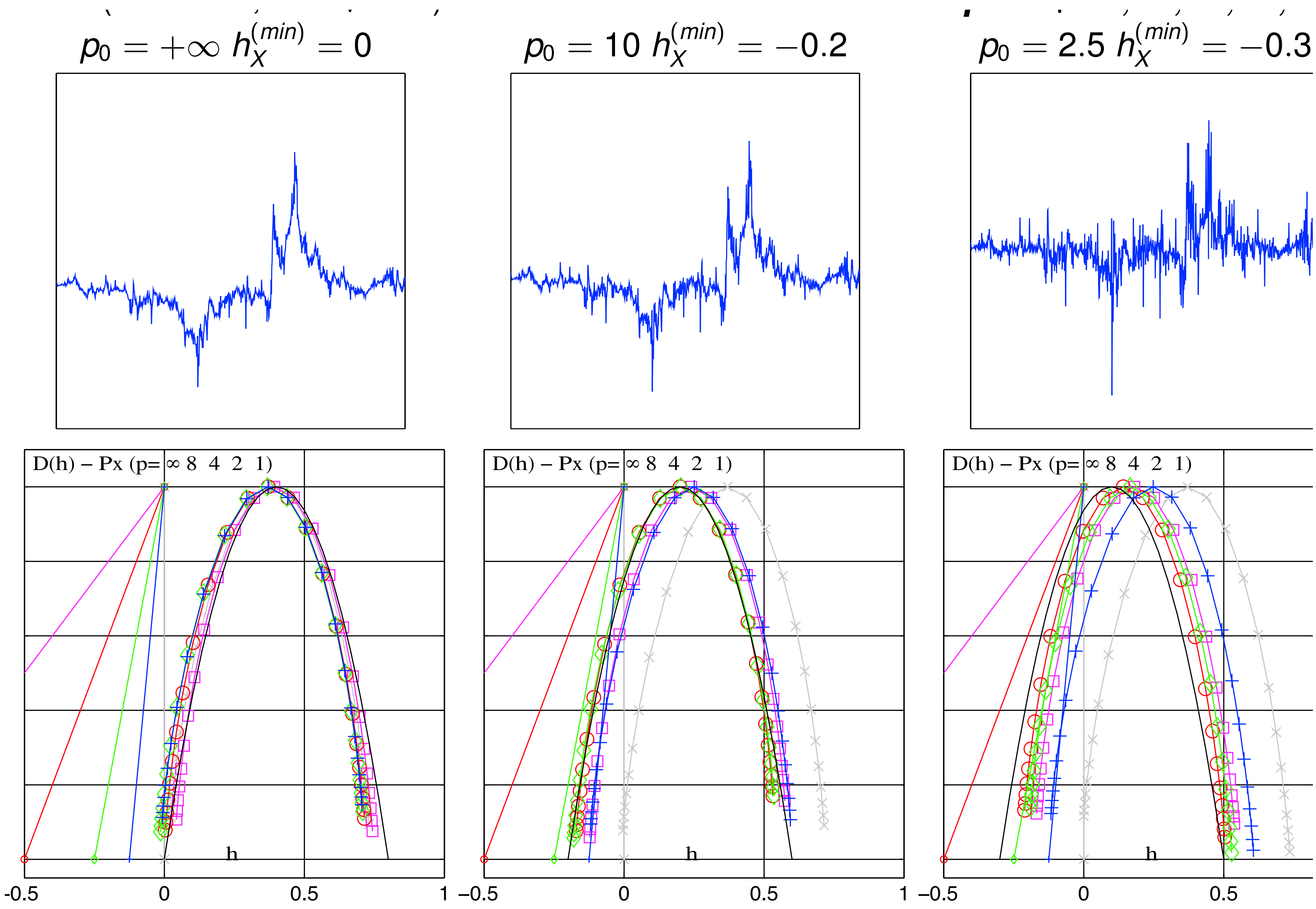
$p_0 = +\infty \; h_X^{(min)} = 0$
$p_0 = 10 \; h_X^{(min)} = -0.2$
$p_0 = 2.5 \; h_X^{(min)} = -0.3$
D(h) − Px (p= ∞ 8 4 2 1)
D(h) − Px (p= ∞ 8 4 2 1)
D(h) − Px (p= ∞ 8)
h
h
h
-0.5
0
0.5
1
−0.5
0
0.5
1
−0.5
0
0.5



$p_0 = +\infty \; h_X^{(min)} = 0$
$p_0 = 10 \; h_X^{(min)} = -0.2$
$p_0 = 2.5 \; h_X^{(min)} = -0.3$
D(h) – Px (p= ∞ 8 4 2 1)
D(h) – Px (p= ∞ 8 4 2 1)
D(h) – Px (p= ∞ 8 4)
h
-0.5 0 0.5 1



