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Coherent states, (discrete) frames and sampling on manifolds: Theory

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Index

Whittaker-Shannon Theorem

If f is a bandlimited function, $f \in U_T$, i.e., the support of its Fourier transform $\hat{f}$ is contained in $[-\omega_N, \omega_N]$, and if $t_n \equiv nT$, $\forall n \in \mathbb{Z}$, $\omega_N \equiv \frac{\pi}{T}$; then f can be reconstructed from its samples

$$f(t) = \sum_{n=-\infty}^{+\infty} f(t_n) h_T(t - t_n),$$

with

$$h_T(t) = sinc(\omega_N t) \equiv \frac{\sin(\omega_N t)}{\omega_N t}.$$

- If f is not bandlimited ($f \notin U_T$), then the previous formula provide us with a function $\check{f} \in U_T$ minimizing $\|\check{f} - f\|$. $\check{f}$ is the orthogonal proyection $P_{U_T} f$ of f over U_T.
- Whittaker-Shannon sampling theorem can be generalized to other spaces $W_{\mathcal{Q}}$, such that $f \in W_{\mathcal{Q}}$ can be recovered from the sampled values $\{f(q_n),\ q_n \in \mathcal{Q}\}$. A signal $f \notin W_{\mathcal{Q}}$ can be aproximated by its orthogonal projection $\check{f} = P_{W_{\mathcal{Q}}} f$ over $W_{\mathcal{Q}}$.

Coherent States. Basic ingridients

Let G a group of "movements". For instance:

- Affine Group $G = \mathbb{R}^+ \times \mathbb{R}$ acting on $\mathbb{R}$:

$$\boxed{x \mapsto gx = ax + b,\ x \in \mathbb{R}, g = (a, b) \in G = \mathbb{R}^+ \times \mathbb{R}}$$

- Rotation Group $G = SO(3)$ acting on $\mathbb{R}^3$

$$\boxed{\vec{x} \mapsto g\vec{x} = \vec{x}',\ \vec{x} \in \mathbb{R}^3, g = R(\alpha, \beta, \gamma) \in G = SO(3)}$$

(α, β, γ) Euler angles.

- Let $\mathcal{H}$ be a Hilbert space of "finite energy signals" ψ and let

$$\begin{aligned} U \quad : \quad & G \to \mathrm{Lin}(\mathcal{H}) \\ & g \mapsto U(g) \end{aligned}$$

be a unitary and irreducible representation of G in $\mathcal{H}$:

$$U(gg') = U(g)U(g'),\ U(g^{-1}) = U^\dagger(g)$$

- Let consider the Hilbert space

$$\boxed{L^2(G, dg) = \{\Psi : G \to \mathbb{C} \,/\, \int_G |\Psi(g)|^2 dg < \infty\}},$$

where $d(g'g) = dg$ is the left invariant Haar measure.

Admissible Vector

Admissible Vector: A non-zero function $\gamma \in \mathcal{H}$ is an admissible (or ”fiducial vector“) if:

$$\Gamma(g) \equiv \langle U(g)\gamma|\gamma\rangle \in L^2(G, dg).$$

That is, if

$$C_\gamma = (\Gamma(g), \Gamma(g)) = \int_G \bar{\Gamma}(g)\Gamma(g)dg = \int_G |\langle U(g)\gamma|\gamma\rangle|^2 dg < \infty \ .$$

Coherent States

Coherent States: Given a unitary and irreducible representation U of G an a nonzero function $\gamma \in \mathcal{H}$ admissible, a system of Coherent States (CS) in $\mathcal{H}$ associated with G is defined as the set of functions in the orbit of γ under G:

$$\gamma_g = U(g)\gamma, \ g \in G.$$

It could well happen that γ is invariant under a nontrivial subgroup $H \subset G$, i.e., $U(h)\gamma = \gamma, \forall h \in H$ (usually up to phase). In these cases, to avoid redundancy, we introduce the concept of "admissibility modulo H".

