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The State of the art in shearlet coorbit space theory contd.

S. Dahlke
Univ. Marburg, Germany

ICTP - TWAS School

on Coherent State

Transforms, Time-Frequency

and Time-Scale Analysis

Applications

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„The State of the Art

in Shearlet Coorbit

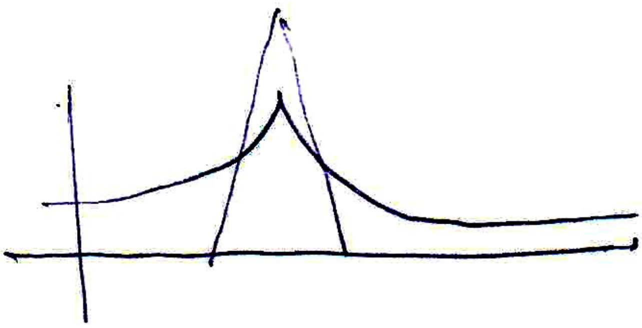
Theory"

0. Motivation

- analyse, approximate, decompose functions $f \in L_2(\mathbb{R}^d)$

- wavelets $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ ONB

$$\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k)$$



- drawbacks: isotropic

ridgelets, curvelets, contourlets, shearlets

2D:

(3)

$$\pi(a,s,t)\psi(x) = |a|^{-3/4}\,\psi(A_a^{-1} S_s^{-1}(x-t))$$

$$A_a = \begin{pmatrix} a & 0 \\ 0 & \operatorname{sign} a\sqrt{|a|} \end{pmatrix}, \quad S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$$SH_\psi f(a,s,t) := \langle f, \underbrace{\pi(a,s,t)\psi}_{\psi_{a,s,t}} \rangle$$

Example 0.1 $\quad x_1 + p x_2 = 0$

$\nu = \delta(x_1 + p x_2)$.

$s = -p, \quad t_1 = -p t_2$

$$SH_\psi \nu(a,s,t) \sim |a|^{-\frac{1}{4}}, \quad |a| \to 0 \qquad (0.1)$$

rapidly decaying else

1. Multivariate Shearlet Transform
2. Coorbit Theory
3. Shearlet Coorbit Spaces

1. Multivariate Shearlet Transform

shear model

$V = W \oplus W'$, $v = w + w'$, $S(v) = w + (w' + Mw')$

$$S = \begin{pmatrix} I & M \\ 0 & I \end{pmatrix} \quad (1.1)$$

$$S_s := \begin{pmatrix} 1 & s^T \\ 0_{d-1} & I_{d-1} \end{pmatrix} \quad (1.2)$$

$$A_a = \begin{pmatrix} a & 0_{d-1}^T \\ 0_{d-1} & \operatorname{sign}(a)|a|^{\frac{1}{d}} I_{d-1} \end{pmatrix} \quad (1.3)$$

$$SH_\psi f(a,s,t) := \langle f, |a|^{\frac{1}{2d}-1} \psi(A_a^{-1} S_s^{-1}(\cdot - t)) \rangle \quad (1.4)$$

<u>Lemma 1.1</u> $\mathbb{R}^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$ with

$$(a,s,t) \circ (a',s',t') := (aa', s + |a|^{1-1/d} s', t + S_s A_a t') \quad (1.5)$$

is a locally compact group FSG.

$$d\mu_\ell(a,s,t) = \frac{1}{|a|^{d+1}} \, da \, ds \, dt \quad (1.6)$$

$$d\mu_r(a,s,t) = \frac{1}{|a|} \, da \, ds \, dt$$

Lemma 1.2

$$\begin{aligned} \pi : \quad FSG &\longrightarrow \mathcal{U}(L_2(\mathbb{R}^d)) \\ (a,s,t) &\longmapsto \pi(a,s,t) \end{aligned} \tag{1.7}$$

$$\pi(a,s,t)f = |a|^{\frac{1}{2d}-1} f(A_a^{-1} S_s^{-1}(x-t))$$

is a continuous unitary representation of FSG in $L_2(\mathbb{R}^d)$

Theorem 1.3 Let $\psi \in L_2$ satisfy

$$C_\psi := \int_{\mathbb{R}^d} \frac{|\hat{\psi}(\xi)|^2}{|\xi_1|^d} \, d\xi < \infty \tag{1.8}$$