$p_0 = +\infty\ h_X^{(min)} = 0$
$p_0 = 10\ h_X^{(min)} = -0.2$
$p_0 = 2.5\ h_X^{(min)} = -0.3$
D(h) – Px (p= ∞ 8 4 2 1)
D(h) – Px (p= ∞ 8 4 2 1)
D(h) – Px (p= ∞ 8 4 2)
h
h
h
-0.5
0
0.5
1
−0.5
0
0.5
1
−0.5
0
0.5

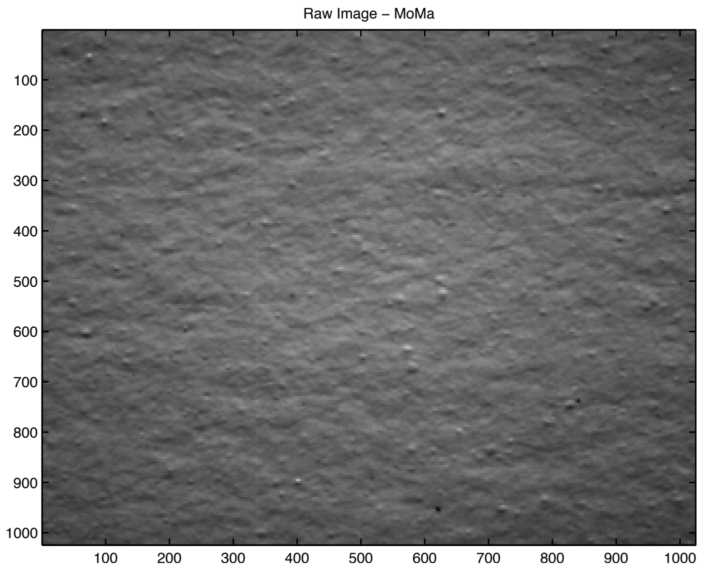


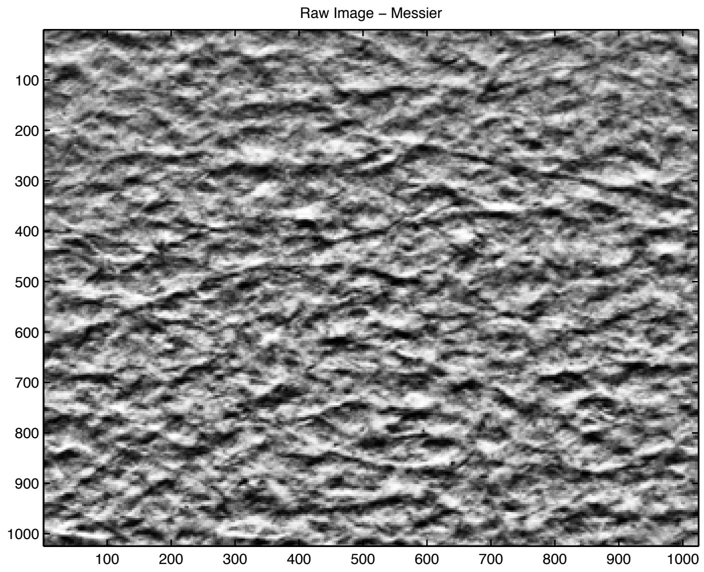
$p_0 = +\infty \; h_X^{(min)} = 0$
$p_0 = 10 \; h_X^{(min)} = -0.2$
$p_0 = 2.5 \; h_X^{(min)} = -0.3$
D(h) – Px (p= ∞ 8 4 2 1)
D(h) – Px (p= ∞ 8 4 2 1)
D(h) – Px (p= ∞ 8 4 2 1)
h
h
h
-0.5
0
0.5
1
−0.5
0
0.5
1
−0.5
0
0.5

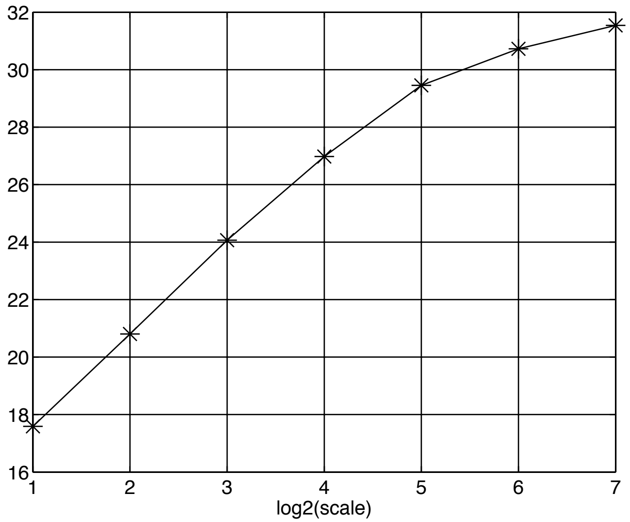
Advantages and drawbacks of multifractal analysis based on the *p*-exponent