Coherent States II

It could also happen that there does not exist admissible vectors since

$$\int_G |\langle U(g)\gamma|\gamma\rangle|^2 dg = \infty \,, \forall\gamma \in \mathcal{H}$$

(for instance, for the continuous series of representations of a non-compact semisimple group). In these cases, we can still define a set of coherent states by restricting ourselves to a quotient space G/H, with H a suitable subgroup of G.

Admissibility $mod(H, \sigma)$

- Consider the homogeneous space $Q = G/H$, with H a closed subgroup. Then the nonzero function γ is admissible $mod(H, \sigma)$ (with $\sigma : Q \longrightarrow G$, a Borel section), and the representation U is square integrable $mod(H, \sigma)$, if the condition

$$0 < \int_Q |\langle U(\sigma(q))\gamma|\psi\rangle|^2 dq < \infty, \ \forall \psi \in \mathcal{H},$$

  holds, where dq is a quasi-invariant measure on Q.
- Coherent states *indexed* by Q:

$$\gamma_{\sigma(q)} = U(\sigma(q))\gamma, \ q \in Q, \ \textit{over-complete} \text{ set in } \mathcal{H}.$$

Resolution operator

- The frame or resolution operator $A_\sigma = \int_Q |\gamma_{\sigma(q)}\rangle\langle\gamma_{\sigma(q)}| dq$ is positive, bounded and invertible.

$$0 < \int_Q |\langle U(\sigma(q))\gamma|\psi\rangle|^2 dq = \langle\psi|A_\sigma|\psi\rangle < \infty, \quad \forall\psi \in \mathcal{H}.$$

- If the operator A_σ^{-1} is bounded, then the set $S_\sigma = \{|\gamma_{\sigma(q)}\rangle,\ q \in Q\}$ is a *frame*, and a *tight frame* if A_σ is proportional to the identity, $A_\sigma = \lambda I,\ \lambda > 0$.
- We shall restrict to the case where γ generates a *frame* (that is, A_σ^{-1} is bounded).

Sampling operator

- Define the sampling or analysis operator, or generalized Bargmann-Fock (GBF) transform :

$$\begin{aligned} T_\gamma : \quad & \mathcal{H} \longrightarrow L^2(Q, dq) \\ & \psi \longmapsto \Psi_\gamma(q) = (T_\gamma \psi)(q) = \langle \gamma_{\sigma(q)} | \psi \rangle . \end{aligned}$$

  $\langle \gamma_{\sigma(q)} | \psi \rangle$: *wavelet* coefficients o *GBF representation* of ψ.
- T_γ is unitary from $\mathcal{H}$ into $L^2_\gamma(Q, dq) \equiv T_\gamma(\mathcal{H})$ (GBF space) which is a Reproducing Kernel Hilbert space with *reproducing kernel* $\langle \gamma_{\sigma(q)} | \gamma_{\sigma(q')} \rangle \equiv B(q, q')$.

Reconstruction Formula

- The inverse map T_γ^{-1} provide us with the reconstruction formula. Given $\Psi_\gamma \in L^2_\gamma(Q, dq)$:

$$A_\sigma|\psi\rangle = \int_Q |\gamma_q\rangle\langle\gamma_q|\psi\rangle dq = \int_Q \Psi_\gamma(q)|\gamma_q\rangle dq \Longrightarrow$$

$$A_\sigma^{-1} A_\sigma|\psi\rangle = |\psi\rangle = T_\gamma^{-1}\Psi_\gamma = \int_Q \Psi_\gamma(q) A_\sigma^{-1}|\gamma_{\sigma(q)}\rangle dq\,,$$

- This formula expands the signal ψ in terms of the *dual frame* $\tilde{S}_\sigma = \{A_\sigma^{-1}|\gamma_{\sigma(q)}\rangle, q \in Q\}$ with coefficients $\Psi_\gamma(q) = (T_\gamma\psi)(q)$.
- These expressions acquire a simpler form when A_σ is proportional to the identity operator (*tight frame*).