Then:

$$\int_{FSG} |\langle f, \psi_{a,s,t} \rangle|^2 \, d\mu = C_\psi \, \|f\|^2_{L_2} \tag{1.9}$$

Remark 1.4 π is

- multiple of an isometry
- irreducible
- square integrable

Theorem 1.5

i.) $$f = SH_\psi^* SH_\psi(f) = C_\psi^{-1} \int_{FSG} \langle f, \psi_{a,s,t} \rangle \psi_{a,s,t} \, d\mu \qquad (1.10)$$

ii.) $C_\psi = 1 \Rightarrow$

$$SH_\psi(f) * SH_\psi(\psi) = SH_\psi(f) \qquad (1.11)$$

2. Coorbit Theory

- new smoothness spaces, decay of the voice transform
- (Banach) frames

General setting

- G locally compact top. group
  dg (left) Haar measure
- π: square integrable representation in $\mathcal{H}$
  $C_\psi = 1$
- $\omega : G \longrightarrow \mathbb{R}^+$

$$\omega(g \circ h) \leq \omega(g)\,\omega(h), \quad \omega \geq 1$$
$$\omega(g) = \omega(g^{-1})\,\Delta(g^{-1}) \tag{2.1}$$

- ψ admissible vector

$$V_\psi : \mathcal{H} \longrightarrow L_2(G, dg)$$
$$f \longmapsto \langle f, \pi(g)\psi \rangle_{\mathcal{H}} \tag{2.2}$$

voice transform

2.1. Coorbit spaces

$$L_{p,\omega}(G) := \left\{ F \,\middle|\, \left(\int_G |F(g)|^p \omega(g)^p dg \right)^{1/p} < \infty \right\} \quad (2.3)$$

Assumption 2.1

$$A_\omega := \left\{ \psi \,\middle|\, \int_G |V_\psi(\psi)(g)|\, \omega(g)\, dg < \infty \right\} \quad (2.4)$$

is nontrivial

$$(V_\psi \psi)(g) =: K(g) \quad (2.5)$$

$$\mathcal{H}_{1,\omega} := \{ f \in \mathcal{H} \mid V_\psi(f) = \langle f, \pi(\cdot)\psi \rangle_{\mathcal{H}} \in L_{1,\omega} \}$$

$$\|f\|_{\mathcal{H}_{1,\omega}} := \|V_\psi f\|_{L_{1,\omega}} \quad (2.6)$$

Proposition 2.2

i.) $\mathcal{H}_{1,\omega}$ is a Banach space

ii.) $$\mathcal{H}_{1,\omega} \hookrightarrow \mathcal{H} \hookrightarrow \tilde{\mathcal{H}}_{1,\omega} \quad (2.7)$$

iii.) $f \in \mathcal{H}_{1,\omega} \Rightarrow \pi(h) f \in \mathcal{H}_{1,\omega} \ \forall h \in G$

iv.) $$V_\psi(\pi(g) f) = L_g V_\psi f \quad (2.8)$$

Proposition 2.2. iii) $\Rightarrow$

$$V_\psi(f)(g) := \langle f, \pi(g)\psi \rangle_{\tilde{\mathcal{H}}_{1,\omega} \times \mathcal{H}_{1,\omega}} \quad (2.9)$$

is well-defined!

Def 2.3

$$m(g \circ h \circ k) \le w(g)\, m(h)\, w(k) \qquad (2.10)$$

Coorbit space $\mathcal{H}_{p,m}$:

$$\mathcal{H}_{p,m} := \{ f \in \mathcal{H}_{1,a}^{\sim} : V_\psi f \in L_{p,m}(G) \}$$

$$\| f \|_{\mathcal{H}_{p,m}} := \| V_\psi f \|_{L_{p,m}}$$

Remarks 2.4 i.) $\mathcal{H}_{p,m}$ independent of ψ

ii.) $(2.10) \Rightarrow$

$$L_{p,m} * L_{1,w} \subseteq L_{p,m}$$
$$L_{1,w} * L_{p,m} \subseteq L_{p,m} \qquad (2.11)$$

$$\mathcal{M}_{p,m} := \{ F \in L_{p,m} \mid F * K = F \} \qquad (2.12)$$

Theorem 2.5 (Correspondence Principle)

