The Abdus Salam International Centre for Theoretical Physics
50th Anniversary 1964–2014

2585-25

Joint ICTP-TWAS School on Coherent State Transforms, Time-Frequency and Time-Scale Analysis, Applications

2 - 20 June 2014

An application of wavelets, curvelets and shearlets to X-ray processing for art conservation

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Wavelets

Basics: algorithm

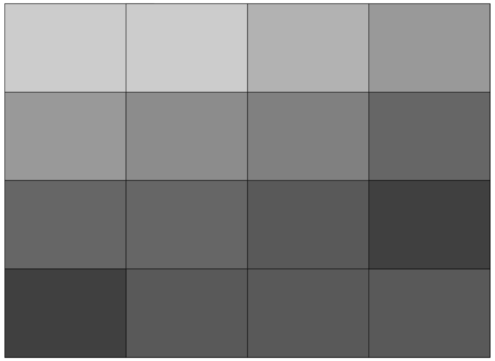
mathematical properties

Application to painting analysis

Wavelets

illustrated via image analysis

Digital images consist of pixels



Small squares, each with constant grey value

Typically: 256 different grey values (from pure white to pure black)

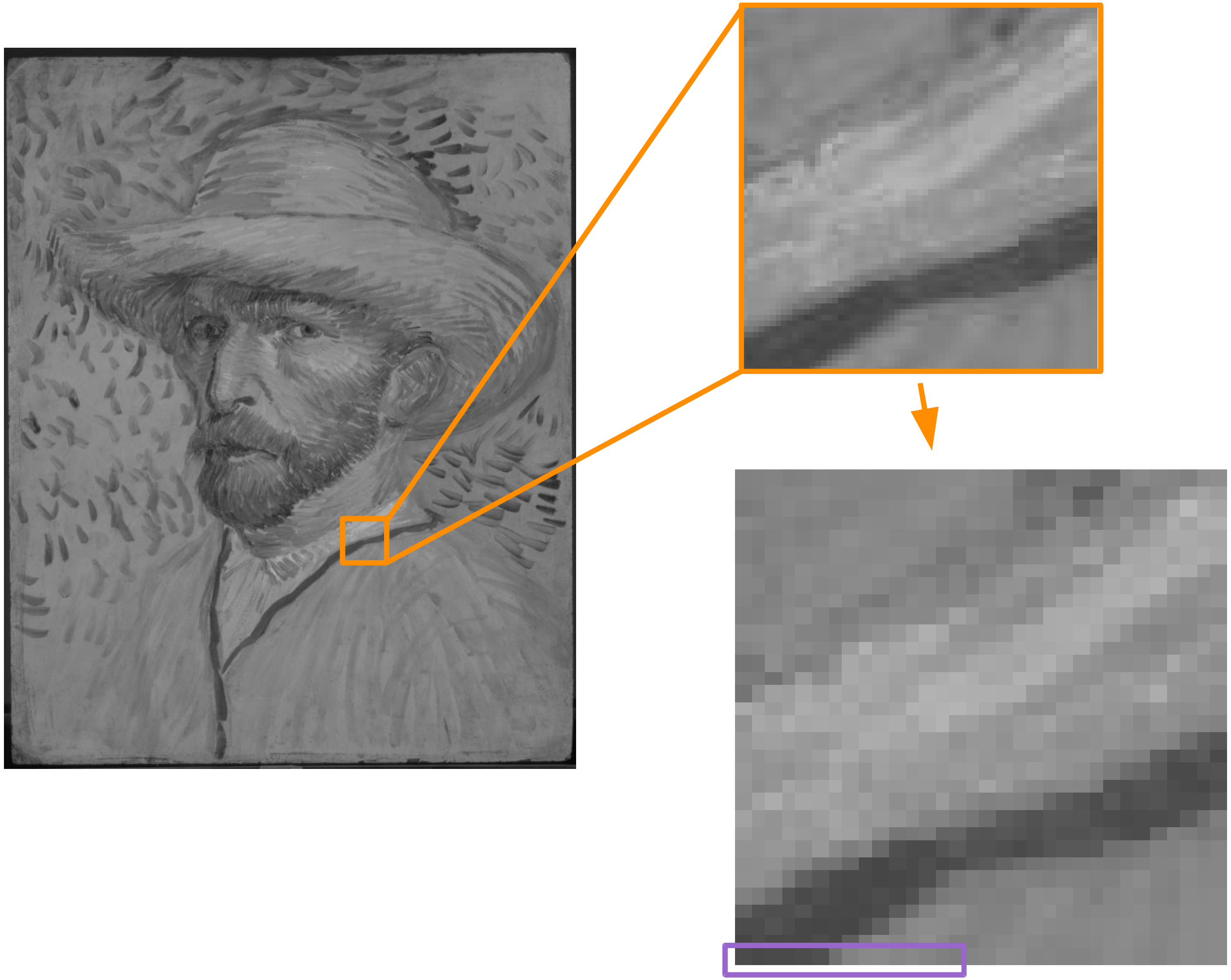
numbered from 0 to 255

Example: a row in a self-portrait by Van Gogh

75 72 74 74 76 80 112 131 137 138 134 137 133 128 126 132 134 →

→ 140 139 132 133 131 131 133 138 138 134 131 135 139 137 138 …

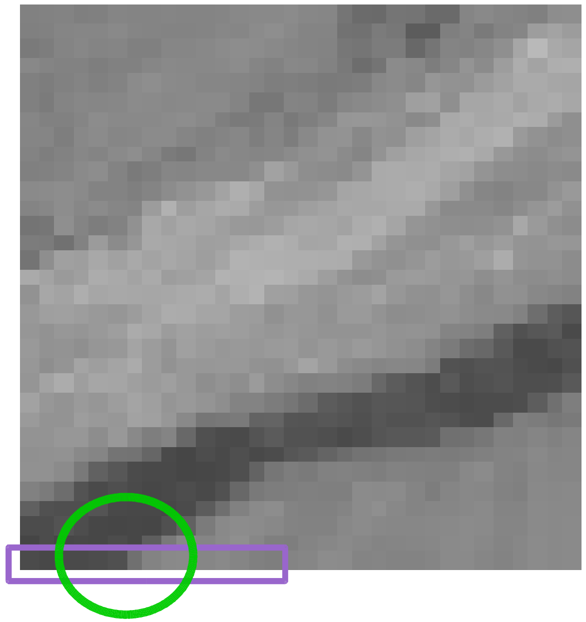
Example: a row in a self-portrait by Van Gogh



75 72 74 74 76 80 112 131 137 138 134 137 133 128 126 132

Example: a row in a self-portrait

by Van Gogh



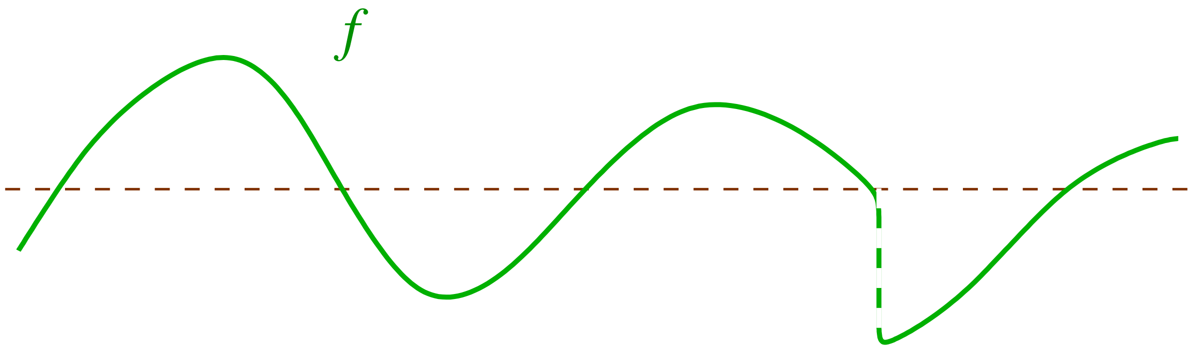
75 72 74 74 76 80 112 131 137 138 134 137 133 128 126 132 …

73.5	74	78	121.5	137.5	135.5	130.5	129
-3	0	4	19	1	3	-5	6

73.75	99.75	136.5	129.75
.5	43.5	-2	-1.5

large differences point to sudden transitions, such as edges

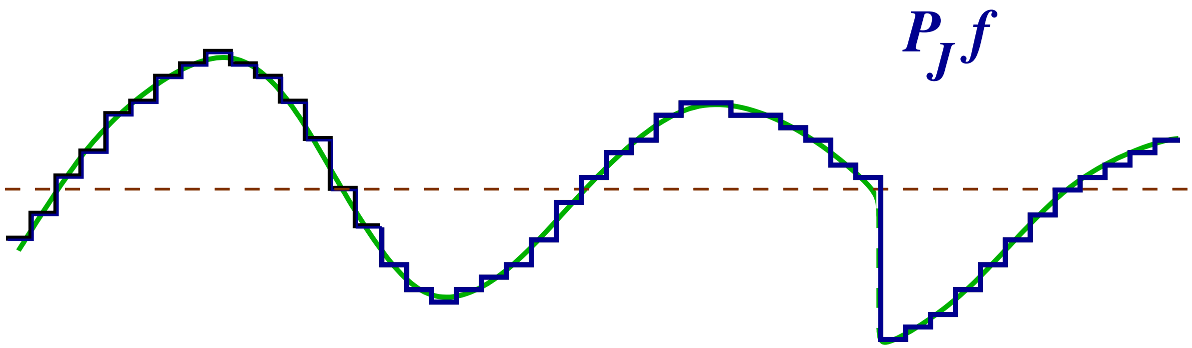
What does this correspond to mathematically?



given a function f

What does this correspond to mathematically?

given a function f

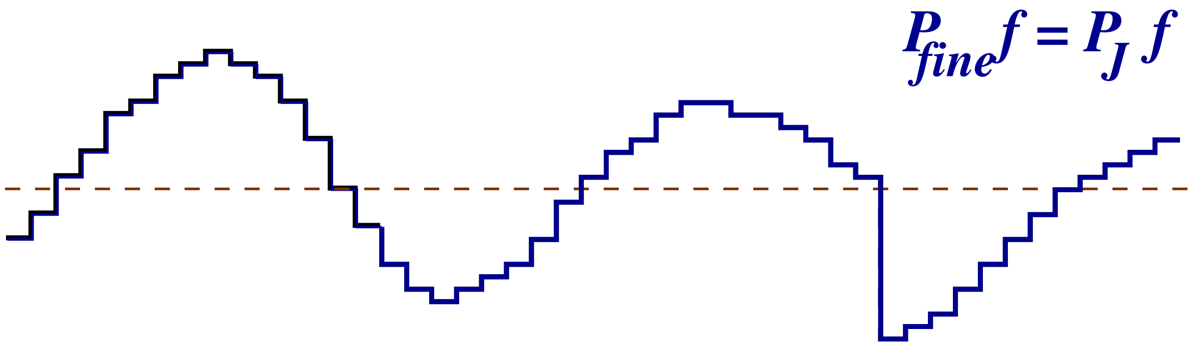


We consider a fine scale approximation of f

$$P_{fine} f = P_J f$$

What does this correspond to mathematically?

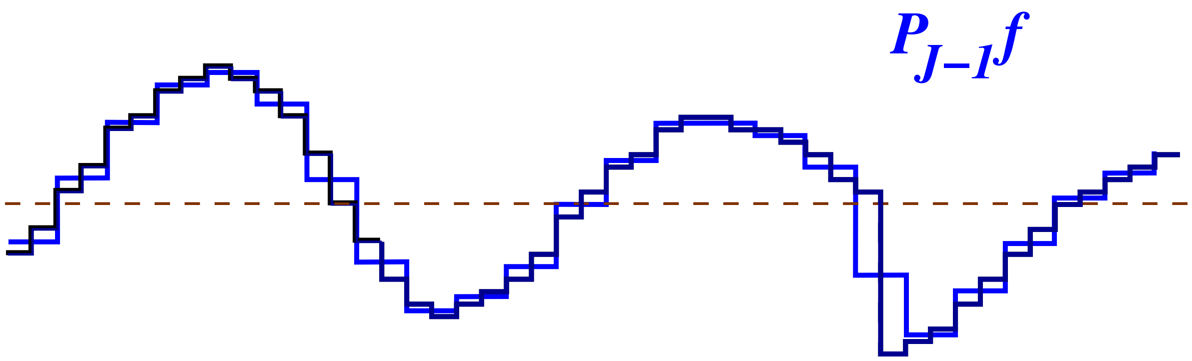
given a function f

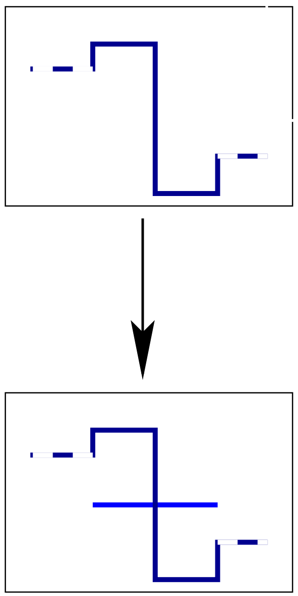


$P_{fine} f = P_J f$ = fine scale approximation to f

What does this correspond to mathematically?

given a function f

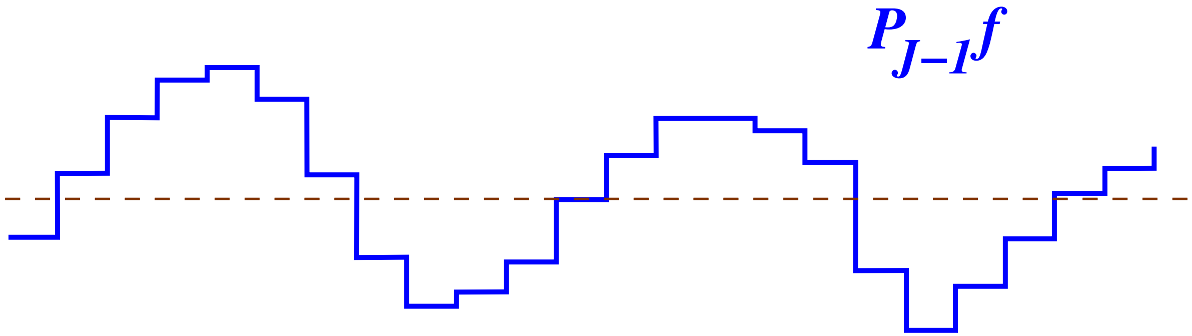




we next determine a coarser approximation to f by replacing pairs of constant levels by their average

What does this correspond to mathematically?

given a function f

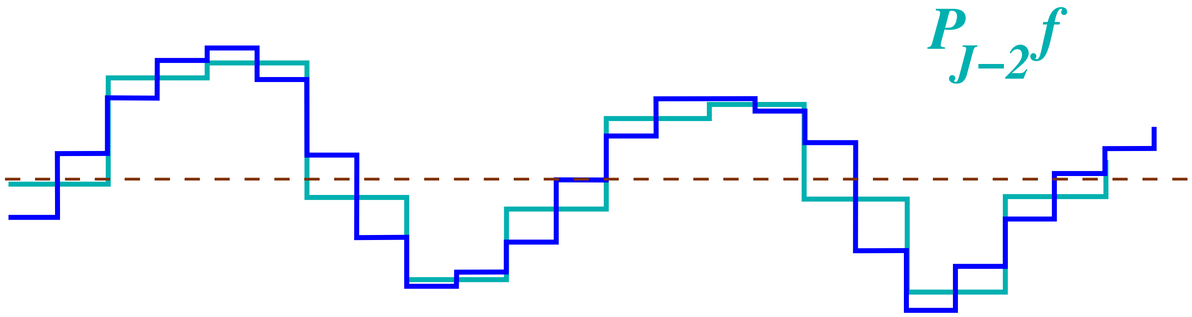


$P_{J-1}f$ = coarser approximation to f

(obtained by "averaging" $P_J f$)

What does this correspond to mathematically?

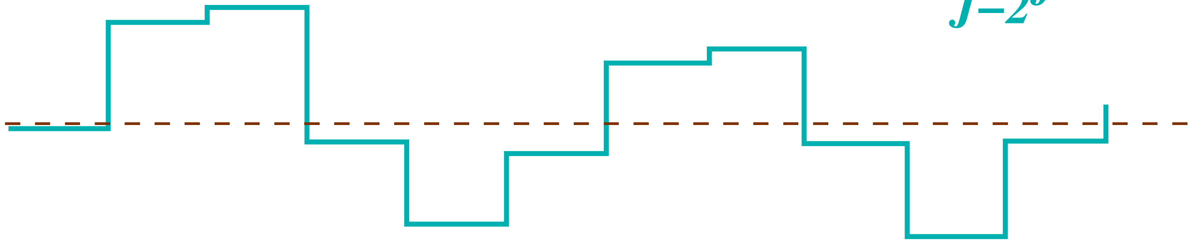
given a function f



we continue this process and replace coarse approximation $P_{J-1}f$ by one that is even coarser

What does this correspond to mathematically?

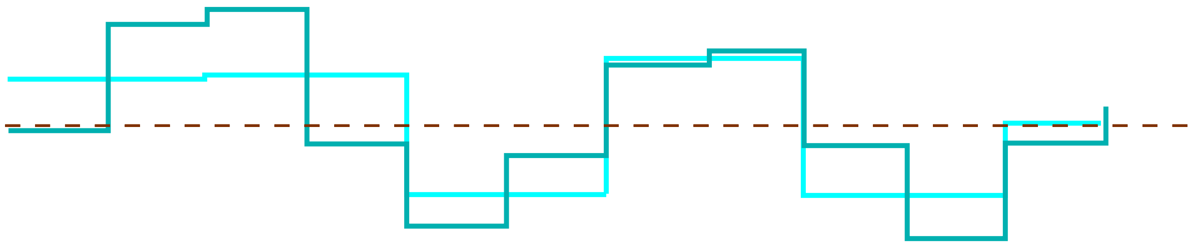
given a function f



the even coarser approximation $P_{J-2}f$

What does this correspond to mathematically?

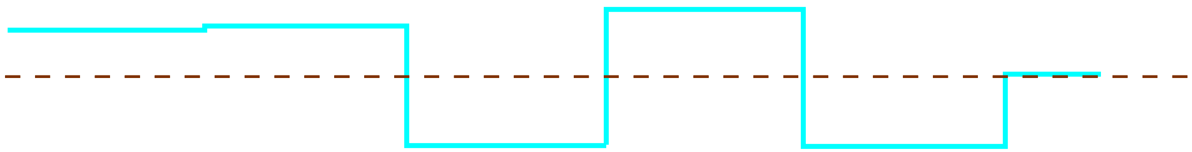
given a function f



and we repeat the process again ...

What does this correspond to mathematically?

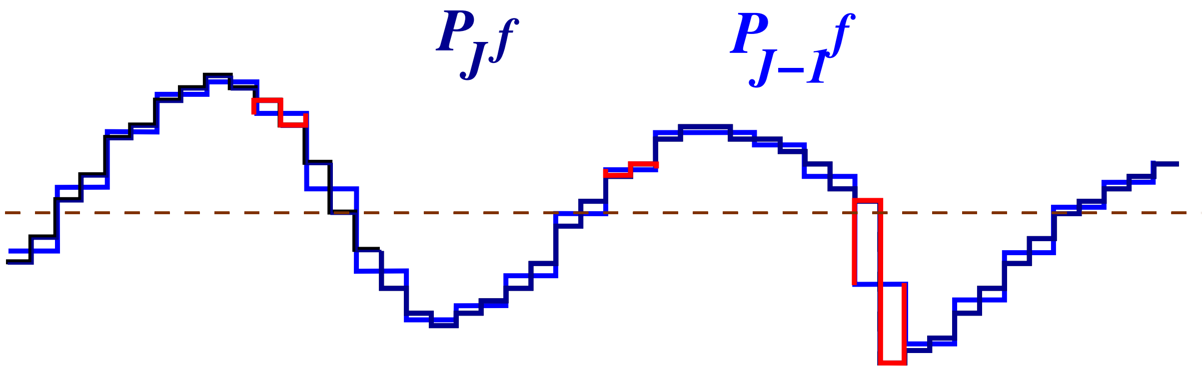
given a function f



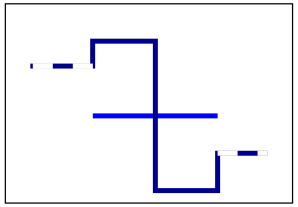
and again ...

What does this correspond to mathematically?

given a function f



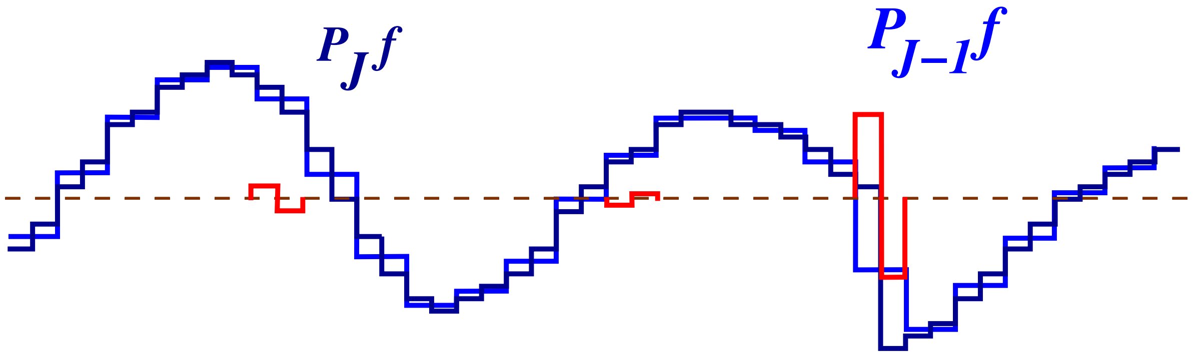
In each of the successive approximations, some detail is lost



we next determine a coarser approximation to f by replacing pairs of constant levels by their average

What does this correspond to mathematically?