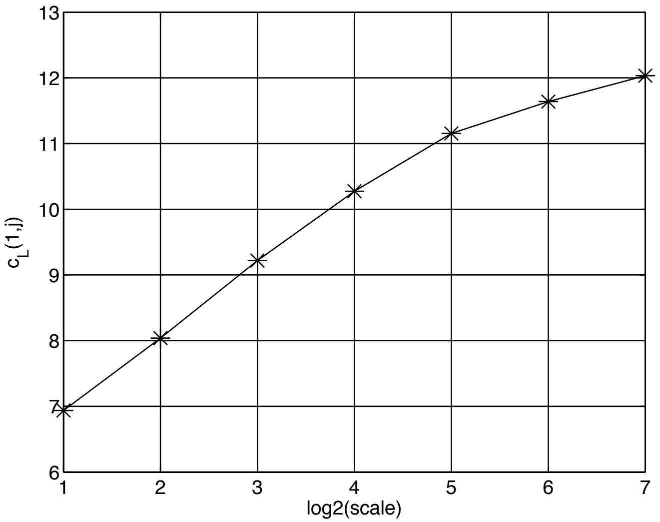
- Allows to deal with larger collections of data
- The estimation is not based on a unique extremal value, but on an l^p average $\Rightarrow$ better statistical properties
- Systematic bias in the estimation of p-leaders

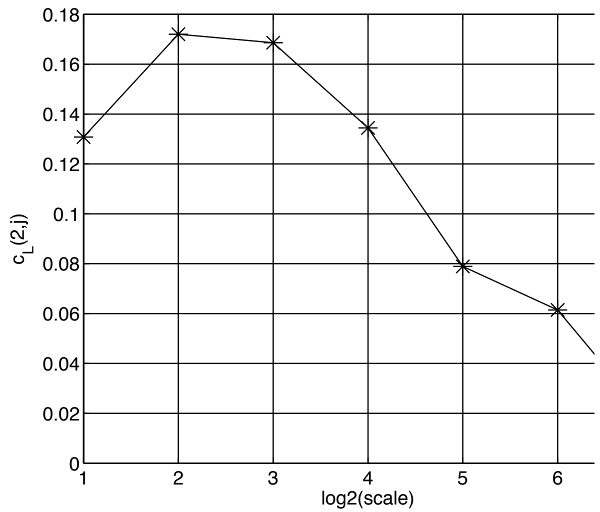
Beyond selfsimilarity ? Historic photographic paper classification











Directions of research and open problems

1. Give an interpretation of the leader scaling function for $p < 0$
2. Obtain genericity results that would take into account the whole scaling function (and not only $p > 0$)
3. Extend multifractality results for solutions of PDEs
4. Develop a multifractal formalism that would take into account directional regularity (M. Clausel, S. Roux, B. Vedel)
5. Develop a local multifractal analysis : Pertinent in theory (Markov processes) and applications (finance)
6. Break the "curse of concavity", and study nonconcave spectra (F. Bastin, C. Esser, T. Kleyntssens , S. Nicolay, L. Simons)
7. Go beyond selfsimilarity

$H_f^{min} < 0$: Spaces of distributions

Let $s > 0$; the fractional integral of order s is the operator defined by

$$\widehat{\mathcal{I}^s(f)}(\xi) = (1 + |\xi|^2)^{-s/2} \, \hat{f}(\xi)$$

The application of $\mathcal{I}^s$ to f shifts H_f^{min} by s : $H_{\mathcal{I}^s}^{min} = H_f^{min} + s$

A first solution if $H_f^{min} < 0$ consists in performing a fractional integration of f of order $s > -H_f^{min}$

In practice one performs a *pseudo-fractional integration* of order $s > -H_f^{min}$ which amounts to replace the wavelet coefficients c_λ by $2^{-sj} c_\lambda$

The new function f_s thus obtained is uniform Hölder, with uniform regularity exponent $H_f^{min} + s > 0$

$\Longrightarrow$ One can compute the wavelet leaders of f_s
$\Longrightarrow$ Mathematical results concerning the multifractal formalism are valid