V_ψ induces isomorphism

$$V_\psi : H_{p,m} \longleftrightarrow \mathcal{M}_{p,m} \qquad (2.13)$$

Example 2.6

i.) wavelet transform, affine group

$\mathcal{H} = L_2(\mathbb{R}) \quad m(a,t) = |a|^{-s} \Rightarrow$

$\mathcal{H}_{p,m} = \dot{B}^{s+\frac{1}{2}-\frac{1}{p}}_{p,p}(\mathbb{R})$ homogeneous Besov space

$$(\Delta_h f)(x) = f(x+h) - f(x)$$

$$\omega_r(f,t)_{L_p} := \sup_{|h|\le t} \|\Delta_h^r f\|_{L_p}$$

$$|f|_{\dot{B}^s_{q,p}} := \left(\int_0^\infty [t^{-s}\omega_r(f,t)_{L_p}]^q \frac{dt}{t}\right)^{1/q}$$

$r \geq \lfloor s \rfloor$

ii.) Gabor Transform, Weyl-Heisenberg group

$\mathcal{H} = L_2(\mathbb{R})$, $\quad m(t,\xi) = (1+|\xi|)^{2s}$

$\mathcal{H}_{p,m} = M^s_{pp}$ modulation space

2.2. Discretisation

Idee: find $X = (g_\lambda)_{\lambda \in \Lambda}$ in G

$$f = \sum_{\lambda \in \Lambda} c_\lambda(f)\, \pi(g_\lambda)\psi \qquad (2.14)$$

X <u>$\mathcal{U}$-dense</u> $\iff \bigcup_{\lambda \in \Lambda} g_\lambda \cdot \mathcal{U} = G, \; e \in \mathcal{U}$ (2.15)

<u>$\mathcal{U}$-oscillation</u>

$$\operatorname{osc}_{\mathcal{U}}(g) := \sup_{u \in \mathcal{U}} |K(ug) - K(g)| \qquad (2.16)$$

$$B_\omega := \{\psi \in L_2 \mid K \in \mathcal{W}(C_0, L_{1,\omega})\}$$

(2.17)

$$\mathcal{W}(C_0, L_{1,\omega}) = \{F \mid \|(L_g \chi_Q) F\|_{L_\infty} \in L_{1,\omega}\}$$

Theorem 2.7 $\psi \in \mathcal{B}_\omega$, $e \subset \mathcal{U}$ sufficiently small, X $\mathcal{U}$-dense

i) Atomic decomposition: $f \in \mathcal{H}_{p,m} \Rightarrow$

$$f = \sum_{\lambda \in \Lambda} c_\lambda(f) \pi(g_\lambda)\psi \qquad (2.18)$$

$$\| (c_\lambda(f))_{\lambda\in\Lambda} \|_{\ell_{p,m}} \lesssim \| f \|_{\mathcal{H}_{p,m}}$$

$((d_\lambda f))_{\lambda\in\Lambda} \in \ell_{p,m} \Rightarrow$

$$f = \sum_{\lambda\in\Lambda} d_\lambda(f) \pi(g_\lambda)\psi \in \mathcal{H}_{p,m} \quad \text{and}$$

$$\| f \|_{\mathcal{H}_{p,m}} \lesssim \| (d_\lambda(f))_{\lambda\in\Lambda} \|_{\ell_{p,m}} \qquad (2.19)$$

ii) Banach frames

$$\| f \|_{\mathcal{H}_{p,m}} \sim \| (\langle f, \pi(g_\lambda)\psi \rangle_{\tilde{\mathcal{H}}_{1,\omega} \times \mathcal{H}_{1,\omega}})_{\lambda\in\Lambda} \|_{\ell_{p,m}} \qquad (2.20)$$