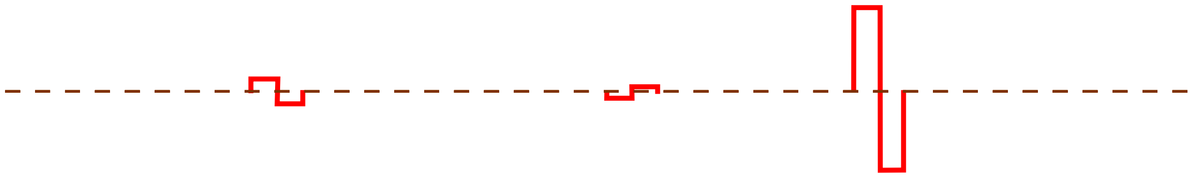
given a function f



In each of the successive approximations, some detail is lost

What does this correspond to mathematically?

given a function f



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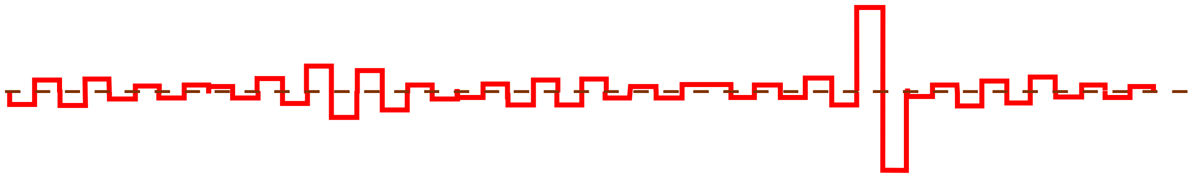
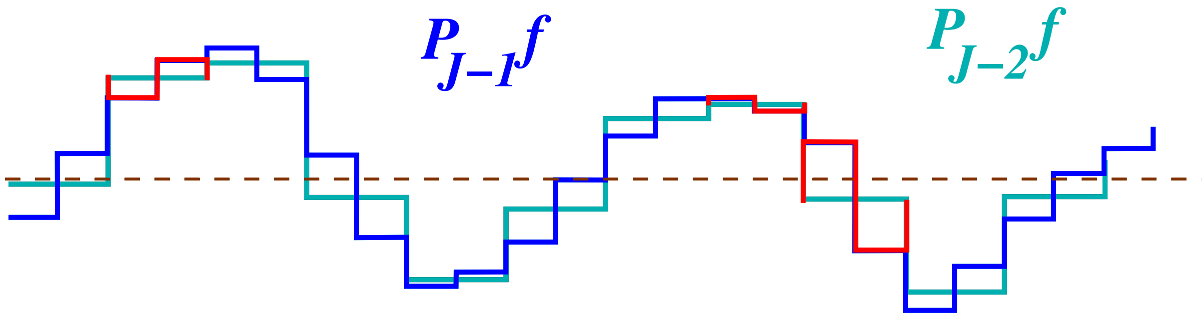


$P_{J-1} f - P_J f$

In each of the successive approximations, some detail is lost

What does this correspond to mathematically?

given a function f



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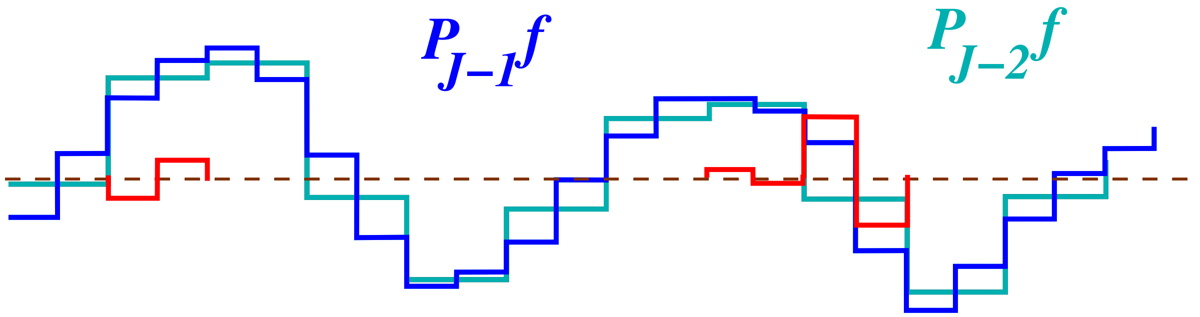

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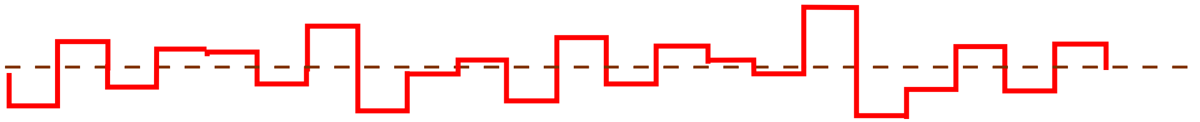
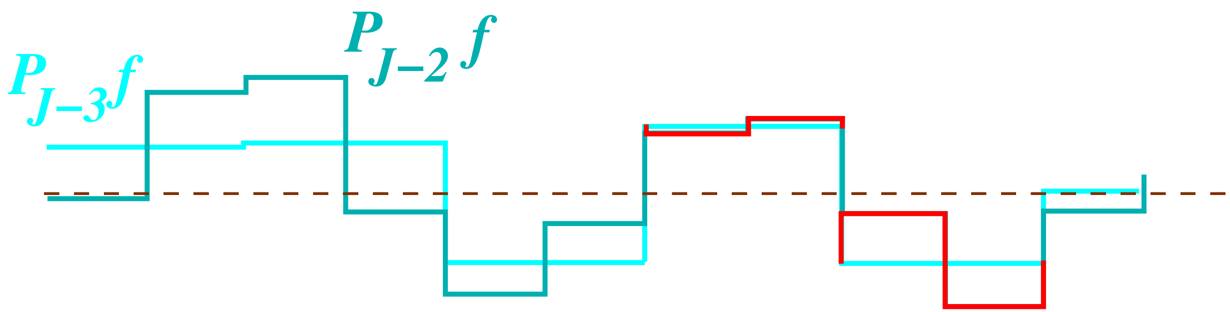


$P_{J-1}f - P_{J-2}f$

In each of the successive approximations, some detail is lost

What does this correspond to mathematically?

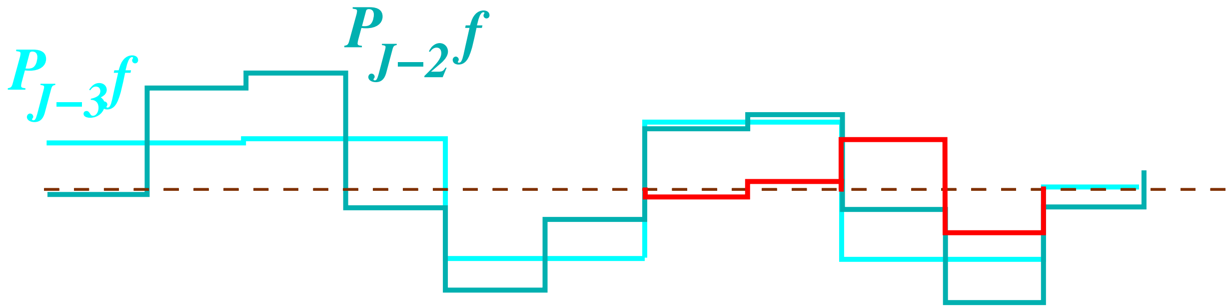
given a function f



In each of the successive approximations, some detail is lost

What does this correspond to mathematically?

given a function f

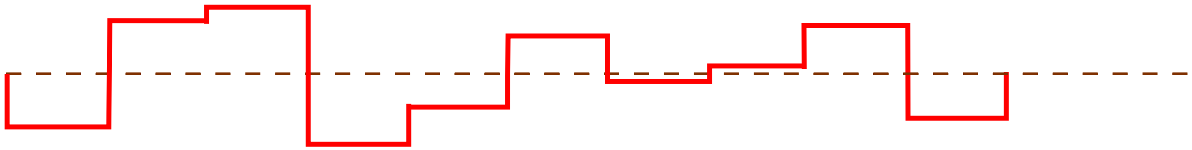


In each of the successive approximations, some detail is lost

What does this correspond to mathematically?

given a function f

$$P_{J-2}f - P_{J-3}f$$



In each of the successive approximations, some detail is lost

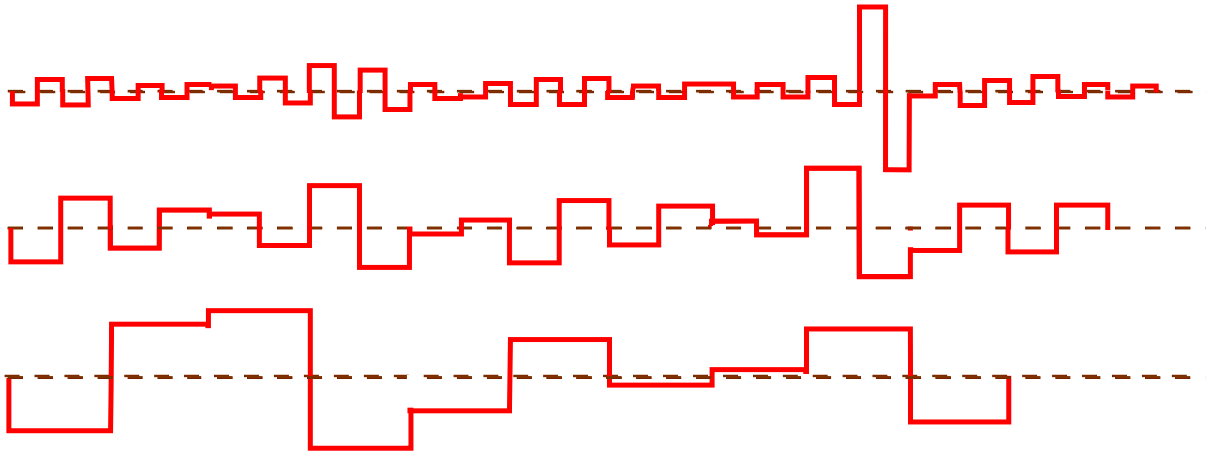
What does this correspond to mathematically?

given a function f

$P_J f - P_{J-1} f$

$P_{J-1} f - P_{J-2} f$

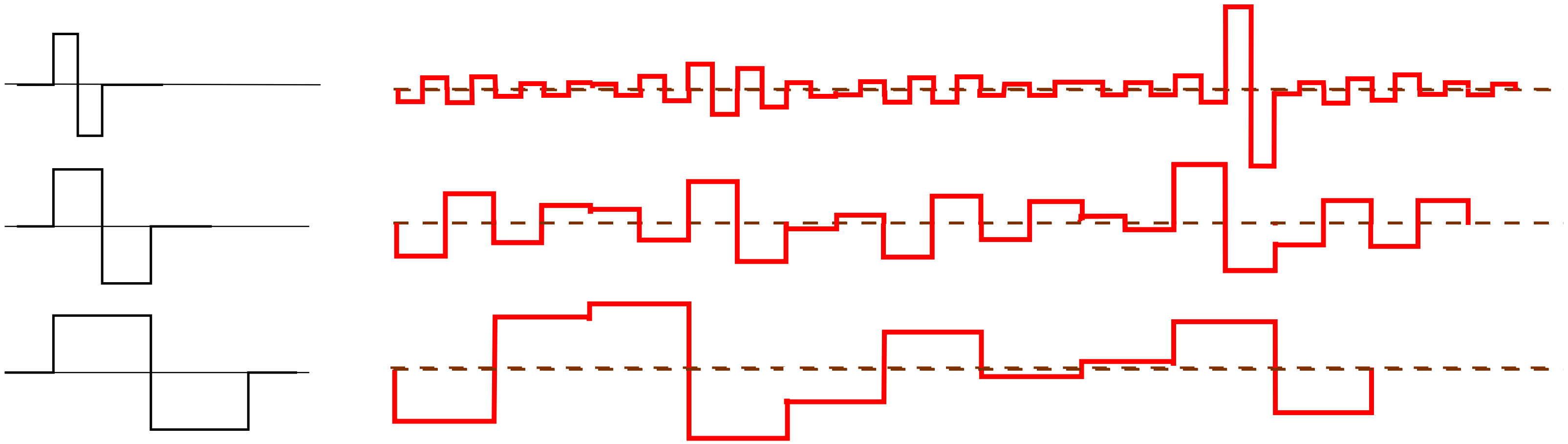
$P_{J-2} f - P_{J-3} f$



In each of the successive approximations, some detail is lost

What does this correspond to mathematically?

given a function f



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What does this correspond to mathematically?

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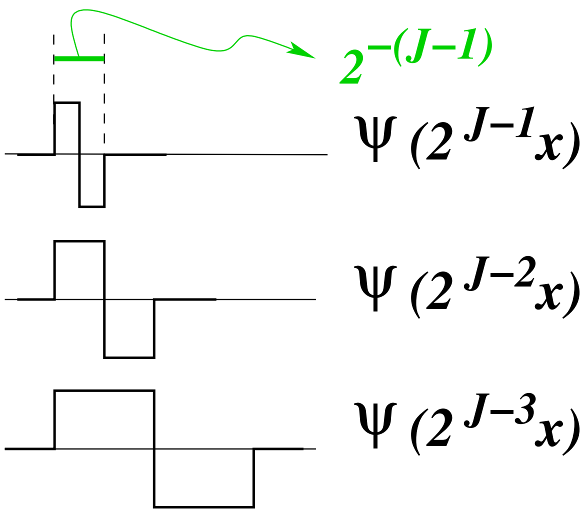


$$P_J(x) - P_{J-1}(x) = \sum_k c_{J-1,k} \Psi(2^{J-1}x - k)$$

$$P_{J-1}(x) - P_{J-2}(x) = \sum_k c_{J-2,k} \Psi(2^{J-2}x - k)$$

$$P_{J-2}(x) - P_{J-3}(x) = \sum_k c_{J-3,k} \Psi(2^{J-3}x - k)$$

In each of the successive approximations, some detail is lost

We have thus

$$P_J f(x) = \sum_{\ell=1}^{L} \left(P_{J-\ell+1} f(x) - P_{J-\ell} f(x) \right) + P_{J-L} f(x)$$

$$= \sum_{\ell=1}^{L} \sum_k c_{J-\ell,k} \, \psi\left(2^{J-\ell} x - k\right) + P_{J-L} f(x).$$

When $J \longrightarrow \infty$, $P_J f \longrightarrow f$; when $J - L \longrightarrow -\infty$, $P_{J-L} f \longrightarrow 0$.

Moreover, the $\psi\left(2^j x - k\right)$ are orthogonal.

The algorithm of averaging and differencing corresponds thus to the decomposition of $f \in L^2(\mathbb{R})$ into the orthonormal basis $2^{j/2} \psi\left(2^j x - k\right) =: \psi_{j,k}(x)$.

The $\psi_{j,k}$ in the example are discontinuous; provided "averaging" and "differencing" are replaced by generalizations (corresponding to higher order approximation schemes), one still has a similar structure, with ψ supported on an interval, but now smoother.

The decomposition

$$f = \sum_{j,k} \langle f , \psi_{j,k} \rangle \psi_{j,k}$$

can be viewed as a particularly convenient form of Calderòn's formula.

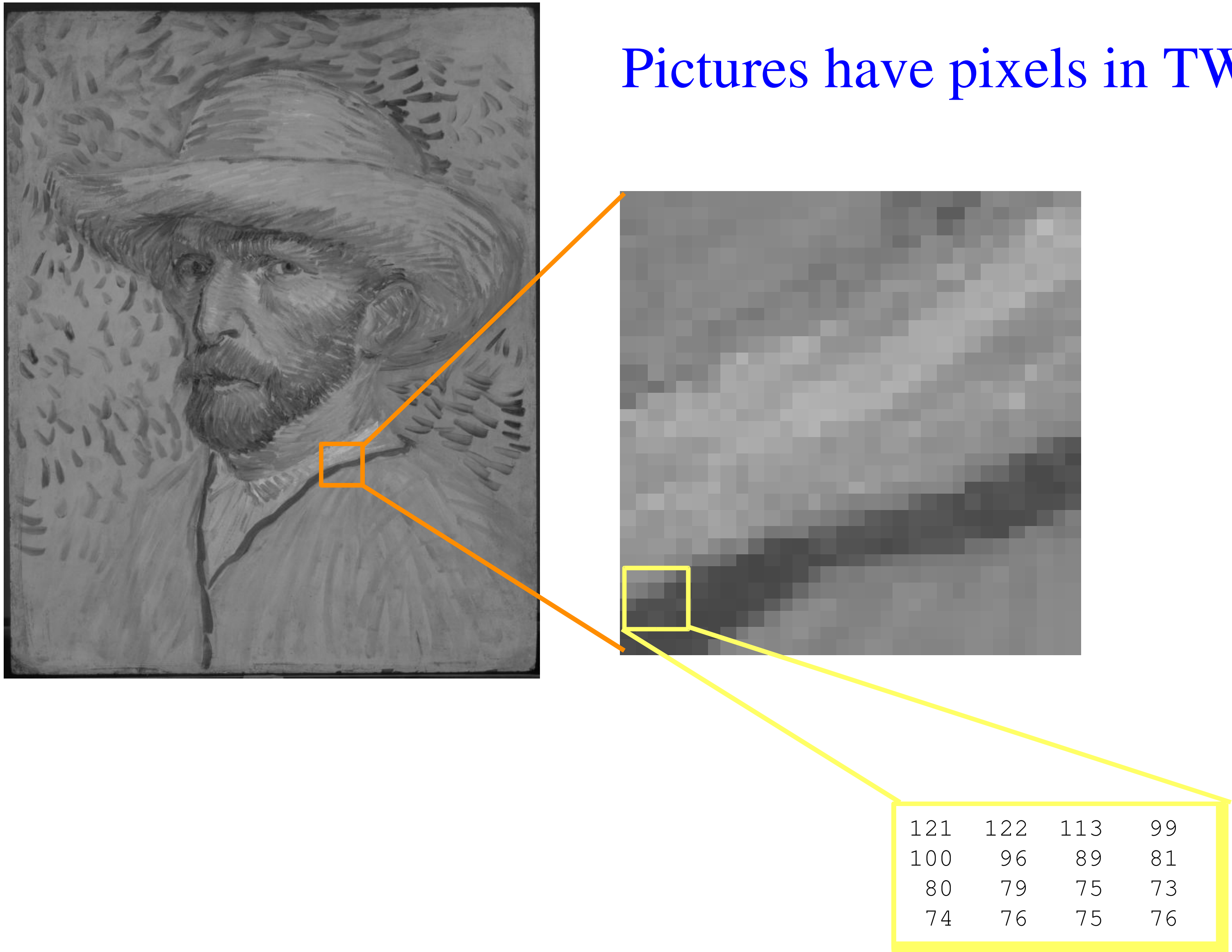
The $(\psi_{j,k})_{j,k \in \mathbb{Z}}$ constitute an unconditional basis for many useful functional spaces.

Examples: $L^p(\mathbb{R})$ (for $1 < p < \infty$), $W^{s,p}(\mathbb{R})$, $B^q_{s,p}(\mathbb{R})$, $C^\alpha(\mathbb{R})$, ...

The 2-dimensional wavelet transform.

So far, we have worked in only 1 dimension

Pictures have pixels in TWO directions!