Discrete *Frame* or *Resolution* operator

- For numerical treatment, the *resolution* operator A_σ is *discretized*, by restricting the integral to a sum over a discrete subset $\mathcal{Q} \subset Q$:

$$A_\sigma = \int_Q |\gamma_{\sigma(q)}\rangle\langle\gamma_{\sigma(q)}| dq \longrightarrow \quad \mathcal{A} = \sum_{q_k \in \mathcal{Q}} |q_k\rangle\langle q_k|,$$
$$\mathcal{S} = \{|q_k\rangle \equiv |\gamma_{\sigma(q_k)}\rangle,\ q_k \in \mathcal{Q}\}$$
$$\mathcal{H}^{\mathcal{S}} = Span(\mathcal{S}).$$

- In general, the operator $\mathcal{A}$ does not coincide with the original A_σ, and $\mathcal{H}^S \neq \mathcal{H}$ (although there are important cases where it does).

Admissibility and *Frame* condition

- The nonzero function γ is *admissible* if:

$$0 < \sum_{q_k \in \mathcal{Q}} |\langle q_k|\psi\rangle|^2 < \infty\,, \quad \forall \psi \in \mathcal{H}.$$

In this case $\mathcal{A}$ is positive, bounded and invertible.

- The set $\mathcal{S}$ is a *frame* if there exist $0 < b \leq B < \infty$ such that:

$$b\|\psi\|^2 \leq \sum_{q_k \in \mathcal{Q}} |\langle q_k|\psi\rangle|^2 \leq B\|\psi\|^2, \quad \forall \psi \in \mathcal{H}.$$

$$\text{i.e.} \quad 0 < b \leq \frac{\langle\psi|\mathcal{A}|\psi\rangle}{\langle\psi|\psi\rangle} \leq B < \infty\,, \quad \forall \psi \in \mathcal{H}$$

In this case $\mathcal{A}^{-1}$ is also bounded, and $\mathcal{H}^S = \mathcal{H}$.

Sampling and synthesis operators

- The sampling operator $\mathcal{T}$ is now:

$$\begin{aligned} \mathcal{T}: \quad & \mathcal{H} \longrightarrow \ell^2 \\ & \psi \longmapsto \mathcal{T}(\psi) = \{\langle q_k|\psi\rangle,\ q_k \in \mathcal{Q}\}\,. \end{aligned}$$

- $\mathcal{T}^* : \ell^2 \longrightarrow \mathcal{H}$ is the *synthesis* operator.
- It turns out that $\mathcal{A} \equiv \mathcal{T}^*\mathcal{T}$.
- The frame condition can be written as:

$$bI \leq \mathcal{T}^*\mathcal{T} \leq BI,$$

where I is the identity operator in $\mathcal{H}$.

Reconstruction formula for the discrete case

- *dual Frame* : $\tilde{\mathcal{S}} = \{|\tilde{q}_k\rangle \equiv \mathcal{A}^{-1}|q_k\rangle \, , \, q_k \in \mathcal{Q}\}$
- Reconstruction formula:

$$\boxed{|\psi\rangle = \sum_{q_k \in \mathcal{Q}} \Psi_k |\tilde{q}_k\rangle,}$$

with $\Psi_k \equiv \langle q_k|\psi\rangle$: *wavelet* coefficients o data.

- The *reproducing kernel* property of the GBF space allows to identify:

$$\text{Data} \quad \Psi(z_k) \quad = \quad \langle z_k|\psi\rangle \quad \textit{wavelet} \text{ coefficients}$$

Resolution of the identity

- *Resolution of the identity*:

$$\mathcal{T}_l^{+}\mathcal{T} = \sum_{q_k \in \mathcal{Q}} |\tilde{q}_k\rangle\langle q_k| = \mathcal{T}^*(\mathcal{T}_l^{+})^* = \sum_{q_k \in \mathcal{Q}} |q_k\rangle\langle \tilde{q}_k| = I$$

  where $\mathcal{T}_l^{+} \equiv (\mathcal{T}^*\mathcal{T})^{-1}\mathcal{T}^*$ is the left-pseudoinverse of $\mathcal{T}$.