$\exists$ reconstruction operator $\mathcal{R}$

$$\mathcal{R}\left((\langle f, \pi(g_\lambda)\psi \rangle_{\tilde{\mathcal{H}}_{1,\omega} \times \mathcal{H}_{1,\omega}})_{\lambda\in\Lambda} \right) = f \qquad (2.21)$$

3. Shearlet Coorbit Space

3.1. Existence

Theorem 3.1 ψ Schwartz

$\operatorname{supp} \hat{\psi} \subseteq [-a_1, a_0] \cup [a_0, a_1] \times [-b_2, b_2] \times \dots \times [-b_d, b_d]$,

$\omega(a,s,t) = \omega(a,s) \Rightarrow$

$$\int_{FSG} |SH_\psi(\psi)(g)|\, \omega(g)\, dg < \infty \qquad (3.1)$$

$\Rightarrow A_\omega$ nontrivial $\Rightarrow$

$$SC_{p,m} := \{ f \in \tilde{\mathcal{H}}_{1,\omega} \mid SH_\psi(f) \in L_{p,m} \} \qquad (3.2)$$

well-defined!

Theorem 3.2 $\operatorname{supp} \psi \subset [-D, D]^d$,

$\omega(a,s,t) = \omega(a) \lesssim |a|^{-\delta_1} + |a|^{\delta_2}$

$$|\hat{\psi}(\xi_1, \tilde{\xi})| \lesssim \frac{|\xi_1|^n}{(1+|\xi_1|)^r} \prod_{k=2}^{d} \frac{1}{(1+|\xi_k|)^r} \qquad (3.3)$$

n, r sufficiently large $\Rightarrow \psi \in B_\omega$!

Remark 3.3 i) $\exists$ compactly supported shearlets!

ii) (3.3) $\Rightarrow$ smoothness + vanishing moments

$\longrightarrow$ admissible shearlet!

3.2. Discretisation

Lemma 3.4 $e \in \mathcal{U} \subset FSG$, $\alpha > 1$, $\beta, \tau > 0$

$$\mathcal{U} \supset [\alpha^{\frac{1}{d}-1}, \alpha^{d}) \times [-\tfrac{\beta}{2}, \tfrac{\beta}{2})^{d-1} \times [-\tfrac{\tau}{2}, \tfrac{\tau}{2})^{d} \quad (3.4)$$

$\Longrightarrow$

$$X := \{ (\varepsilon \alpha^{-j}, \beta \alpha^{-j(1-\frac{1}{d})} k, S_{\beta\alpha^{-j(1-\frac{1}{d})}k} A_{\alpha^{-j}} \tau \ell) \; j \in \mathbb{Z}, k \in \mathbb{Z}^{d-1}, \ell \in \mathbb{Z}^{d}, \varepsilon \in \{-1, 1\} \} \quad (3.5)$$

is $\mathcal{U}$-dense

Lemma 3.4, Theorem 3.2, theorem 2.7 $\Longrightarrow$ atomic decomposition + Banach frames for $SC_{p,m}$ exist!

3.3. Density

theorem 3.5

$$\mathcal{S}_0(\mathbb{R}^d) := \{ f \in \mathcal{S}(\mathbb{R}^d) \mid |\hat{f}(\xi)| \le \frac{\xi_1^{2\alpha}}{(1+|\xi|^2)^{2\alpha}} \; \forall \alpha > 0 \} \quad (3.6)$$

is <u>dense</u> in $\mathcal{H}_{m,p}$,

$$m(a,s,t) = m(a,s) = |a|^{-r} \left(\frac{1}{|a|} + |a| + |s| \right)^{n} \quad r \in \mathbb{R}, \; n \ge 0 \quad (3.7)$$

3.4. Embeddings

3.4.1 Embeddings into Coorbit Spaces

$$m(a,s,t) = m(a) = |a|^{-r}, \quad SC_{p,m} =: SC_{p,r} \qquad (3.8)$$

Theorem 3.6 $1 \le p_1 \le p_2 \le \infty$

i.) $SC_{p_1,r} \subset SC_{p_2,r}$

ii.) $\mathcal{G}_p^{r} := SC_{p,\, r+d(\frac{1}{2}-\frac{1}{p})}$

$$\mathcal{G}_{p_1}^{r_1} \subset \mathcal{G}_{p_2}^{r_2}, \quad r_1 - \frac{d}{p_1} = r_2 - \frac{d}{p_2}$$