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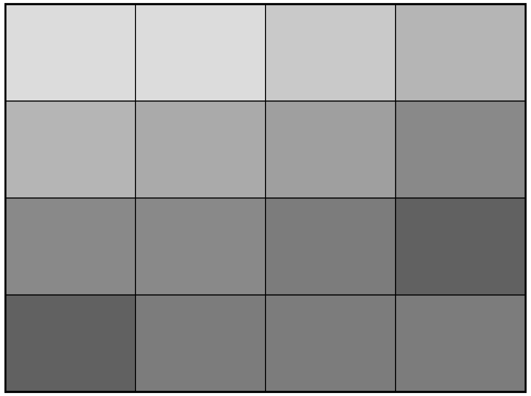
100	96	89	81
80	79	75	73
74	76	75	76



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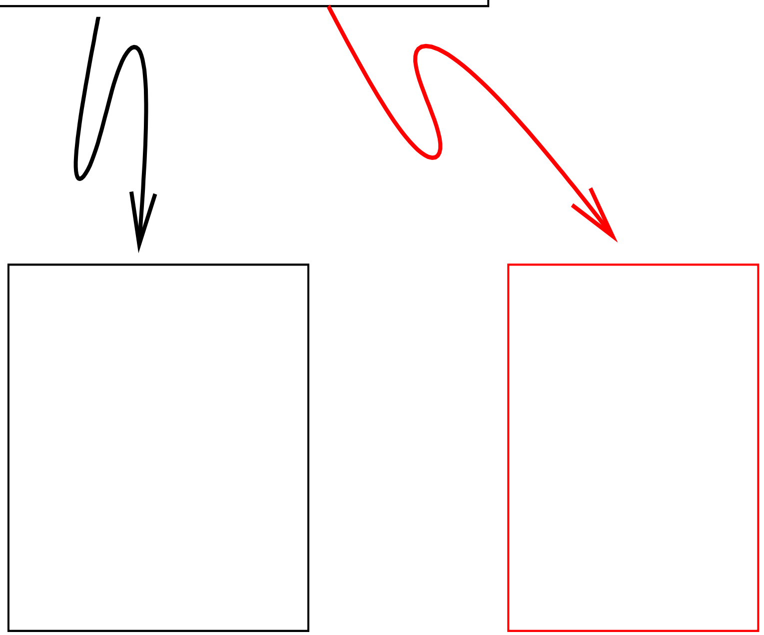
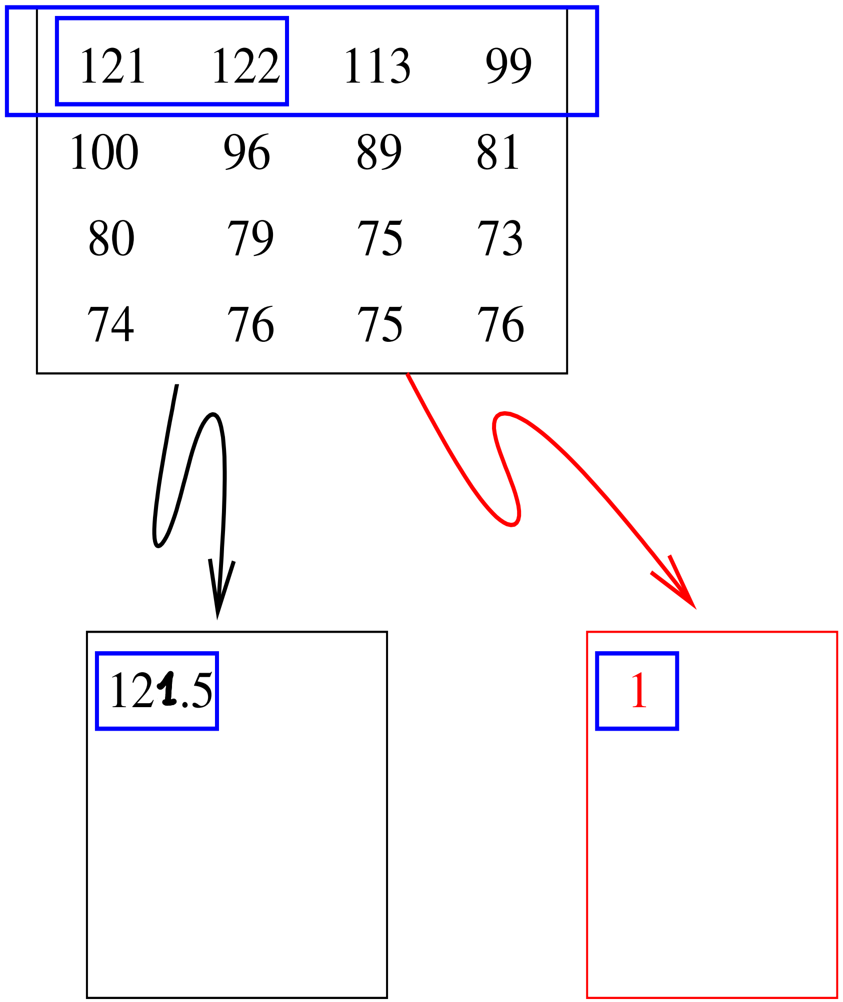


averaging pairwise   differencing pairwise

in each row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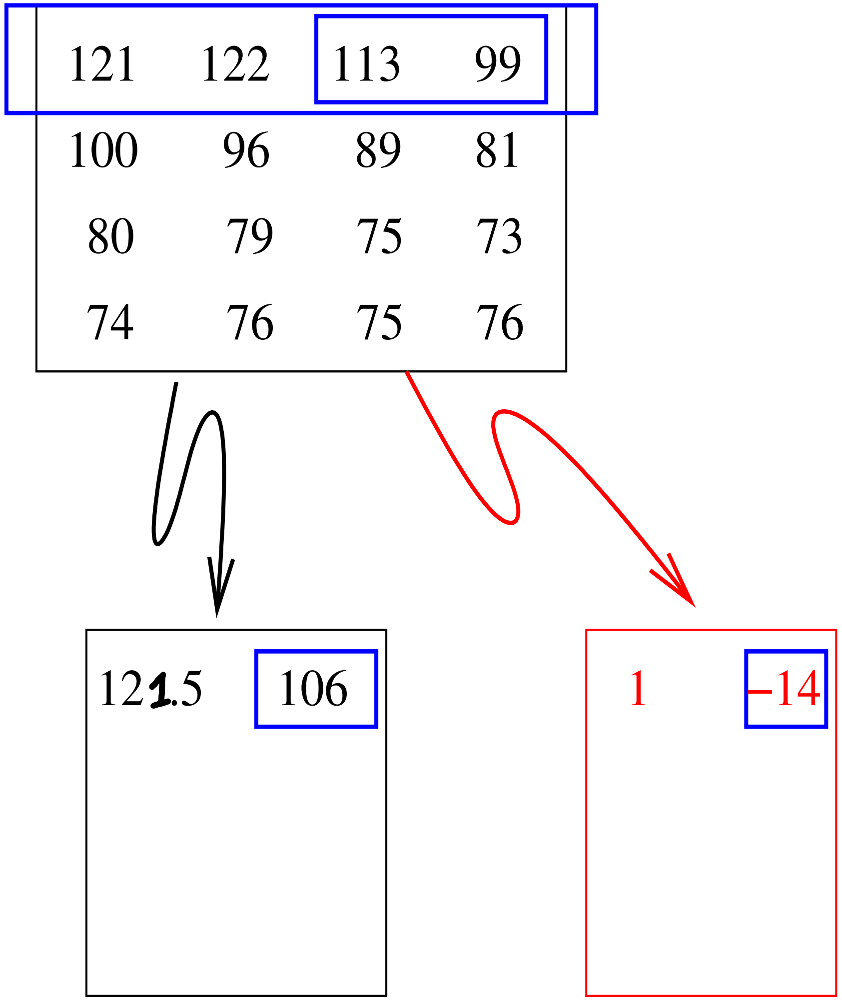
100 96 89 81
80 79 75 73
74 76 75 76
121.5
1



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100 96 89 81
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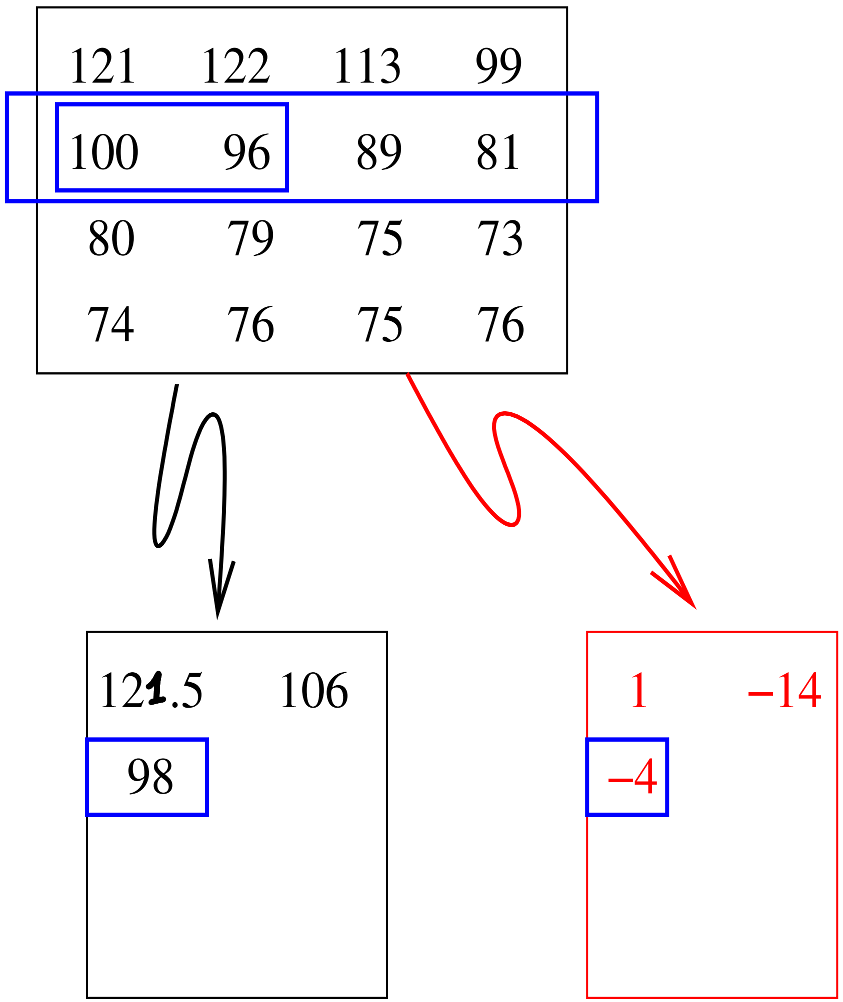
74 76 75 76
121.5
106
1
−14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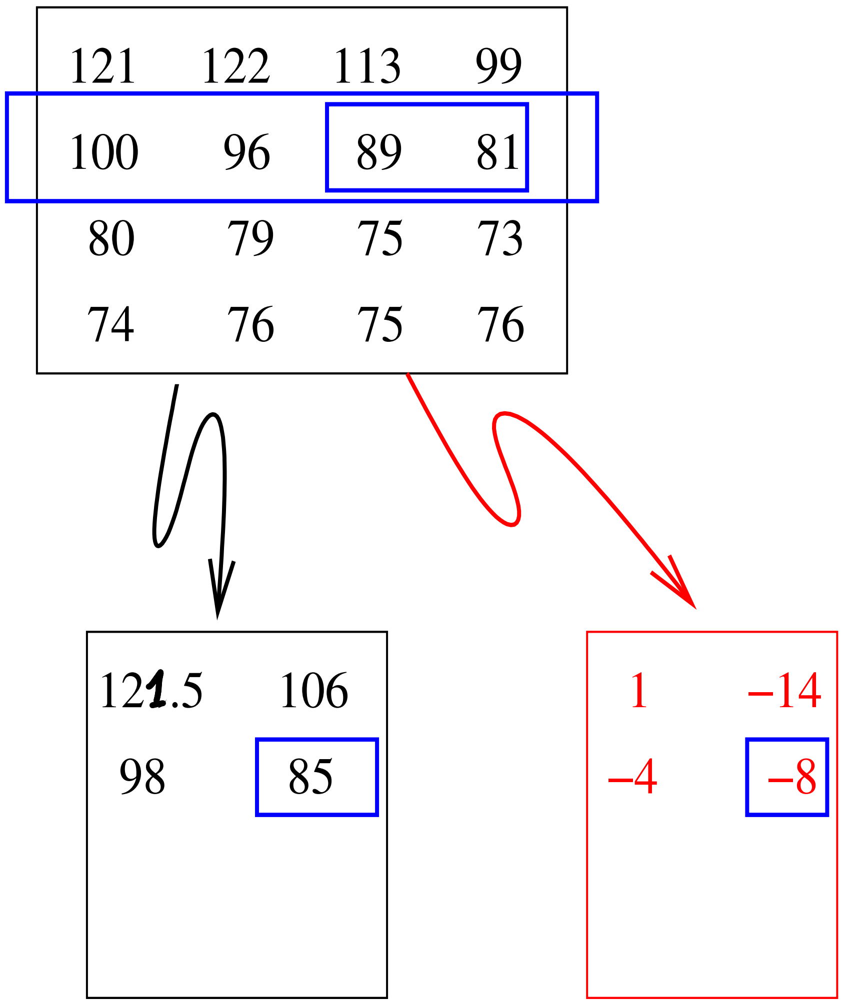
80 79 75 73
74 76 75 76
121.5 106
98
1 −14
−4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74 76 75 76
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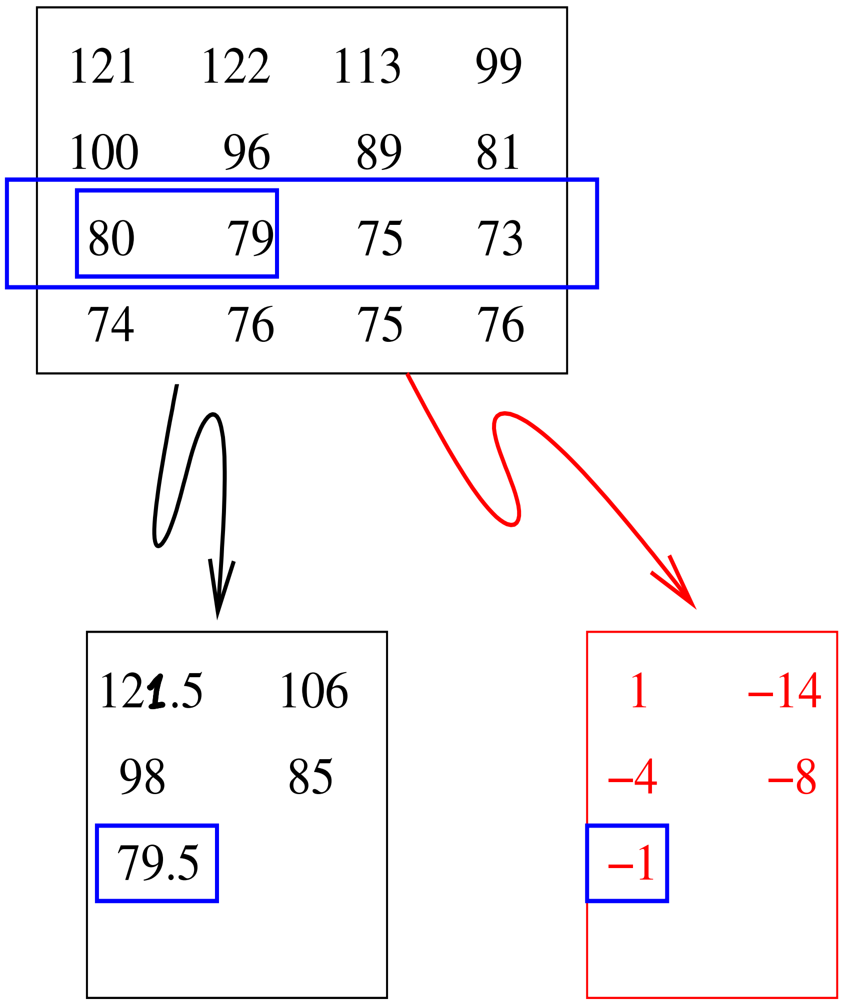
98 85
1 −14
−4 −8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74 76 75 76
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98 85
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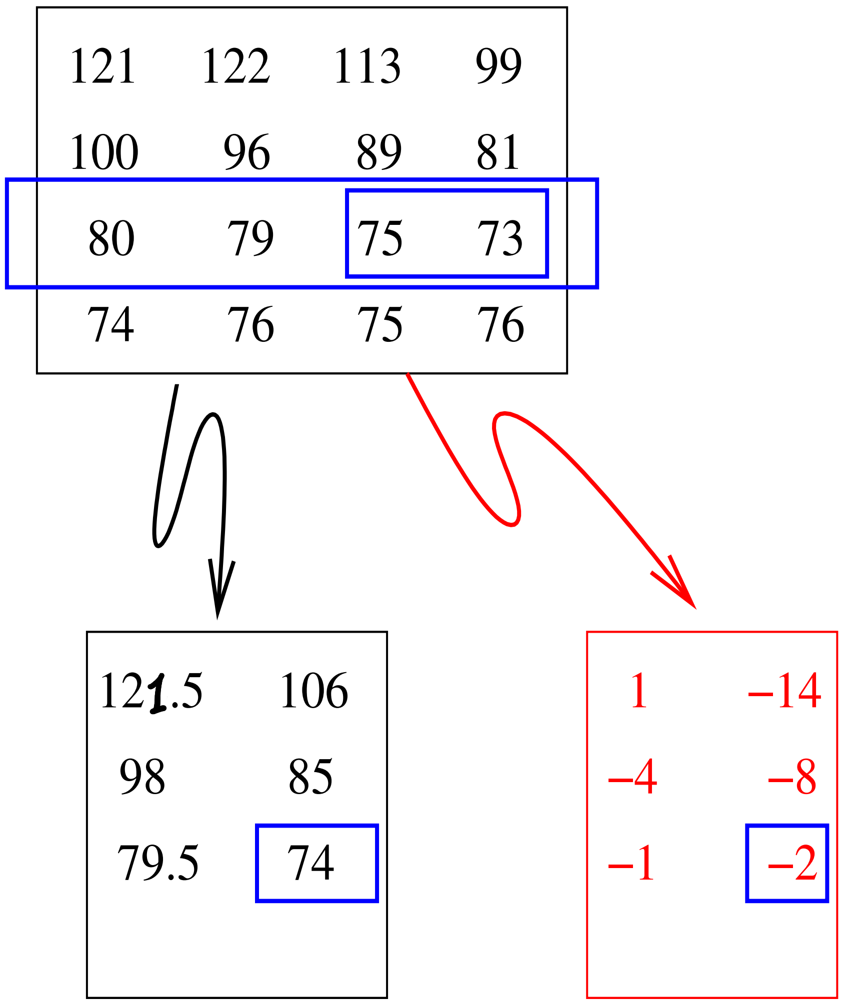
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98 85
79.5 74
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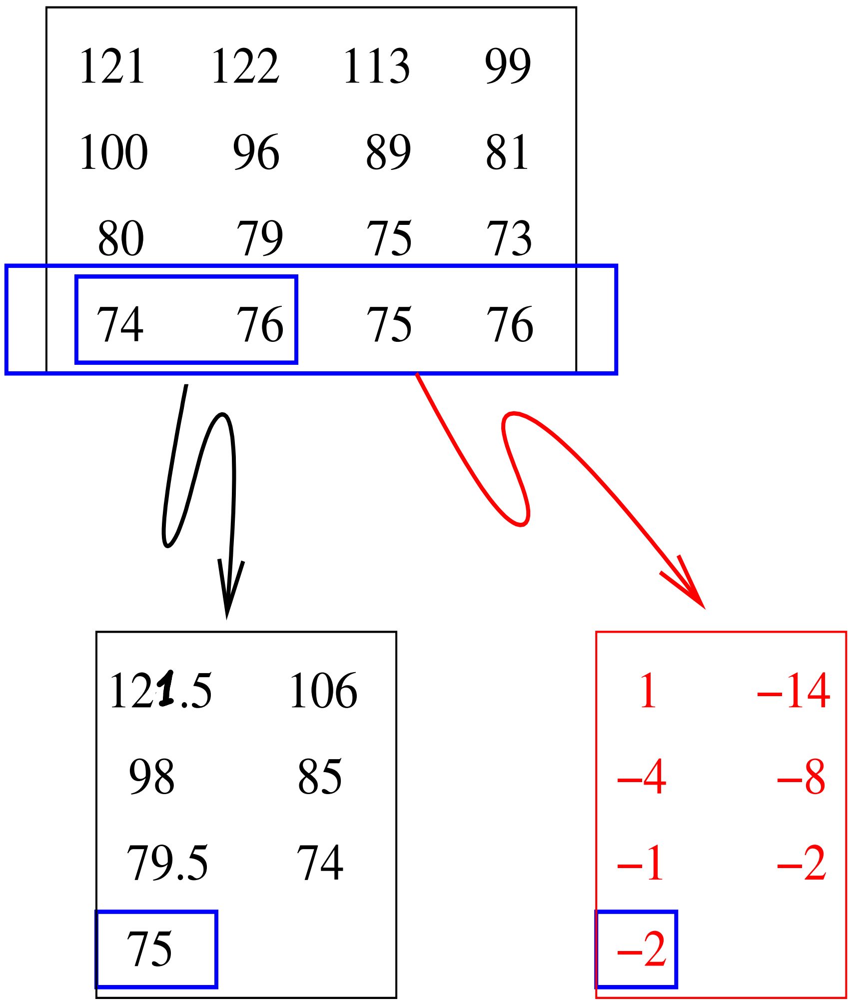
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79.5 74
75
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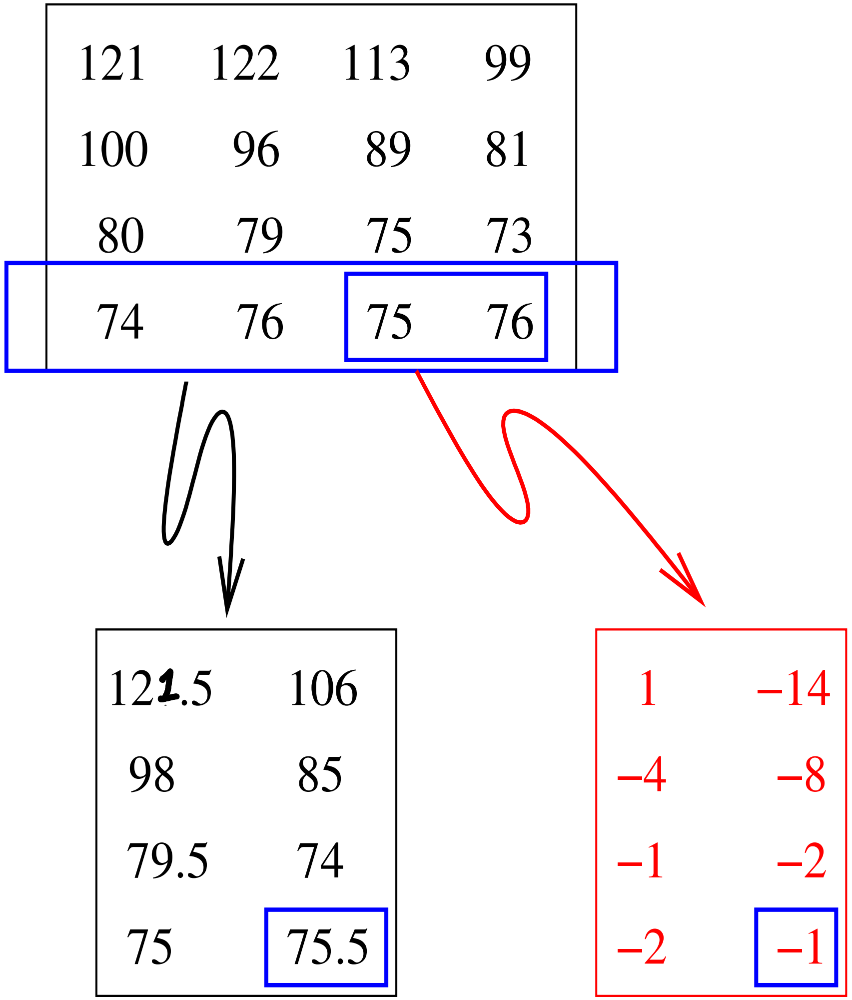
−4 −8
−1 −2
−2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79.5 74
75 75.5
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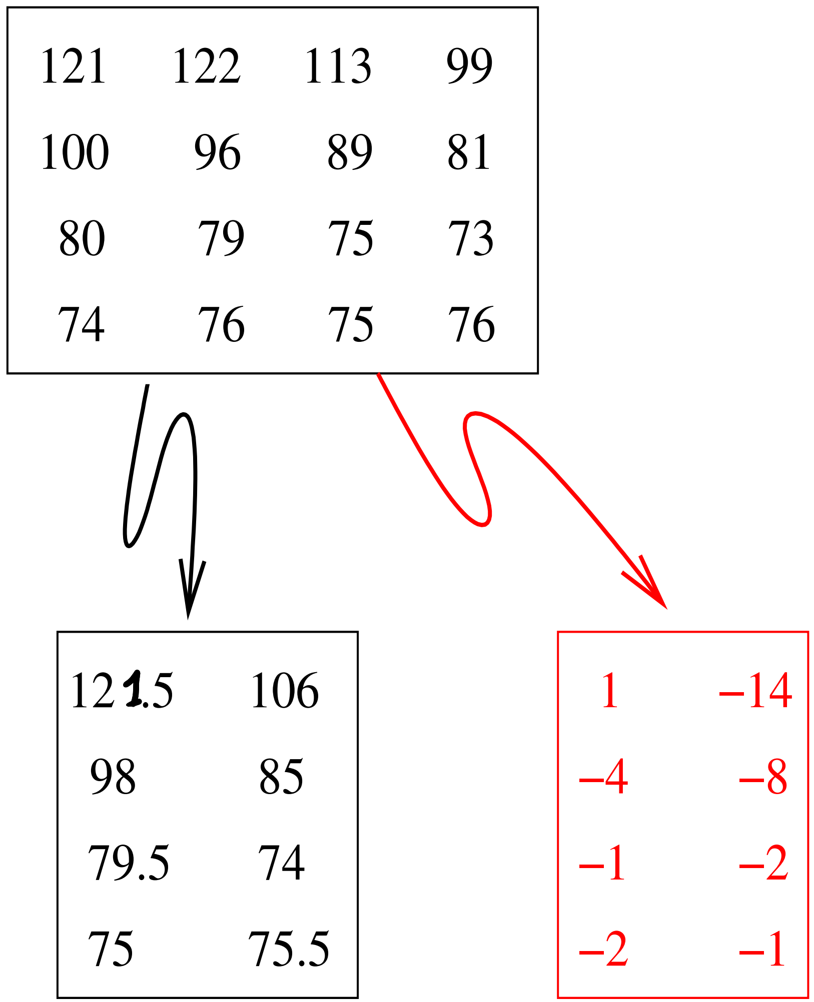
−2 −1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98 85
79.5 74
75 75.5
1 −14
−4 −8
−1 −2
−2 −1