- The operator $P = \mathcal{T}\mathcal{T}_l^{+}$ acting on ℓ^2 is an orthogonal projector into the range of $\mathcal{T}$.

sinc-type function ($\Xi_k(q)$)

- In the GBF space, the reconstruction formula reads:

$$\Psi(q) \equiv \langle q|\psi\rangle = \sum_{q_k \in \mathcal{Q}} \langle q|\tilde{q}_k\rangle \Psi_k \equiv \sum_{q_k \in \mathcal{Q}} \Xi_k(q)\Psi_k$$

- This is a *sinc*-type reconstruction formula, with *sinc*-type function $\Xi_k(q) = \langle q|\tilde{q}_k\rangle$.

- The formulas obtained correspond to *Oversampling*, when we have more data than necessary to exactly recover the original function.

Undersampling

- The case when there are not enough data to fully reconstruct the original signal is named *Undersampling*. In this case only a partial reconstruction is possible.

- $\mathcal{S}$ does not generate a discrete *frame*, and the operator $\mathcal{A} = \mathcal{T}^*\mathcal{T}$ is not invertible. But we can buid another operator from $\mathcal{T}$:
$$\mathcal{B} = \mathcal{T}\mathcal{T}^*.$$
- If the subset $\mathcal{S}$ is free (made of linearly independent vectors), then $\mathcal{B}$ is invertible.
- Discrete *Reproducing kernel* ($\mathcal{B}$):

$$\mathcal{B}_{kl} = \langle q_k | q_l \rangle \quad \text{Gram Matrix.}$$

Undersampling II

- We need the Right-Pseudoinverse of $\mathcal{T}$:
  $\mathcal{T}_r^+ \equiv \mathcal{T}^*(\mathcal{T}\mathcal{T}^*)^{-1} \Longrightarrow \mathcal{T}\mathcal{T}_r^+ = I_{\ell^2}$.

- $P_{\mathcal{S}} = \mathcal{T}_r^+\mathcal{T}$ is the orthogonal projector onto the subspace $\mathcal{H}^S$.

- Dual *pseudo-frame* : $|\tilde{q}_k\rangle = \sum_{q_l \in \mathcal{Q}} \mathcal{B}_{lk}^{-1} |q_l\rangle$.

Resolution of the proyector $P_{\mathcal{S}}$

The dual pseudo-frame provides a *resolution* of the projector $P_{\mathcal{S}}$.

$$\mathcal{T}_r^{+}\mathcal{T} = \sum_{q_k \in \mathcal{Q}} |\tilde{q}_k\rangle\langle q_k| = \mathcal{T}^*(\mathcal{T}_r^{+})^* = \sum_{q_k \in \mathcal{Q}} |q_k\rangle\langle\tilde{q}_k| = P_{\mathcal{S}}.$$

Partial reconstruction $\check{\psi}$ of the signal ψ

- Using the resolution of the projector $P_{\mathcal{S}}$, acting on the signal ψ on the GBF space:
- $|\check{\psi}\rangle = P_{\mathcal{S}}|\psi\rangle$

$$\check{\Psi}(q) \equiv \langle q|\check{\psi}\rangle = \sum_{q_k\in\mathcal{Q}} \langle q|\tilde{q}_k\rangle \Psi_k \equiv \sum_{q_k\in\mathcal{Q}} L_k(q)\Psi_k.$$

- where $L_k(q)$ are Lagrange-type interpolating functions: $L_k(q) = \langle q|\tilde{q}_k\rangle,\ L_k(q_l) = \delta_{kl}$.
- The quadratic error is: $E_\psi(\mathcal{S})^2 = \frac{\|\psi - \check{\psi}\|^2}{\|\psi\|^2} = \frac{\langle\psi|I - P_{\mathcal{S}}|\psi\rangle}{\|\psi\|^2}$.

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