Proof: Theorem 2.7 $\Rightarrow$

$$\|f\|_{SC_{p_2,r}} \lesssim \Big(\sum_{j\in\mathbb{Z}} d^{j r p_2} \sum_{k,\ell,\varepsilon} |c_\varepsilon(j,k,\ell)|^{p_2} \Big)^{1/p_2}$$

$\ell_{p_1} \subset \ell_{p_2}$

$$\lesssim \Big(\sum_{j\in\mathbb{Z}} d^{j r p_2} \Big(\sum_{k,\ell,\varepsilon} |c_\varepsilon(j,k,\ell)|^{p_1} \Big)^{p_2/p_1} \Big)^{1/p_2}$$

$$\lesssim \Big(\sum_{j\in\mathbb{Z}} d^{j r p_1} \Big(\sum |c_\varepsilon(j,k,\ell)|^{p_1} \Big)^{1/p_1}$$

$$\lesssim \|f\|_{SC_{p_1,r}} \qquad \blacksquare$$

3.4.2 Embedding into Besov spaces

$d = 2$!

Def 3.7

$$SCC_{p,r} := \{ f \in SC_{p,r} \mid \tag{3.9}$$

$$f(x) = \sum_{j,k,e} c(j,k,e)\, \psi_{j,k,e}(x) \mid c(j,k,e) = 0,\ |k| > 2^{\frac{2j}{3}} \}$$

is called generalised cone-adapted shearlet coorbit space

Theorem 3.8 $SCC_{p,r} \subset \dot{B}^{\sigma_1}_{p,p}(\mathbb{R}^2) + \dot{B}^{\sigma_2}_{p,p}(\mathbb{R}^2)$

$$\sigma_1 + \lfloor \sigma_1 \rfloor \leq 2r - \frac{9}{2} + \frac{4}{p} \tag{3.10}$$

$$\sigma_2 - \frac{\lfloor \sigma_2 \rfloor}{2} = r + \frac{3}{2p} + \frac{1}{4} \tag{3.11}$$

3.5 Traces

- What is $SC_{p,r}\big|_{M}$?
- $d = 3$, $M :=$ coordinate plane
- $Tr\big|_{\partial\Omega} : B^{s}_{p,p}(\Omega) \longrightarrow B^{s-1/p}_{p,p}(\partial\Omega)$, $\Omega \subset \mathbb{R}^d$ smooth

$$SCC^{(\eta)}_{p,r} := \{ f \in SC_{p,r} \mid \tag{3.12}$$

$$f = \sum_{j,k,e} c(j,k,e)\, \psi_{j,k,e},\ c(j,k,e) = 0,\ |k_i| > d^{\frac{2}{3}j},\ \eta_i = 1,\ \eta \in \{0,1\} \}$$

Theorem 3.9

$$Tr_{x_1}\left(SCC^{(1,1)}_{p,r}(\mathbb{R}^3)\right) \subset \dot{B}^{\sigma_1}_{p,p}(\mathbb{R}^2) + \dot{B}^{\sigma_2}_{p,p}(\mathbb{R}^2) \tag{3.13}$$

$$\sigma_1 + 2\lfloor \sigma_1 \rfloor = 3r - \frac{21}{2} + \frac{8}{p} \tag{3.14}$$

$$\sigma_2 = 3r - 3 + \frac{2}{p} \tag{3.15}$$

Théorème 3.10

$$Tr_{x_3}\left(SCC^{(0,1)}_{p,r}(\mathbb{R}^3)\right) \subset SC_{p,r_1}(\mathbb{R}^2) + SC_{p,r_2}(\mathbb{R}^2) \quad (3.16)$$

$$r_1 = r - \frac{5}{6} + \frac{2}{3p} \quad (3.17)$$

$$r_2 = r - \frac{1}{6} \quad (3.18)$$