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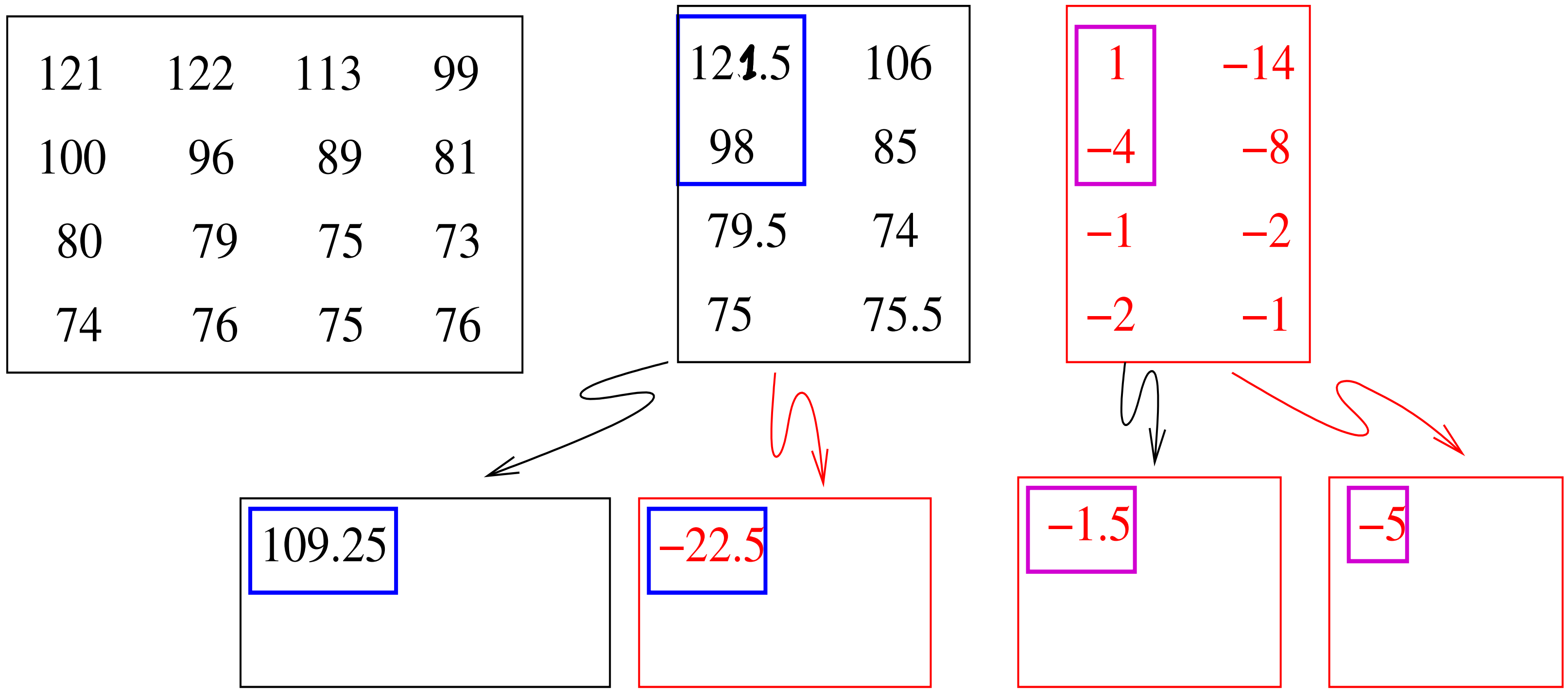
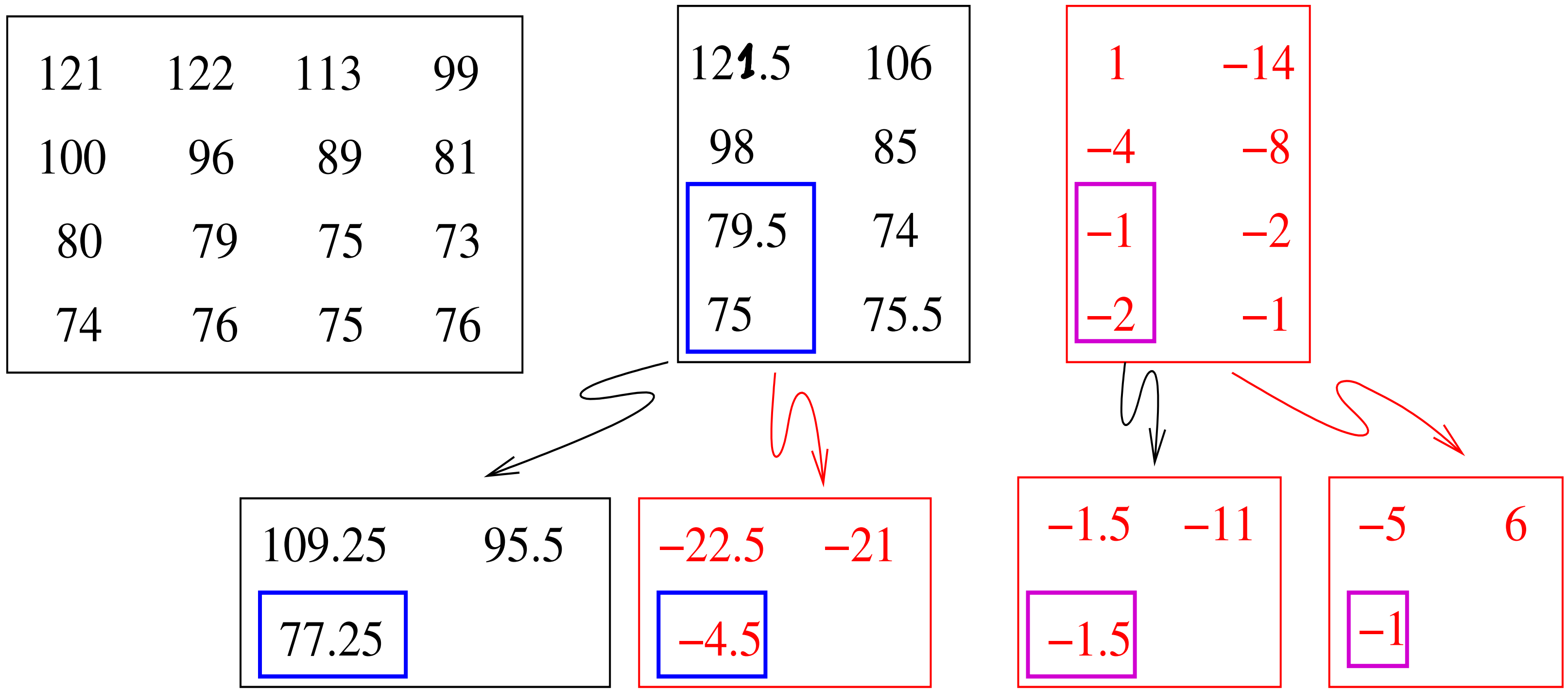
121.5	106
98	85
79.5	74
75	75.5

1	−14
−4	−8
−1	−2
−2	−1



averaging and differencing averaging and differencing

vertically, in each of the two tables



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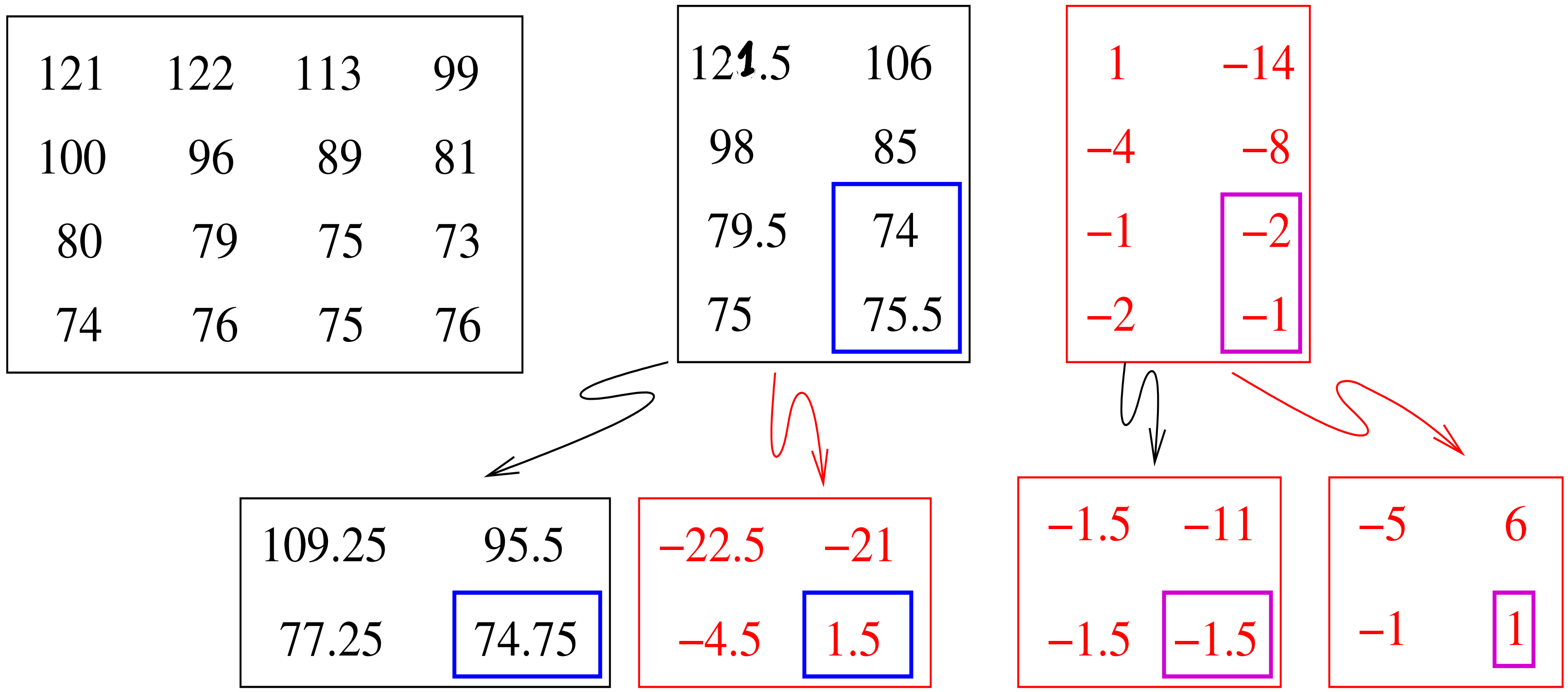
100 96 89 81
80 79 75 73
74 76 75 76
121.5 106
98 85
79.5 74
75 75.5
1 −14
−4 −8
−1 −2
−2 −1
109.25 95.5
77.25
−22.5 −21
−4.5
−1.5 −11
−1.5
−5 6
−1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98 85
79.5 74
75 75.5
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−2 −1
109.25 95.5
77.25 74.75
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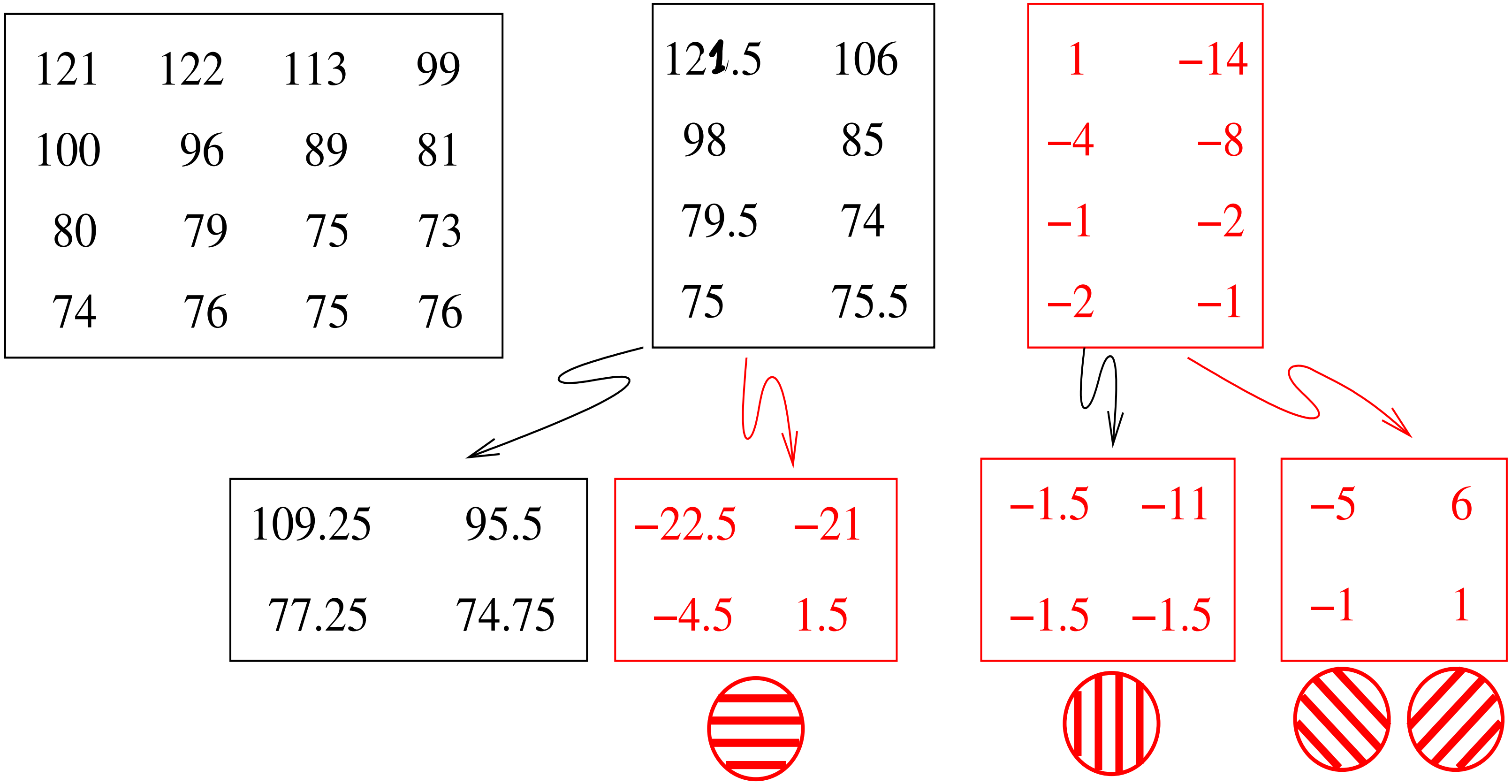
−1.5 −1.5
−5 6
−1 1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75 75.5
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−4 −8
−1 −2
−2 −1
109.25 95.5
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−1.5 −11
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−5 6
−1 1

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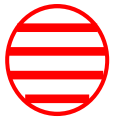
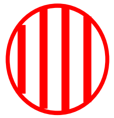
100	96	89	81
80	79	75	73
74	76	75	76

109.25	95.5
77.25	74.75

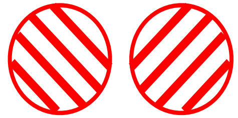
−22.5	−21
−4.5	1.5

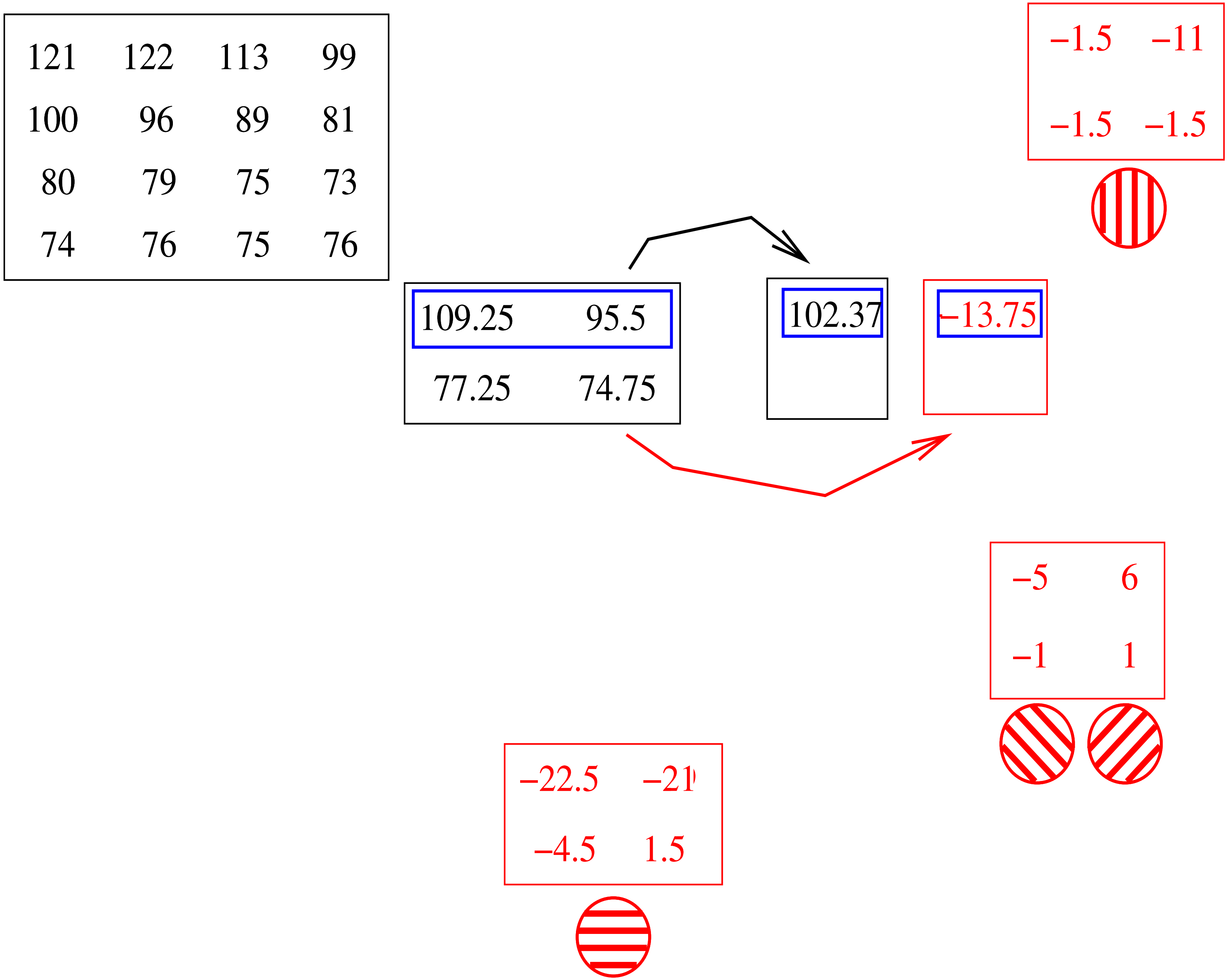


−1.5	−11
−1.5	−1.5



−5	6
−1	1

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−13.75
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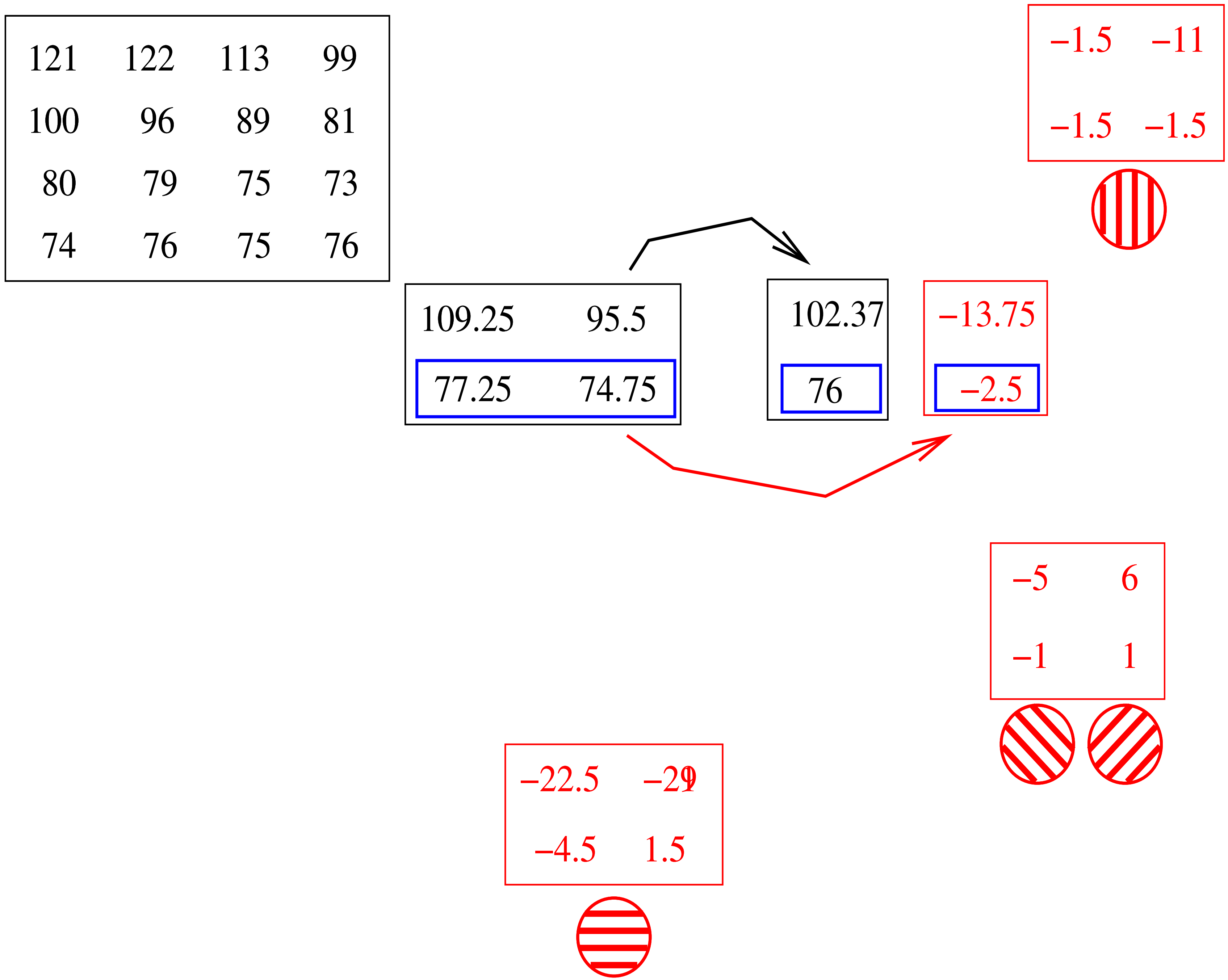
−1.5 −1.5
−5 6
−1 1
−22.5 −21
−4.5 1.5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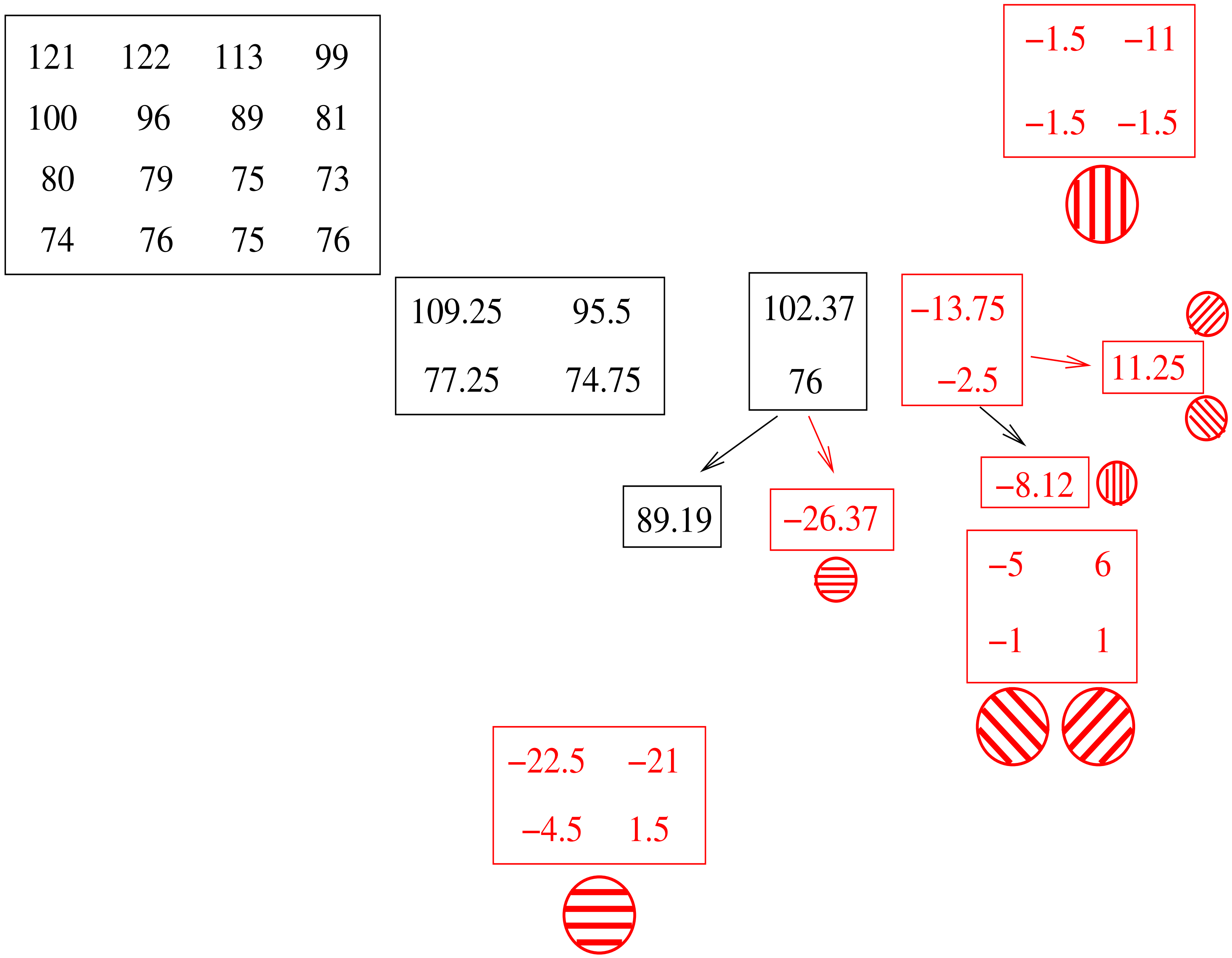
80 79 75 73
74 76 75 76
109.25 95.5
77.25 74.75
102.37
76
−13.75
−2.5
−1.5 −11
−1.5 −1.5
−5 6
−1 1
−22.5 −29
−4.5 1.5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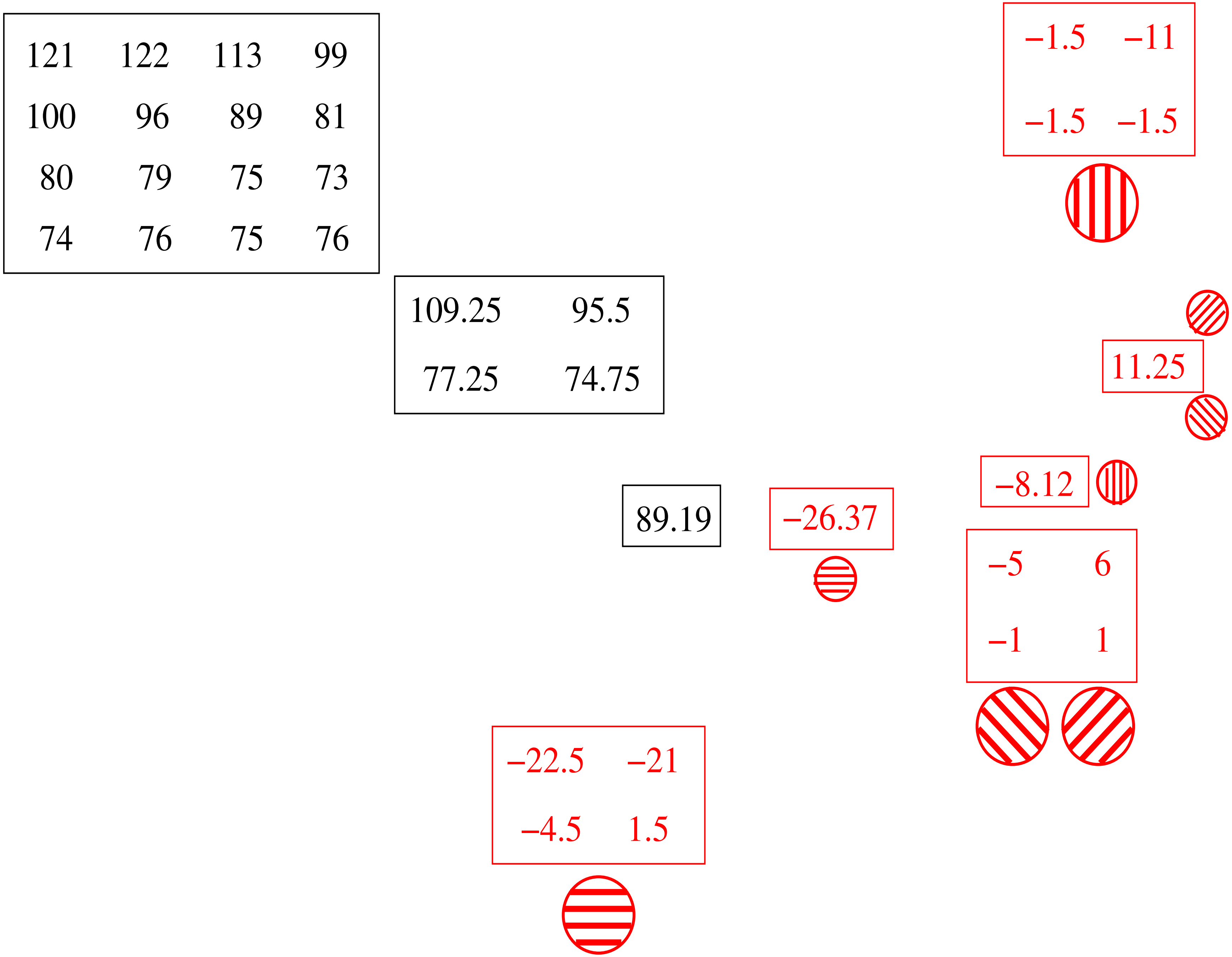
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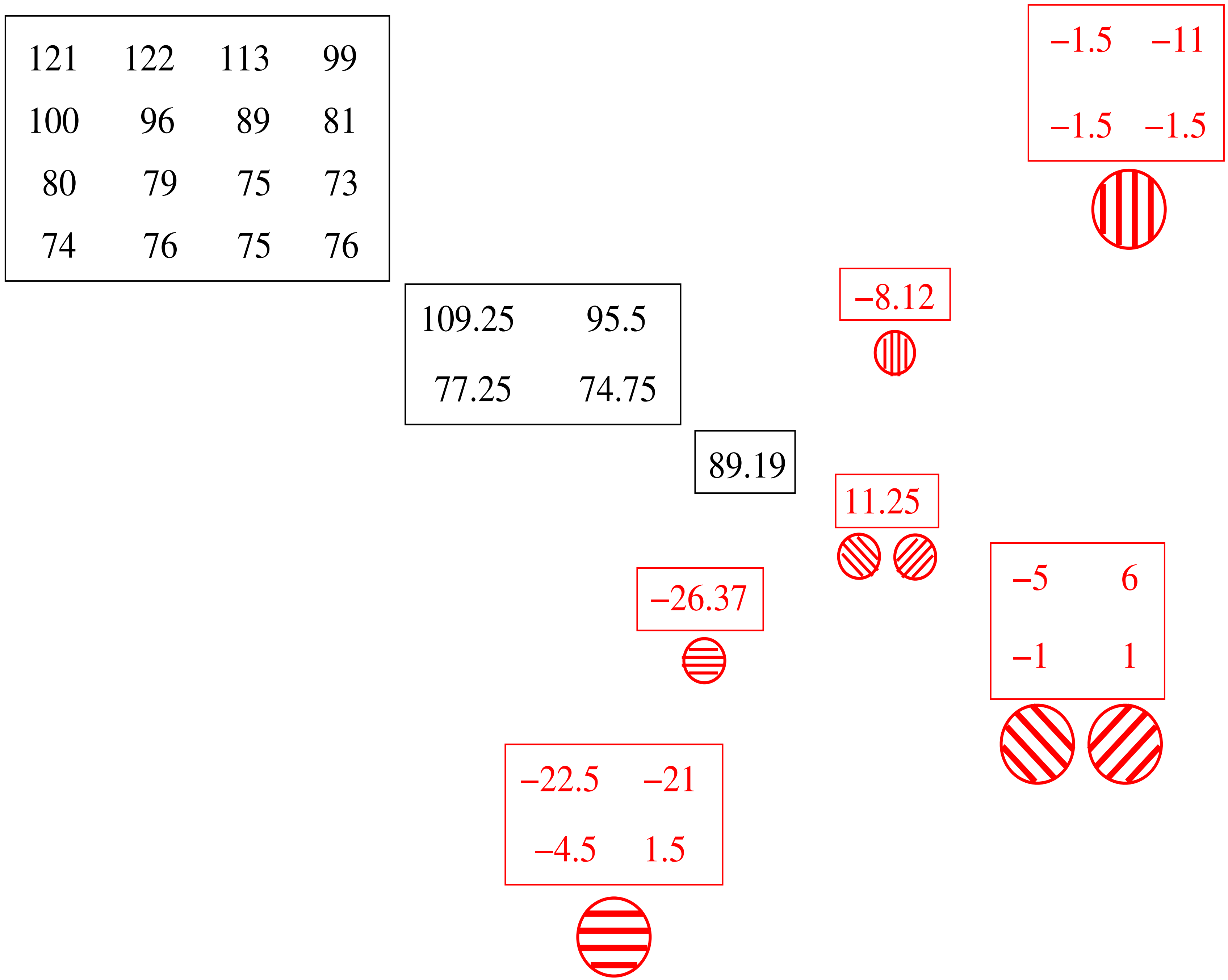
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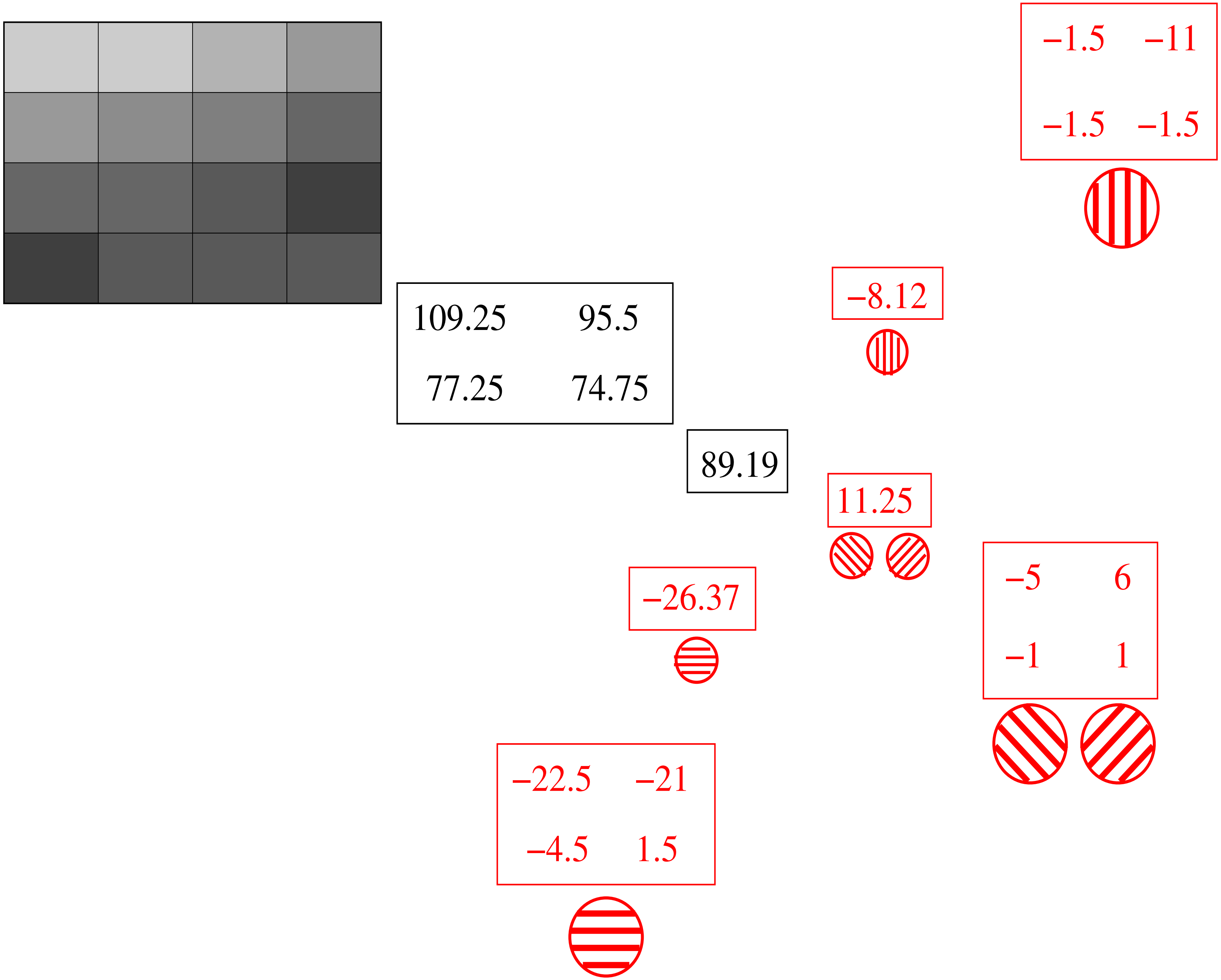

−1.5 −11
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−8.12
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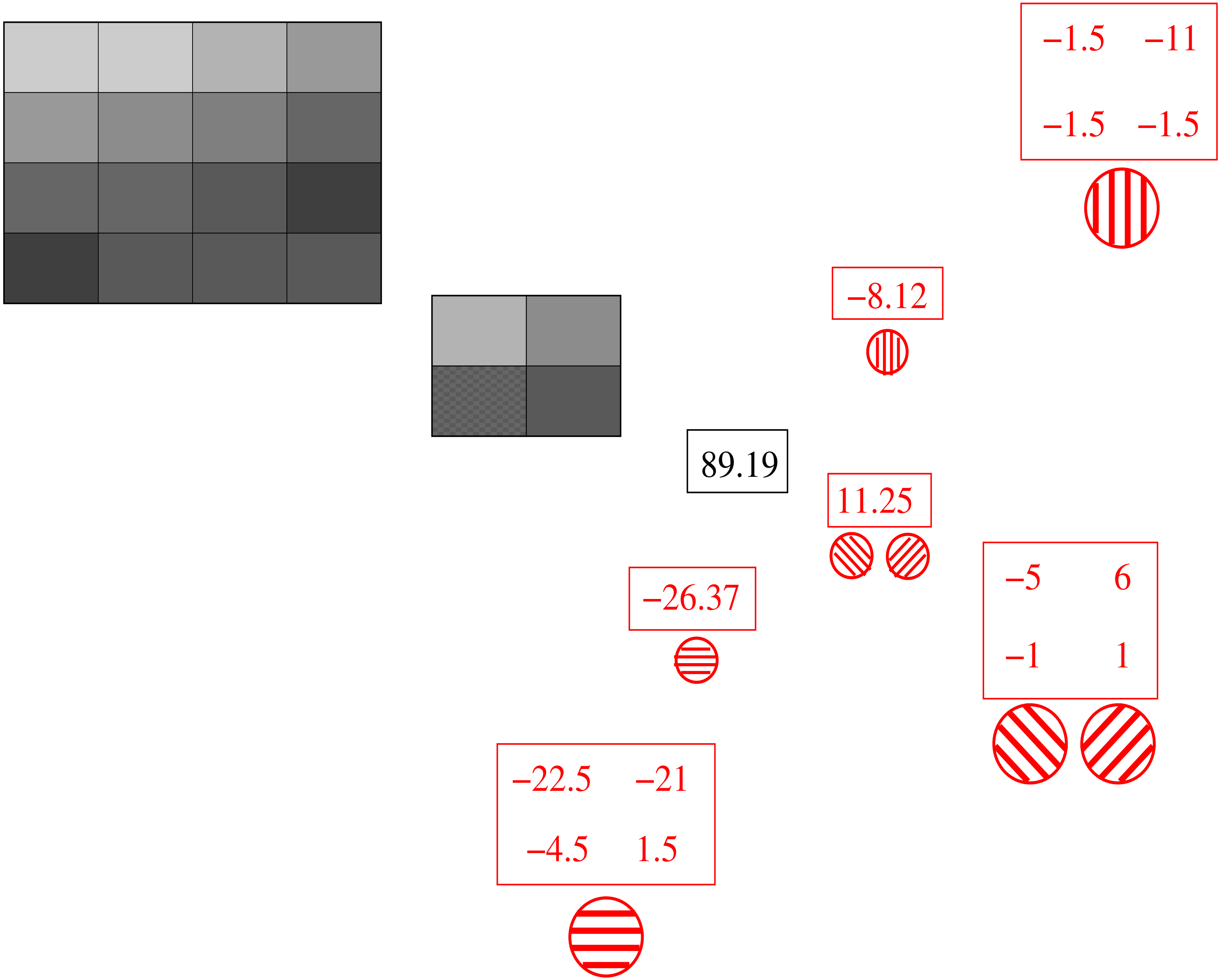

−1.5 −11
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−8.12
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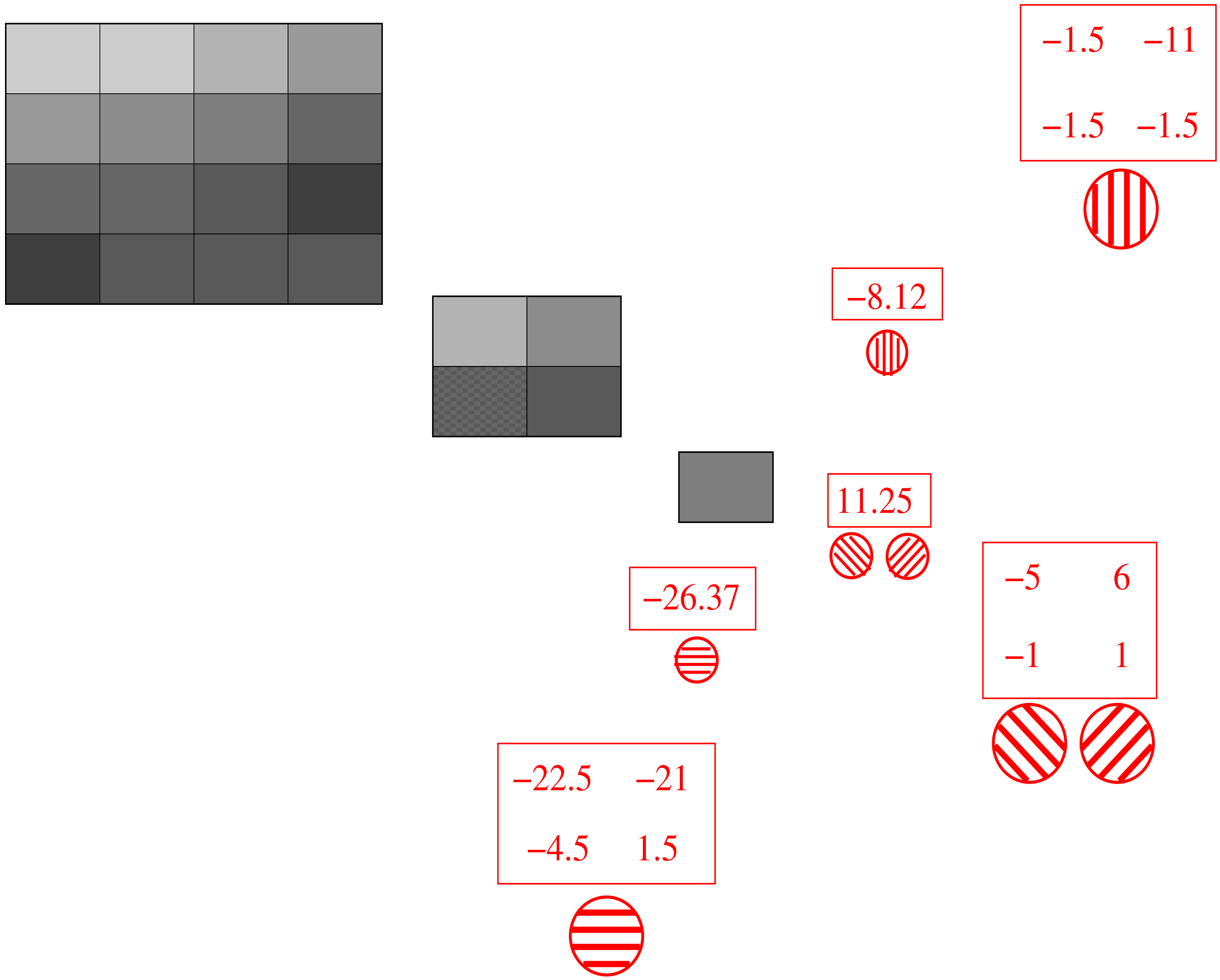


−1.5 −11
−1.5 −1.5
−8.12
11.25
−26.37
−5 6
−1 1
−22.5 −21
−4.5 1.5

Wavelet decomposition: graphical illustration of the algorithm and its properties

“Average” horizontally and vertically


14255

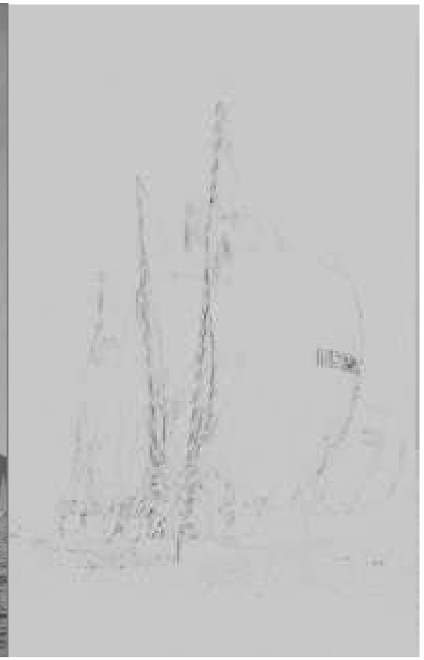
14255

“Difference” horizontally

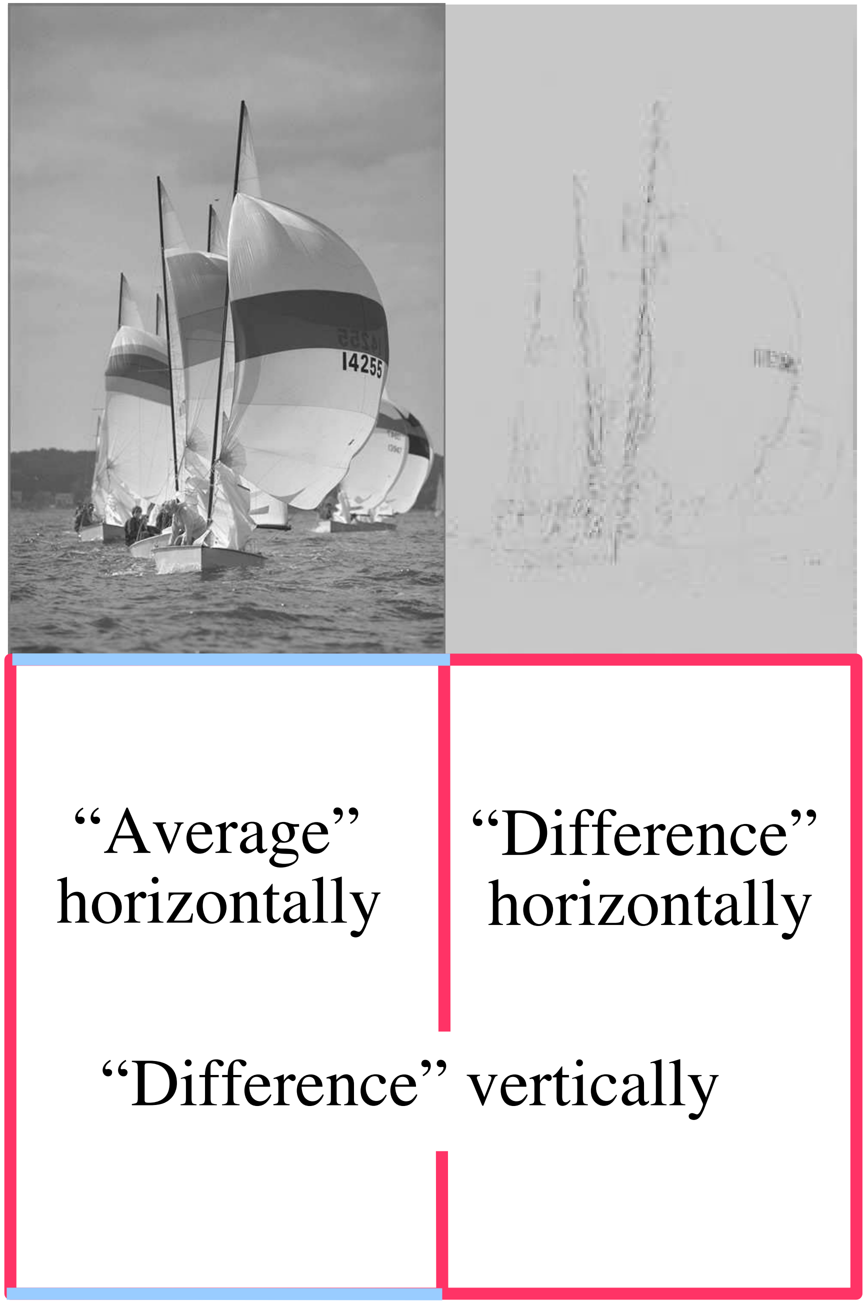
“Average” vertically


14255


14255



“Difference” vertically


"Average" horizontally
"Difference" horizontally
"Difference" vertically


14255


14255

Repeat
at the next
scale

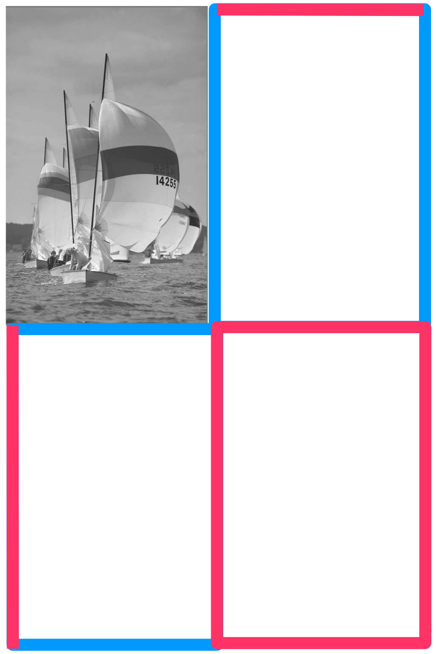

14255

14255


14255


14255


14255




14255

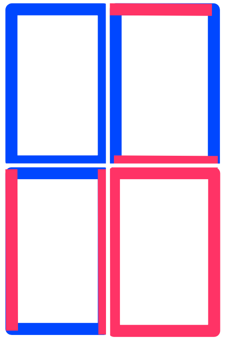

14255

Repeat


14255


14255


14255

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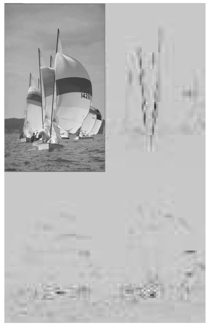


14255


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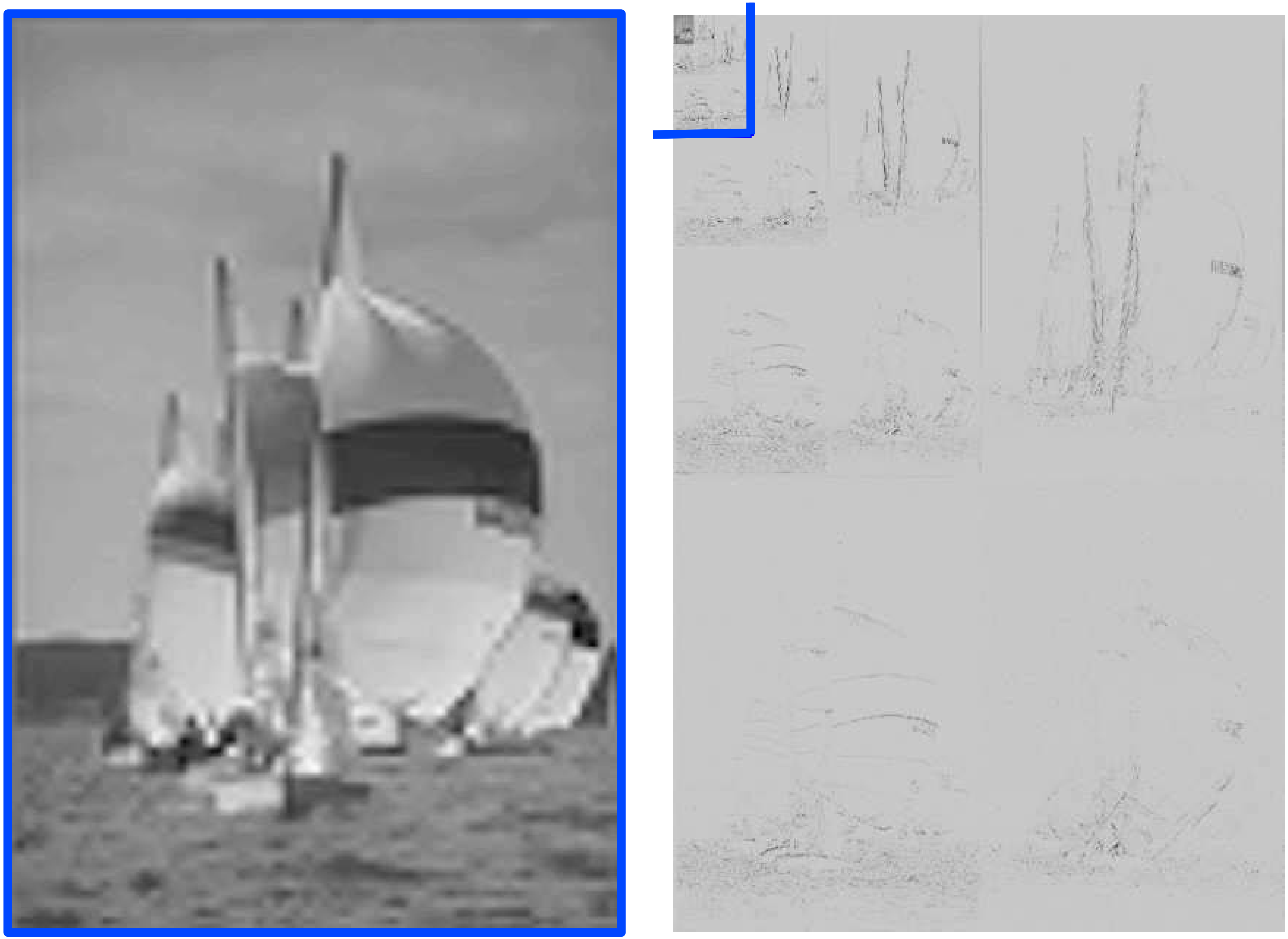
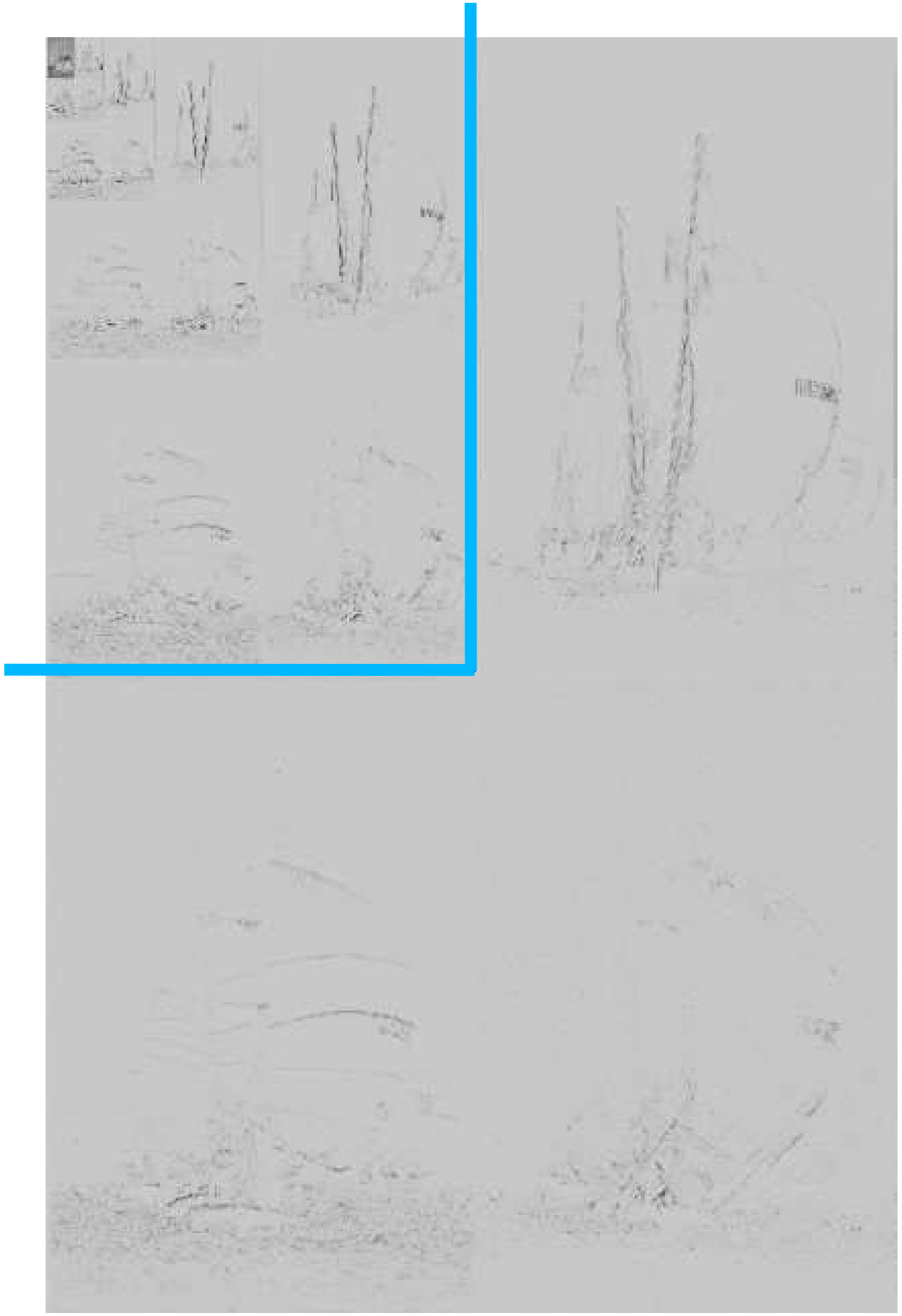
Meaning of the successive approximations

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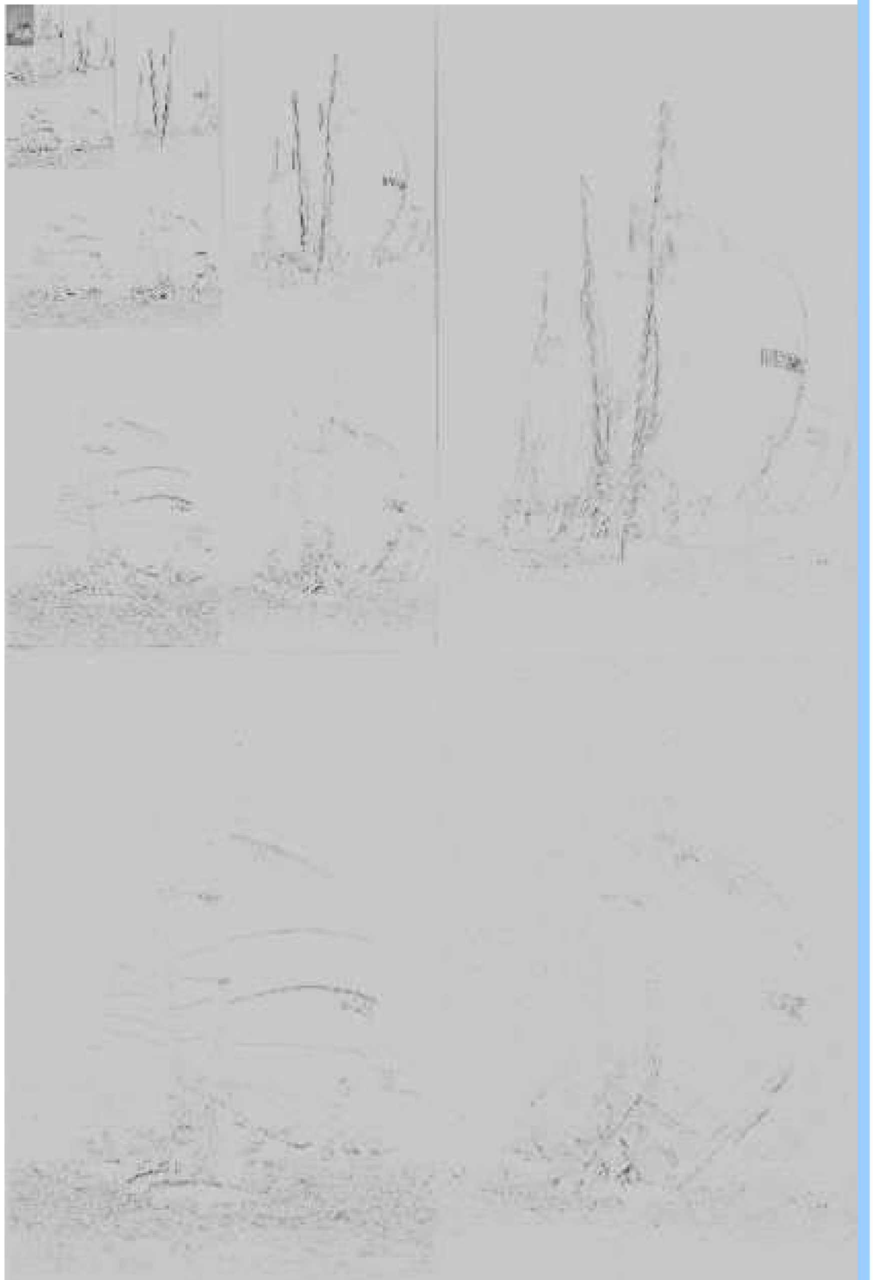
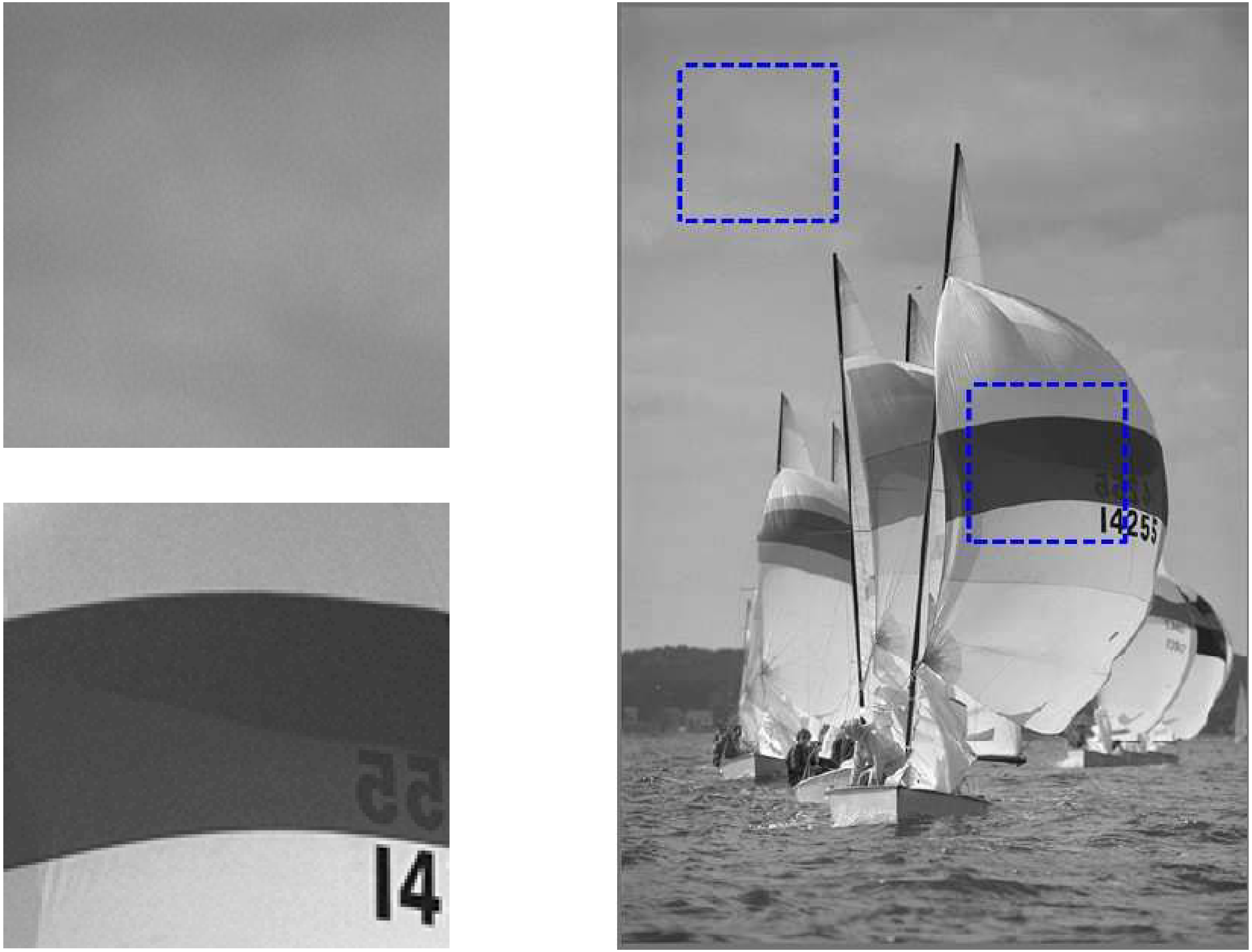


Local properties reflected by the successive approximations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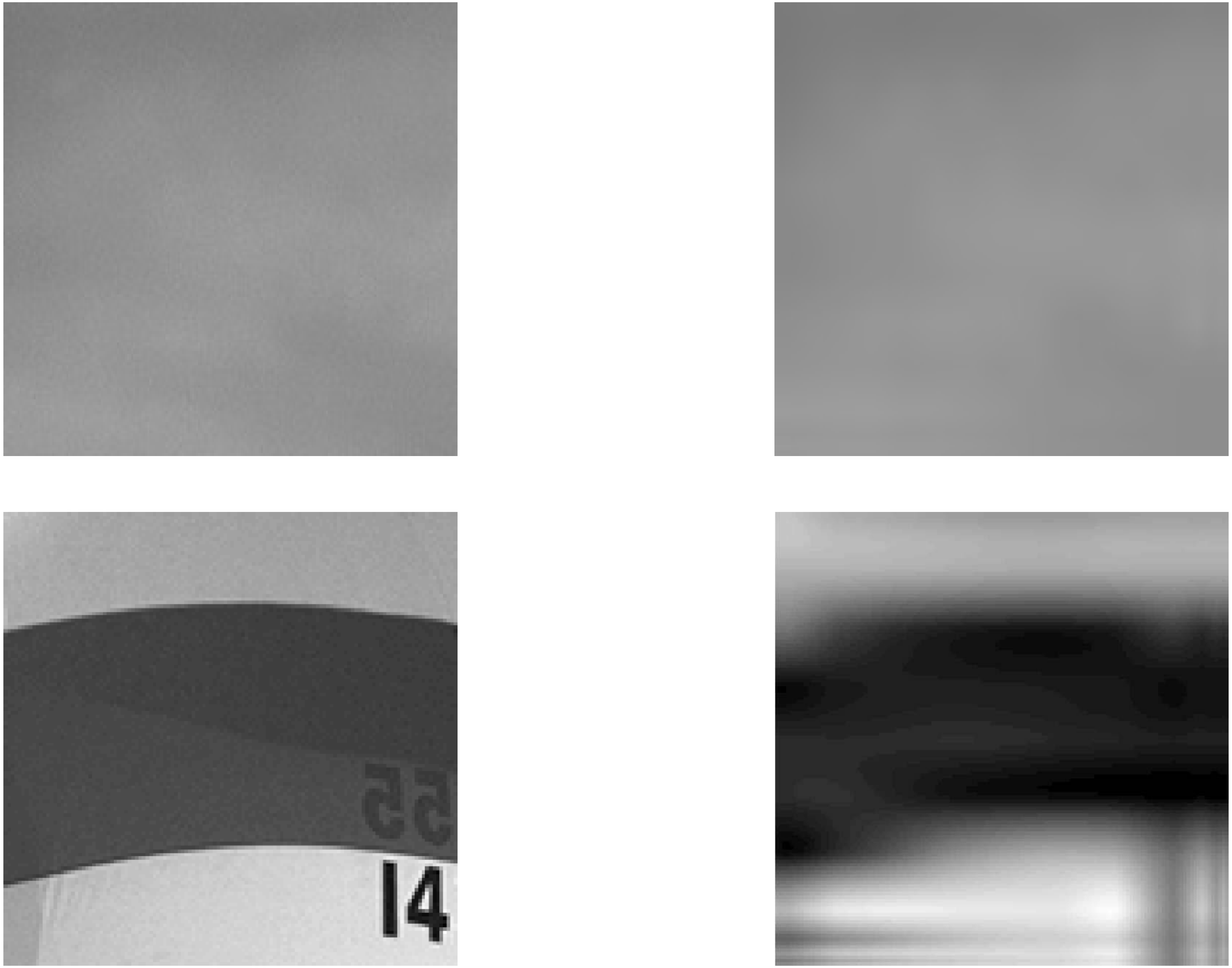
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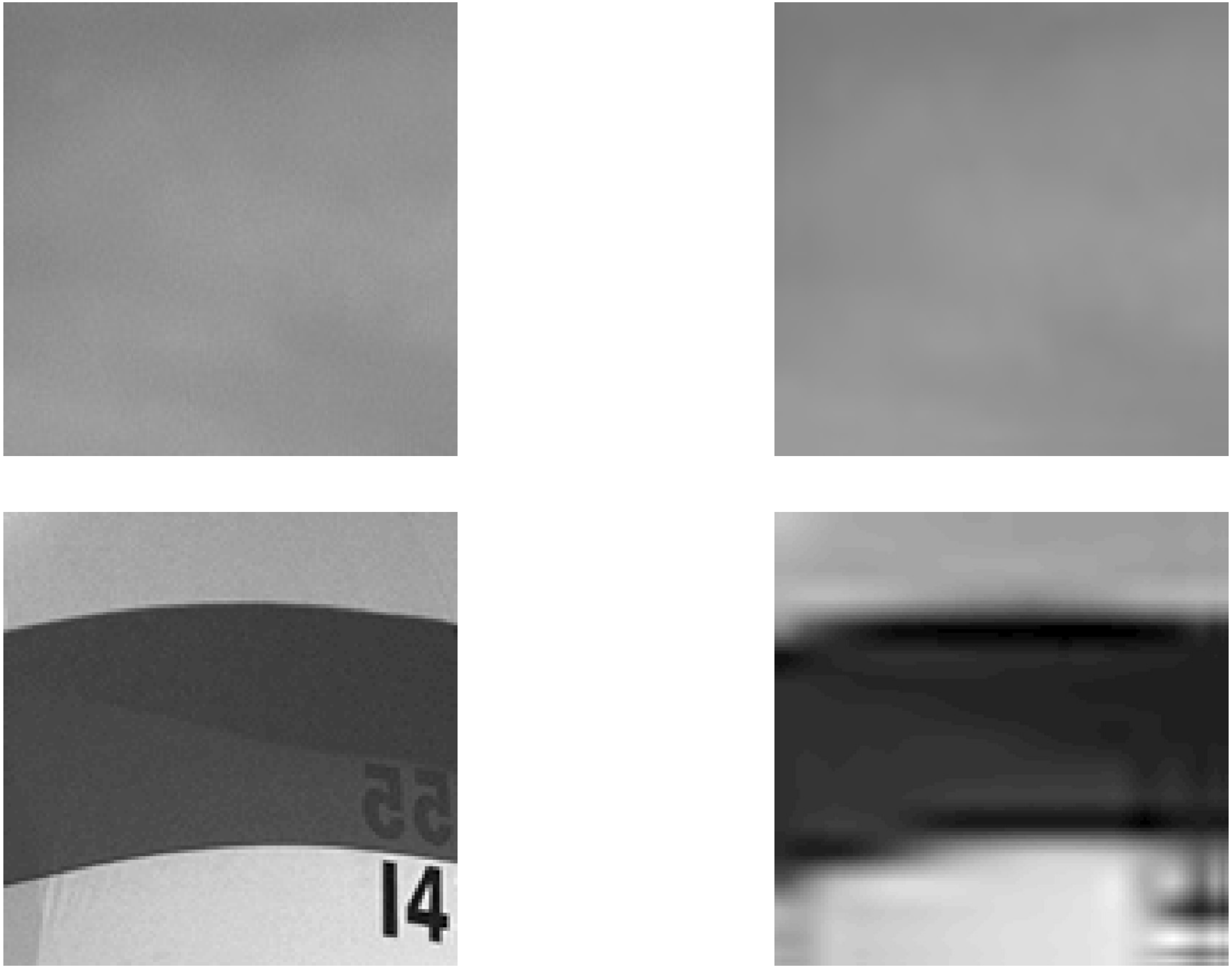
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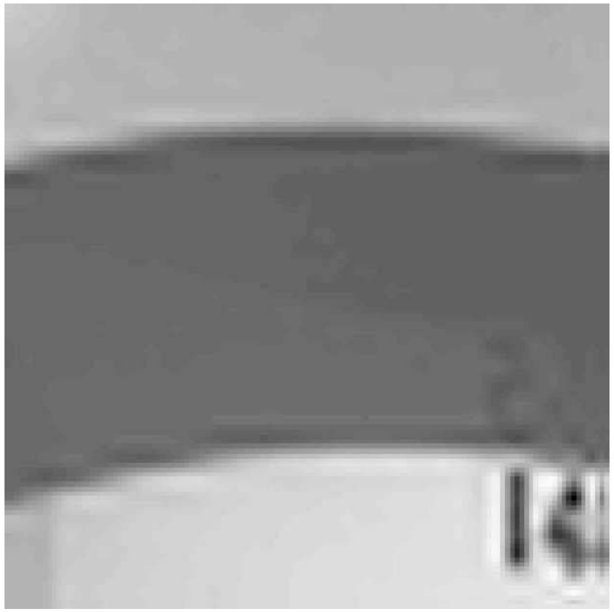

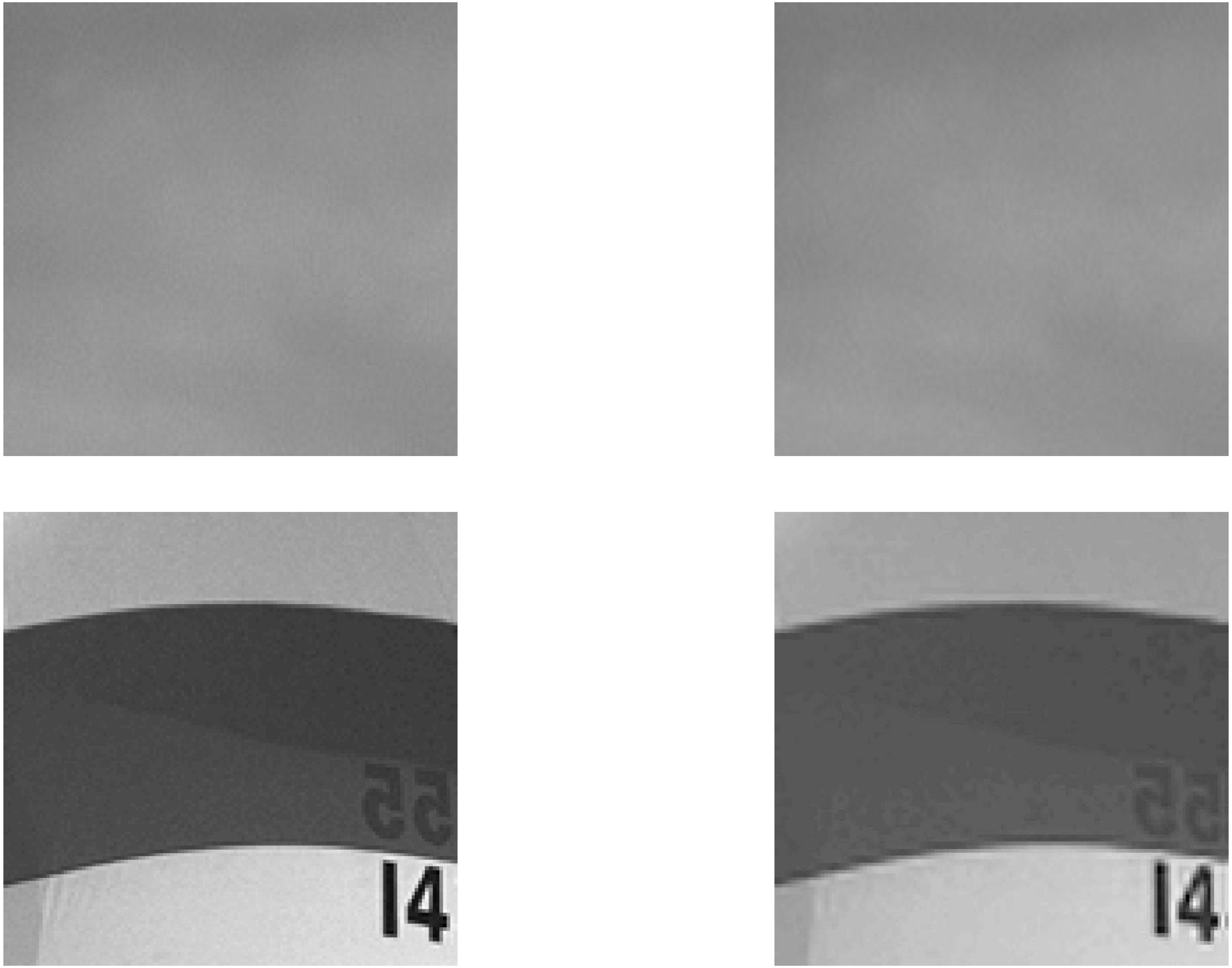
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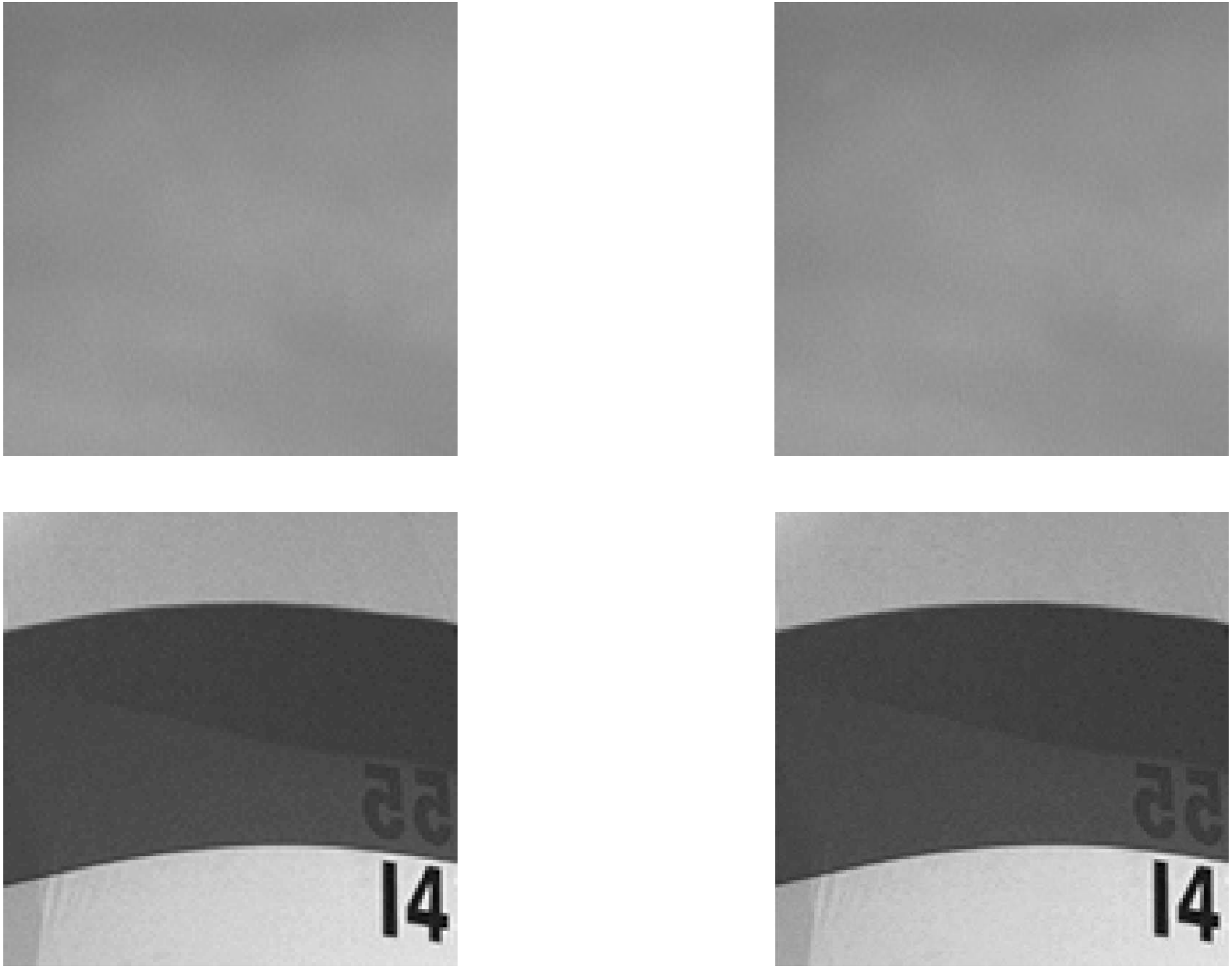
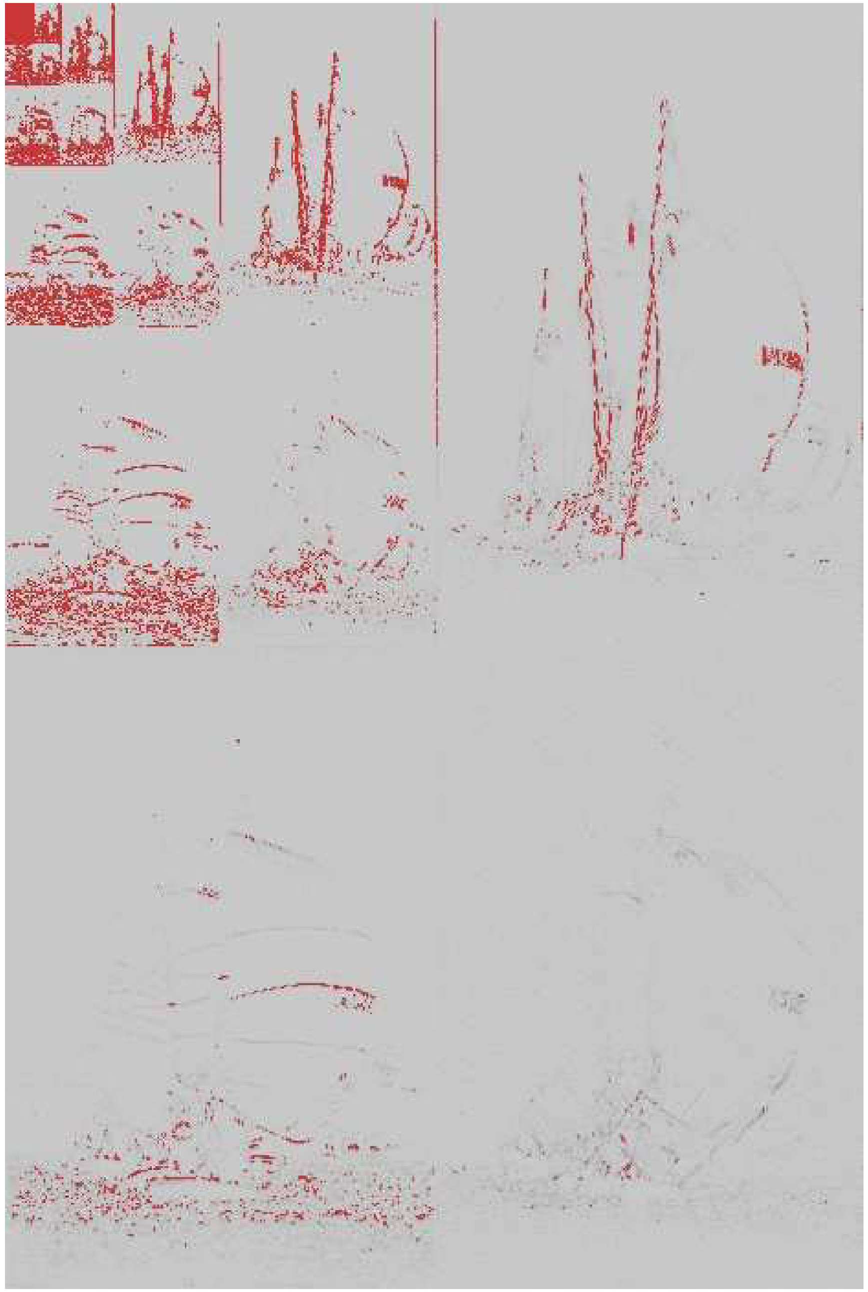
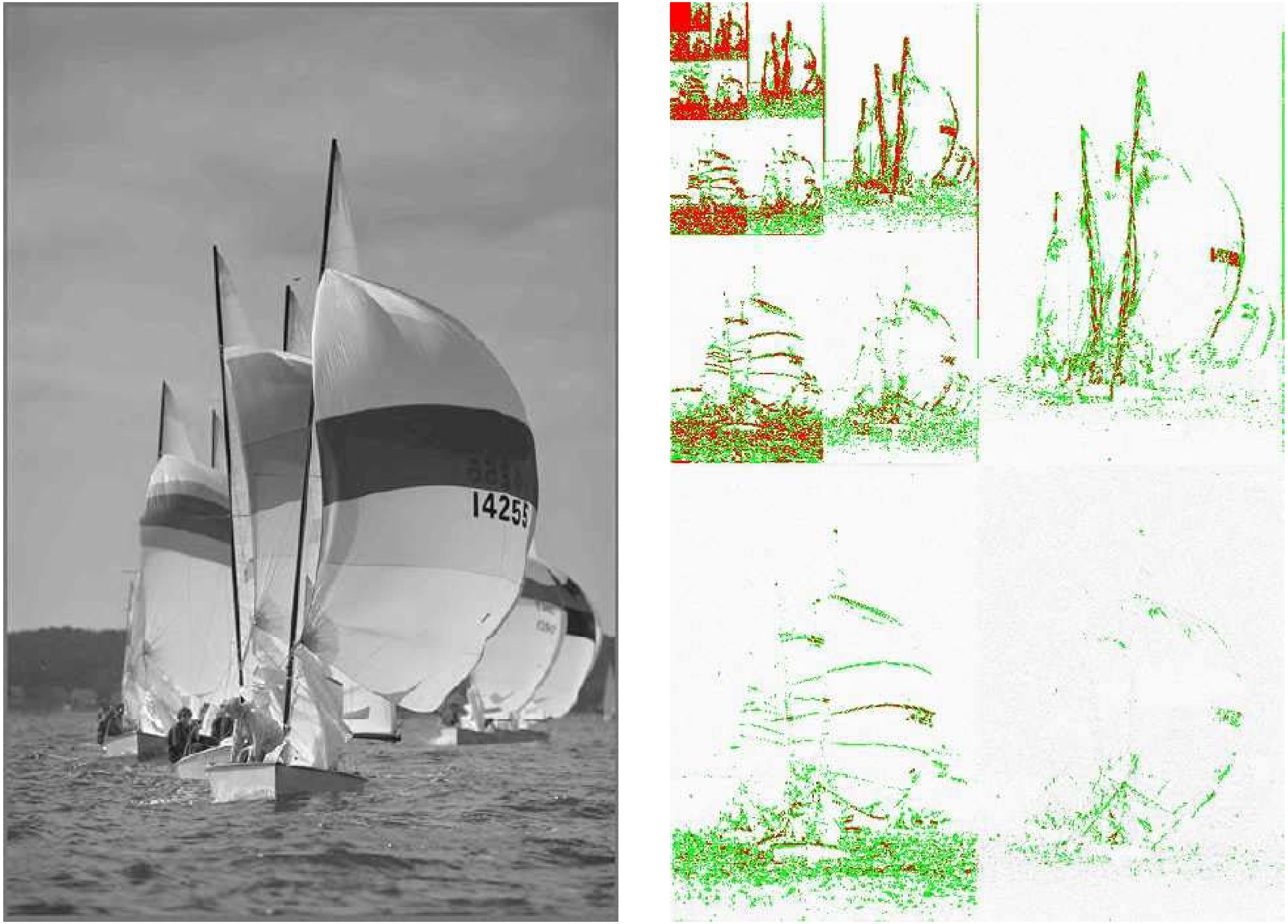

14
14

Compression


14255

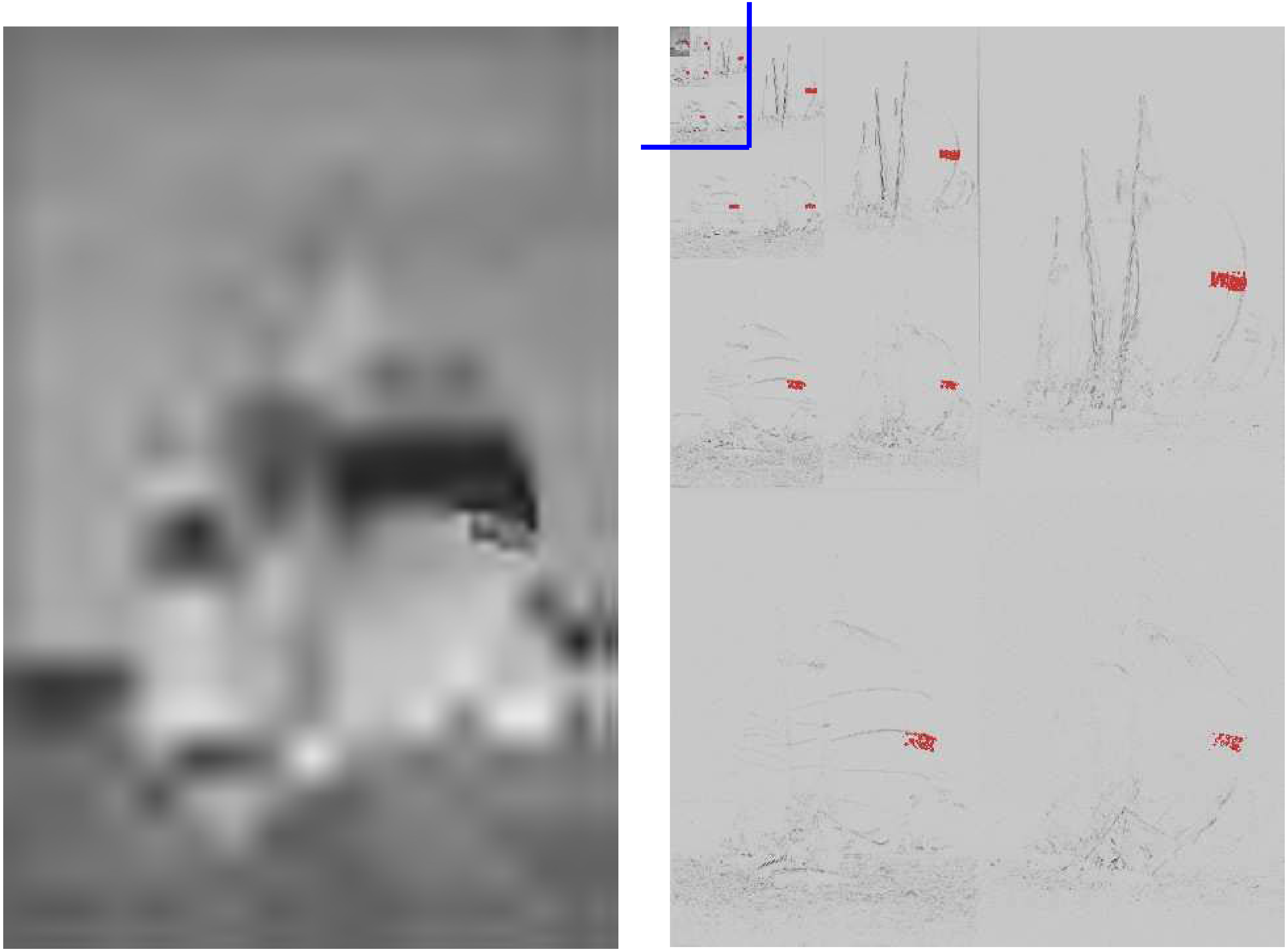


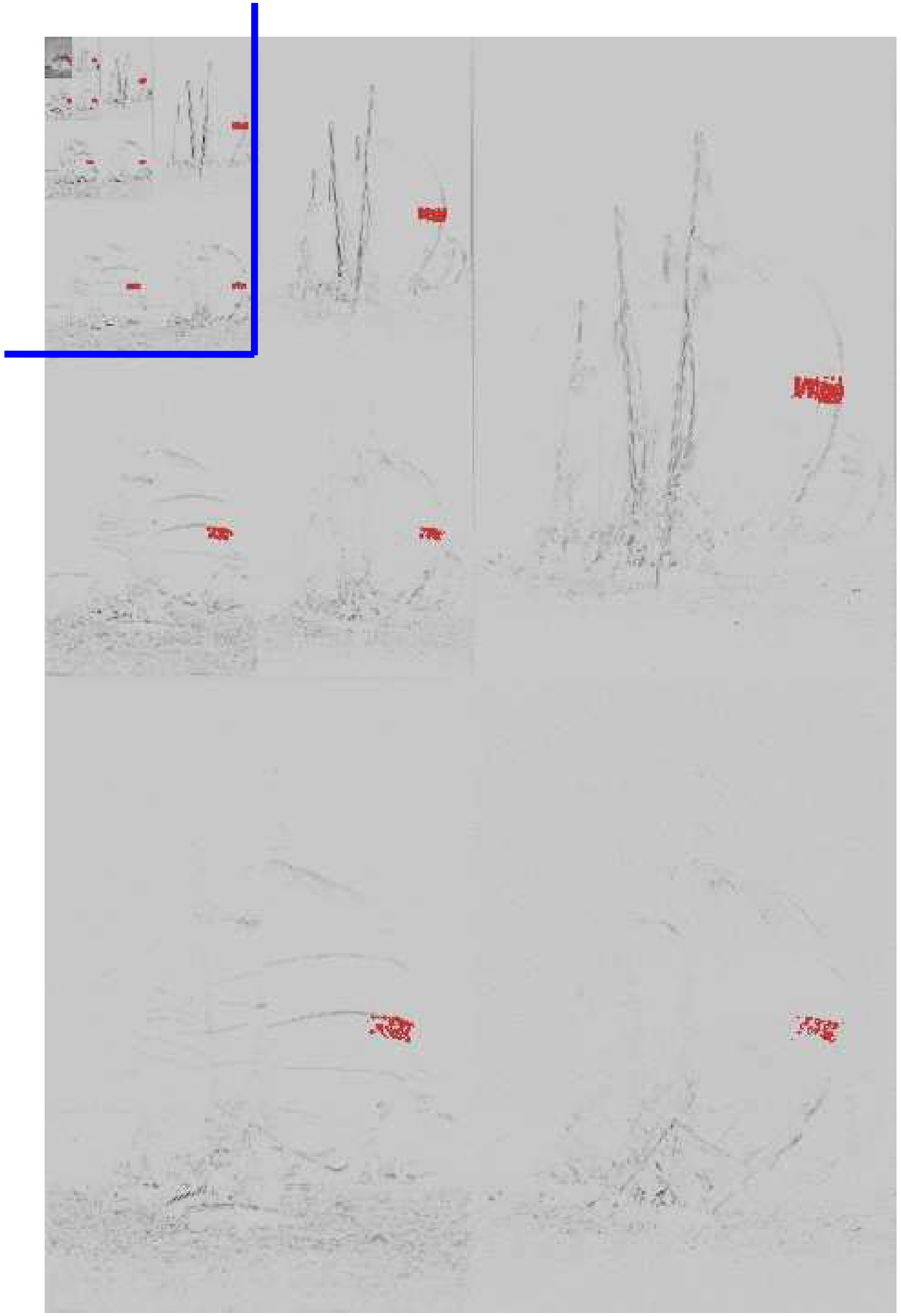
Compression Ratio: 3.3%



Compression Ratio: 10%

Localization:
fast, interactive retrieval of data



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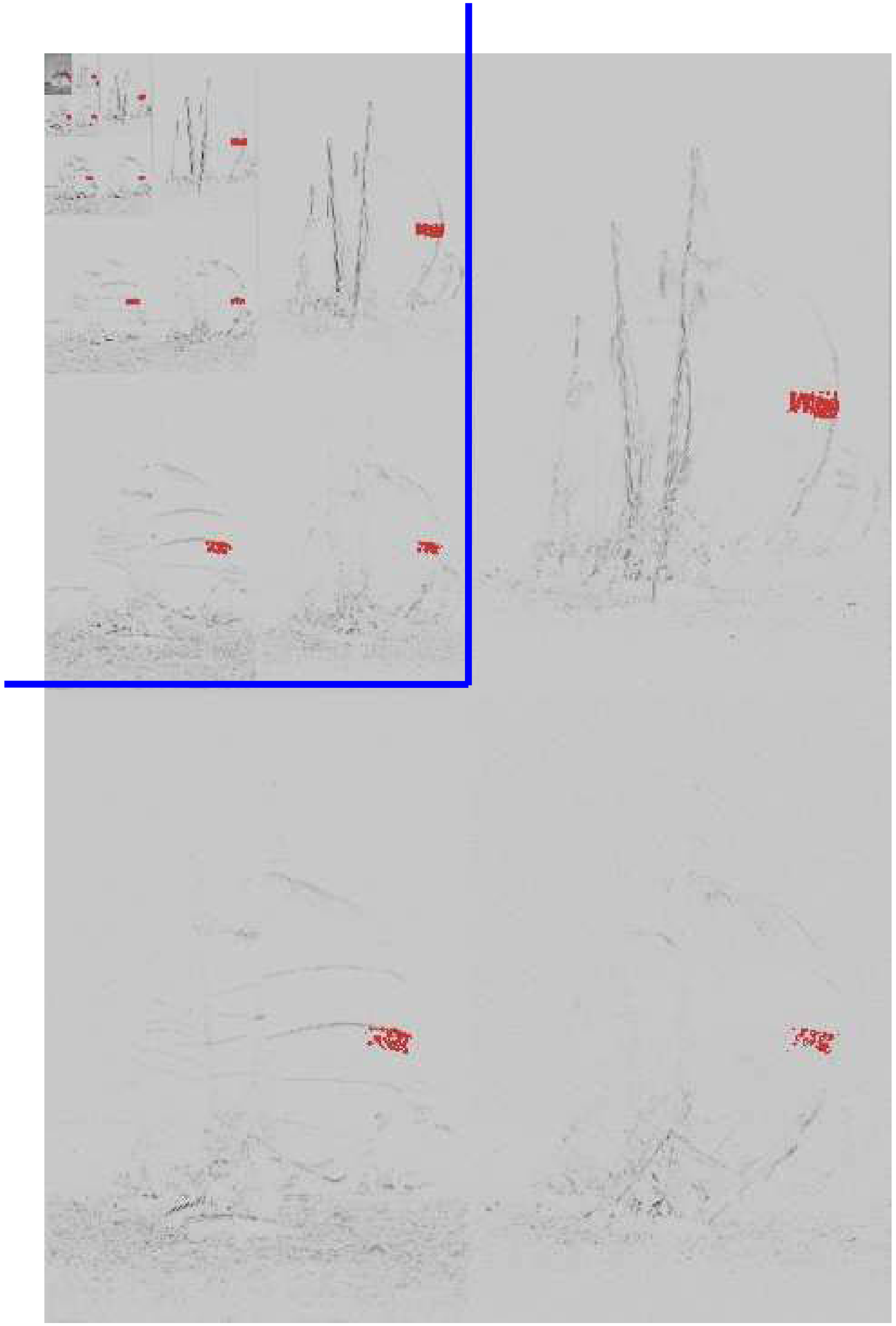


14255